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A note on singular integrals with angular integrability

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Abstract

In this note we study the rough singular integral

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy,$$

where $n \geq 2$ and Ω is a function in $L\log L(\mathrm{S}^{n-1})$ with vanishing integral. We prove that T_Ω is bounded on the mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$, on the vector-valued mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n,\ell^{\tilde{p}})$ and on the vector-valued function spaces $L^p(\mathbb{R}^n,\ell^{\tilde{p}})$ if $1<\tilde{p}\leq p<\tilde{p}n/(n-1)$ or $\tilde{p}n/(\tilde{p}+n-1)<p\leq\tilde{p}<\infty$. The same conclusions hold for the well-known Riesz transforms and directional Hilbert transforms. It should be pointed out that our proof is based on the Calderón–Zygmund's rotation method.

MSC: Primary 42B20; secondary 42B25

Keywords: Singular integrals; Rough kernels; Directional Hilbert transforms; Riesz transforms; Mixed radial-angular spaces

1 Introduction

Singular integral theory was initiated in the seminal work of Calderón and Zygmund [1] and since then has been an active area of research. A celebrated work was due to Calderón and Zygmund [2] who first studied the rough singular integral

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy, \tag{1.1}$$

where Ω is a function in $L\log L(\mathrm{S}^{n-1})$ with vanishing integral,

$$\int_{\mathrm{S}^{n-1}} \Omega(y)\,d\sigma(y) = 0, \tag{1.2}$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ $(n\geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. By introducing the "method of rotations", Calderón and Zygmund [2] showed that T_Ω is bounded on the Lebesgue spaces $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Here the function class $L\log L(\mathrm{S}^{n-1})$ denotes the set of all functions $\Omega:\mathrm{S}^{n-1}\to\mathbb{R}$ which satisfy

$$\|\Omega\|_{L\log L(\mathrm{S}^{n-1})} := \int_{\mathrm{S}^{n-1}} \bigl|\Omega(\theta)\bigr|\log\bigl(2+\bigl|\Omega(\theta)\bigr|\bigr)\,d\sigma(\theta) < \infty.$$

The same conclusion was obtained independently by Coifman and Weiss [3] and Connett [4] under the less restrictive condition that Ω lies in the Hardy space $H^1(\mathrm{S}^{n-1})$. The weak type $(1,1)$ bounds of T_Ω were proved by many authors under the condition that $\Omega \in L\log L(\mathrm{S}^{n-1})$ (see [5, 6]). For other developments on this topic we can consult [7–15], among others.

It is well known that the mixed radial-angular space $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ is merely a formal extension of the Lebesgue space L^p, but over the last several years it has been successfully used in studying Strichartz estimates and dispersive equations (see [16–28]). Recall that the mixed radial-angular spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$, $1 \le p, \tilde{p} \le \infty$, consist of all functions u satisfying $\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} := \left(\int_0^\infty \|u(\rho\cdot)\|^p_{L^{\tilde{p}}(\mathrm{S}^{n-1})}\rho^{n-1}\,d\rho\right)^{1/p} \quad \text{and}$$

$$\|u\|_{L^\infty_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} := \sup_{\rho>0}\|u(\rho\cdot)\|_{L^{\tilde{p}}(\mathrm{S}^{n-1})}.$$

It is clear that the spaces $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ have the following easy properties.

(i) If $p = \tilde{p}$ and $1 \le p \le \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}. \tag{1.3}$$

(ii) If u is a radial function on $\mathbb{R}^n$ and $1 \le p, \tilde{p} \le \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \simeq \|u\|_{L^p(\mathbb{R}^n)}.$$

(iii) If $1 \le \tilde{p}_1 \le \tilde{p}_2 \le \infty$ and $1 \le p \le \infty$, then

$$\|u\|_{L^p_{|x|}L^{\tilde{p}_1}_\theta(\mathbb{R}^n)} \le C_{n,p,\tilde{p}_1,\tilde{p}_2}\|u\|_{L^p_{|x|}L^{\tilde{p}_2}_\theta(\mathbb{R}^n)}.$$

Here the notation $A \simeq B$ means that there are two positive constants C, C' such that $A \le CB$ and $B \le C'A$.

