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Generalizations of Troisi's inequality in weighted p-Sobolev spaces with singularities

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Abstract
We extend classical Troisi's inequality to the weighted p-Sobolev spaces on stretched cone, edge, and corner respectively. The results here can be used to investigate anisotropic elliptic equations involving cone degeneracy, edge degeneracy, and corner degeneracy, which will be studied in our forthcoming papers.

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1 Introduction and main results

In 1969, M. Troisi (cf. [1]) found an important inequality. Its classical form can be described as: given $1 \le p_i < \infty$, $i = 1, \dots, n$, for a smooth function u compactly supported in $\mathbb{R}^n$, the following inequality holds:

$$\|u\|_s \le C \prod_{i=1}^{n} \|\partial_{x_i} u\|_{p_i}^{\frac{1}{n}}, \quad \sum_{i=1}^{n} \frac{1}{p_i} > 1, s = \frac{n}{\sum_{i=1}^{n} \frac{1}{p_i} - 1}, \tag{1.1}$$

where $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q \,dx)^{\frac{1}{q}}$ with $1 \le q < \infty$ and C is independent of u. It is the well-known Troisi's inequality that can be used to study the existence of multiple nonnegative solutions to the anisotropic critical problem (cf. [2])

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right) = |u|^{s-2} u, \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $1 < p_i < \infty$ for $i = 1, 2, \dots, n$, $\sum_{i=1}^{n} \frac{1}{p_i} > 1$, $s = n/(\sum_{i=1}^{n} \frac{1}{p_i} - 1)$ is anisotropic critical exponent and $\max_{1 \le i \le n}\{p_i\} < s$. Applications of (1.1) also can be found in [3] to study the existence of fundamental solutions to anisotropic elliptic equations. Another generalization of (1.1) in [4] is used to prove regularity of the weak solution to the Navier–Stokes equations based on one component of velocity. By arithmetic and geometric mean inequality, (1.1) becomes an anisotropic Sobolev inequality presented as

$$\|u\|_s \le \frac{C}{n} \sum_{i=1}^{n} \|\partial_{x_i} u\|_{p_i}, \quad \sum_{i=1}^{n} \frac{1}{p_i} > 1, s = \frac{n}{\sum_{i=1}^{n} \frac{1}{p_i} - 1}. \tag{1.3}$$

In particular, if $p_i = p$ in (1.3) for $i = 1,\dots,n$, then (1.1) finally reduces to the classical Gagliardo–Nirenberg–Sobolev inequality

$$\|u\|_s \le \frac{C}{n}\sum_{i=1}^{n}\|\partial_{x_i}u\|_p, \quad 1 \le p < n, s = \frac{np}{n-p}. \tag{1.4}$$

Here the methods to prove (1.3) and (1.4) are similar as that in Adams and Vancouver [5] by using mixed norms and permutation inequalities and that in Kružkov [6, p. 282] to establish a new proof based on fundamental theorem of calculus.

Motivations of this paper mainly come from the attention to studying the anisotropic elliptic equations as (1.2) with conical singularity, edge singularity, and corner singularity respectively. For instance, (1.2) including conical singularity corresponds to

$$-\sum_{i=1}^{n} D_{c,i}\big(|D_{c,i}u|^{p_i-2}D_{c,i}u\big) = |u|^{s-2}u \quad \text{in } \mathbb{R}_+^{n+1}, \tag{1.5}$$

where $D_{c,0} = t\partial_t$, $D_{c,i} = \partial_{x_i}$, $1 < p_i < \infty$ for $i = 0,1,2,\dots,n$, and anisotropic critical exponent $s := (n+1)/(\sum_{i=0}^{n}\frac{1}{p_i} - 1)$. As indicated above, the anisotropic elliptic equations with singularities of edge type or corner type parallel to (1.2) can be formulated as well.

Considering the pivotal role of Troisi's inequality in studying such kinds of singular anisotropic elliptic equations like (1.5) (e.g., see the results in our upcoming papers), we need, in the first place, to deduce it being of different forms in different weighted p-Sobolev spaces. To be specific, we will generalize (1.1) to some singular weighted p-Sobolev spaces (see Sect. 2 below) in which the usual gradient operator $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$ becomes the cone type, edge type, and corner type gradient operators such as $D_c = (t\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$ in $\mathbb{R}_+^{n