Recently the mixed radial-angular spaces also played an active role in singular integral theory. A good start in this direction was due to Córdoba [29] who proved that T_Ω is bounded on $L^p_{|x|}L^2_\theta(\mathbb{R}^n)$ for all $1 < p < \infty$, provided that $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$. Later on, D'Ancona and Lucà [30] used the same argument in [29, Theorem 2.1] to extend the above result to the following.

Theorem A ([30]) *Let $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$ satisfy (1.2) and $1 < p, \tilde{p} < \infty$. Then*

$$\|T_\Omega f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{\Omega,p,\tilde{p}}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}.$$

Very recently, Cacciafesta and Lucà [31] extended Theorem A to the weighted setting (see [31, Theorem 1.1]). It should be pointed out that

$$\mathcal{C}^1(\mathrm{S}^{n-1}) \subsetneq L\log L(\mathrm{S}^{n-1}) \subsetneq H^1(\mathrm{S}^{n-1}) \subsetneq L^1(\mathrm{S}^{n-1}). \tag{1.4}$$

The main focus of the current note is to consider the $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ boundedness of T_Ω without assuming that Ω is in $\mathcal{C}^1(\mathrm{S}^{n-1})$ with mean value zero. Actually, we want to improve Theorem A to $\Omega \in L\log L(\mathrm{S}^{n-1})$. To be more precisely, our main result can be formulated as follows.

Theorem 1.1 *Let $\Omega \in L\log L(\mathrm{S}^{n-1})$ and satisfy (1.2). If $1 < \tilde{p} \le p < \tilde{p}n/(n-1)$ or $\tilde{p}n/(\tilde{p}+n-1) < p \le \tilde{p} < \infty$, then the following are valid*:

$$\|T_\Omega\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{p,\tilde{p},\Omega}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}; \tag{1.5}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}|T_\Omega f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{p,\tilde{p},\Omega}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}; \tag{1.6}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}|T_\Omega f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)} \le C_{p,\tilde{p},\Omega}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)}. \tag{1.7}$$

We would like to remark that Theorem 1.1 is based on the Calderón–Zygmund rotation method. In order to prove Theorem 1.1, let us introduce the direction Hilbert transforms and Riesz transforms. For a $w \in \mathbb{R}^n$, we define the directional Hilbert transform $\mathcal{H}_w$ in the direction w as

$$\mathcal{H}_w f(x) = \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} f(x-tw)\frac{dt}{t}, \tag{1.8}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class on $\mathbb{R}^n$). For $1 \le j \le n$, the jth Riesz transform is given by

$$\mathcal{R}_j f(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\mathrm{p.v.}\int_{\mathbb{R}^n} f(x-y)\frac{y_j}{|y|^n}\,dy, \tag{1.9}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$.

Recently, Córdoba [29] proved the following.

Theorem B ([29]) *Let $w \in \mathrm{S}^{n-1}$. Then $\mathcal{H}_w$ is bounded on $L^p_{|x|}L^2_\theta(\mathbb{R}^n)$ if and only if $2n/(n+1) < p < 2n/(n-1)$.*

In this paper we shall extend Theorem B to the following.

Theorem 1.2 *Let $w \in \mathrm{S}^{n-1}$. Then $\mathcal{H}_w$ defined as (1.8) is bounded on $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ if and only if $1 < \tilde{p} \le p < \tilde{p}n/(n-1)$ or $\tilde{p}n/(\tilde{p}+n-1) < p \le \tilde{p} < \infty$. Moreover, the following are valid*:

$$\|\mathcal{H}_w f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{p,\tilde{p}}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}; \tag{1.10}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{H}_w f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{p,\tilde{p}}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}, \tag{1.11}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{H}_w f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)} \le C_{p,\tilde{p}}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)}. \tag{1.12}$$

The above constants $C_{p,\tilde{p}}$ are independent of w.

Theorem 1.2 together with the rotation method yields the following.

Theorem 1.3 *Let Ω be odd and integrable over S^{n-1}. If $1 < \tilde{p} \le p < \tilde{p}n/(n-1)$ or $\tilde{p}n/(\tilde{p}+n-1) < p \le \tilde{p} < \infty$, then the following are valid*:

$$\|T_\Omega\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{p,\tilde{p}}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}; \tag{1.13}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}|T_\Omega f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{p,\tilde{p}}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}, \tag{1.14}$$

$$\left\|\left(\sum_{j\in\mathbb{Z}}|T_\Omega f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)} \le C_{p,\tilde{p}}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)}. \tag{1.15}$$

Here the above constants $C_{p,\tilde{p}} > 0$ are independent of Ω. The same conclusions hold for the Riesz transforms $\mathcal{R}_j$ for all $1 \le j \le n$.

Remark 1.1 When $\tilde{p} = 2$, the part result of Theorem 1.2 implies Theorem A. On the other hand, Córdoba [29] proved the following, Meyer's lemma: Given a countable family of directions $\{\theta_j\}_{j\in\mathbb{Z}}$ in $\mathbb{R}^n$ and set $\mathcal{H}_j f = \mathcal{H}_{\theta_j} f$. Then the following inequality holds:

$$\left\|\left(\sum_{j\in\mathbb{Z}}|H_j f_j|^2\right)^{1/2}\right\|_{L^p_{|x|}L^2_\theta(\mathbb{R}^n)} \le C_p\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^2\right)^{1/2}\right\|_{L^p_{|x|}L^2_\theta(\mathbb{R}^n)} \tag{1.16}$$

for $2n/(n+1) < p < 2n/(n-1)$. By using the arguments as in deriving (1.11) and (1.12), we find that if $1 < \tilde{p} \le p < \tilde{p}n/(n-1)$ or $\tilde{p}n/(\tilde{p}+n-1) < p \le \tilde{p} < \infty$, the following inequalities hold:

$$\begin{aligned}&\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{H}_j f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \le C_{p,\tilde{p}}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)},\\ &\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{H}_j f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)} \le C_{p,\tilde{p}}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^{\tilde{p}}\right)^{1/\tilde{p}}\right\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{1.17}$$

It is clear that (1.17) yields (1.16) when $\tilde{p} = 2$.

Throughout the paper, we use $C_{\alpha,\beta,\ldots}$ to denote positive constants that depend on the parameters $\alpha, \beta, \ldots$.

2 Proofs of main results

Let us begin with the proof of Theorem 1.2.

Proof of Theorem 1.2 We only prove (1.10) since (1.11) and (1.12) are analogous. We shall adopt the method of deriving the proof in [30, Theorem 2.6] to prove (1.8). Let $1 < \tilde{p} < p < \tilde{p}n/(n-1)$ and $t = p/(p-\tilde{p})$. It is obvious that $t > n$. Fix a number s in the interval $(1, t/n)$. Denote by X the set of all $g \in \mathcal{S}(\mathbb{R})$ with $\int_0^\infty g^t(r)r^{n-1}\,dr \le 1$. By polar coordinates, we have

$$\begin{aligned} \|\mathcal{H}_w\|^{\tilde{p}}_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} &= \left(\int_0^\infty \left(\int_{S^{n-1}} \left|\mathcal{H}_w f(r\theta)\right|^{\tilde{p}}\, d\sigma(\theta)\right)^{p/\tilde{p}} r^{n-1}\, dr\right)^{\tilde{p}/p} \\ &= \sup_{g\in X} \int_0^\infty \int_{S^{n-1}} \left|\mathcal{H}_w f(r\theta)\right|^{\tilde{p}} g(r) r^{n-1}\, d\sigma(\theta)\, dr \\ &= \sup_{g\in X} \int_{\mathbb{R}^n} \left|\mathcal{H}_w f(x)\right|^{\tilde{p}} g\big(|x|\big)\, dx. \end{aligned} \tag{2.1}$$

Fix $g \in X$ and set $h(x) = g(|x|)$. It is well known that

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\mathcal{H}_w f(x)\right|^{\tilde{p}} g\big(|x|\big)\, dx &\le C_{\tilde{p},s} \int_{\mathbb{R}^n} \left|f(x)\right|^{\tilde{p}} \left(\mathcal{M}_w h^s(x)\right)^{1/s} dx \\ &\le C_{\tilde{p},s} \int_{\mathbb{R}^n} \left|f(x)\right|^{\tilde{p}} \left(\mathcal{U} h^s(x)\right)^{1/s} dx \end{aligned} \tag{2.2}$$

for all $p \in (1,\infty)$ and $s \in (1,\infty)$. Here $\mathcal{M}_w$ denotes the one-dimensional Hardy–Littlewood maximal function in the direction of w and $\mathcal{U}$ is the universal Kakeya maximal function defined by

$$\mathcal{U} f(x) = \sup_{\substack{a,b>0\\ w\in S^{n-1}}} \frac{1}{a+b} \int_{-a}^{b} \left|f(x+tw)\right|\, dt.$$

It was shown in [32] (also see [29]) that if f is a radial function, then

$$\|\mathcal{U} f\|_{L^\nu(\mathbb{R}^n)} \le C_\nu \|f\|_{L^\nu(\mathbb{R}^n)}, \quad \text{for } \nu > n. \tag{2.3}$$

Notice that $t/s > n$ and h^s is a radial function. It follows from (2.2)–(2.3) that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left|\mathcal{H}_w f(x)\right|^{\tilde{p}} g\big(|x|\big)\, dx \\ &\quad\le C_{\tilde{p},s} \int_0^\infty \int_{S^{n-1}} \left|f(r\theta)\right|^{\tilde{p}}\, d\sigma(\theta) \big(\mathcal{U} h^s(r)\big)^{1/s} r^{n-1}\, dr \\ &\quad\le C_{\tilde{p},s} \int_0^\infty \left(\int_{S^{n-1}} \left|f(r\theta)\right|^{\tilde{p}}\, d\sigma(\theta)\right)^{p/\tilde{p}} r^{n-1}\, d\rho)^{\tilde{p}/p} \left(\int_0^\infty \big(\mathcal{U} h^s(r)\big)^{t/s} r^{n-1}\, dr\right)^{1/t} \\ &\quad\le C_{\tilde{p},s} \|f\|^{\tilde{p}}_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \left\| \big(\mathcal{U} h^s\big)^{1/s} \right\|_{L^t(\mathbb{R}^n)} \\ &\quad\le C_{p,\tilde{p},s} \|f\|^{\tilde{p}}_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}, \end{aligned}$$

which together with (2.1) implies that (1.8) holds for $1 < \tilde{p} < p < \tilde{p}n/(n-1)$. By the duality we get the case $\tilde{p}n/(\tilde{p}+n-1) < p < \tilde{p} < \infty$. The trivial case $1 < p = \tilde{p} < \infty$ follows easily from the L^p bounds for $\mathcal{H}_w$ and (1.3).

To prove the "only if" part we take $f = \chi_{B(0,1)}$, where $B(0,1)$ is the unit cube in $\mathbb{R}^n$. Without loss of generality we only consider the case $w = (0,0,\dots,0,1)$ because of the rotational symmetry. One can easily check that

$$\left|\mathcal{H}_w f(x_1,x_2,\dots,x_{n-1},x_n)\right| \ge C\frac{1}{|x_n|},$$

whenever $|x_i| \le \frac{1}{2}$, $i = 1, 2, \dots, n-1$ and $|x_n| \ge 2$. An elementary computation yields

$$\begin{aligned}&\int_0^\infty \left(\int_{S^{n-1}} \left|\mathcal{H}_w f(r\theta)\right|^{\tilde{p}} d\sigma(\theta)\right)^{p/\tilde{p}} r^{n-1}\,dr \\ &\quad \ge C\int_2^\infty \left(r^{-\tilde{p}} r^{-(n-1)}\right)^{p/\tilde{p}} r^{n-1}\,dr \ge C\int_2^\infty r^{(n-1)(1-p/\tilde{p})-p}\,dr,\end{aligned}$$

which together with the $L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)$ boundedness for $\mathcal{H}_w$ yields $\tilde{p}n/(\tilde{p}+n-1) < p \le \tilde{p} < \infty$. The case $1 < \tilde{p} \le p < \tilde{p}n/(n-1)$ follows by duality. □

Proof of Theorem 1.3 We shall prove (1.13) and (1.14)–(1.15) are analogous. By the method of rotations, it was shown in [33] that

$$T_\Omega f(x) = \frac{\pi}{2}\int_{S^{n-1}} \Omega(w)\mathcal{H}_w f(x)\,d\sigma(w). \tag{2.4}$$

By (2.4) and Minkowski's inequalities, one has

$$\begin{aligned}&\|T_\Omega f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)} \\ &\quad = \left(\int_0^\infty \left(\int_{S^{n-1}} \left|T_\Omega f(r\theta)\right|^{\tilde{p}} d\sigma(\theta)\right)^{p/\tilde{p}} r^{n-1}\,dr\right)^{1/p} \\ &\quad = \frac{\pi}{2}\left(\int_0^\infty \left(\int_{S^{n-1}} \left|\int_{S^{n-1}} \Omega(w)\mathcal{H}_w f(r\theta)\,d\sigma(w)\right|^{\tilde{p}} d\sigma(\theta)\right)^{p/\tilde{p}} r^{n-1}\,dr\right)^{1/p} \\ &\quad \le \frac{\pi}{2}\left(\int_0^\infty \left(\int_{S^{n-1}} \left|\Omega(w)\right|\left(\int_{S^{n-1}} \left|\mathcal{H}_w f(r\theta)\right|^{\tilde{p}} d\sigma(\theta)\right)^{1/\tilde{p}} d\sigma(w)\right)^{p} r^{n-1}\,dr\right)^{1/p} \\ &\quad \le \frac{\pi}{2}\int_{S^{n-1}} \left|\Omega(w)\right|\left(\int_0^\infty \left(\int_{S^{n-1}} \left|\mathcal{H}_w f(r\theta)\right|^{\tilde{p}} d\sigma(\theta)\right)^{p/\tilde{p}} r^{n-1}\,dr\right)^{1/p} d\sigma(w) \\ &\quad \le C_{p,\tilde{p}}\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p_{|x|}L^{\tilde{p}}_\theta(\mathbb{R}^n)}\end{aligned}$$

if $1 < \tilde{p} \le p < \tilde{p}n/(n-1)$ or $\tilde{p}n/(\tilde{p}+n-1) < p \le \tilde{p} < \infty$. This finishes the proof of Theorem 1.3. □

Proof of Theorem 1.1 We first prove that the conclusions of Theorem 1.1 hold for T_Ω if Ω is an even function and $\Omega \in L\log L(S^{n-1})$ satisfies (1.2). By [33, Proposition 4.1.16], we obtain

$$T_\Omega = -\sum_{j=1}^n \mathcal{R}_j\mathcal{R}_j T_\Omega, \tag{2.5}$$

where $\mathcal{R}_j$ is the jth Riesz transform defined by (1.9). Let $T_j = \mathcal{R}_j T_\Omega$. Fix $1 \le j \le n$. By the idea in Grafakos's book [33, pp. 274–278], we see that there exists an odd integrable kernel Ω_j such that

$$T_j = T_{\Omega_j}. \tag{2.6}$$

We get from (2.5) and (2.6)

$$T_\Omega = -\sum_{j=1}^{n} R_j T_{\Omega_j}. \tag{2.7}$$

Then (1.5)–(1.7) follow easily from Theorem 1.3 and (2.7).

Let Ω be given as in Theorem 1.1. We can simply write $\Omega = \Omega_e + \Omega_o$, where $\Omega_e(x) = \frac{\Omega(x)+\Omega(-x)}{2}$ and $\Omega_o(x) = \frac{\Omega(x)-\Omega(-x)}{2}$. Then T_Ω can be written as $T_\Omega = T_{\Omega_e} + T_{\Omega_o}$. One can easily check that Ω_e is even and $\Omega_e \in L\log L(\mathrm{S}^{n-1})$ satisfies (1.2). Ω_o is odd and $\Omega_o \in L^1(\mathrm{S}^{n-1})$. Applying the proved claim for T_Ω with even kernel Ω and Theorem 1.3, we get (1.5)–(1.7). □

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Competing interests
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Authors' contributions
The author was the only one to work in drafting and approving the final manuscript.

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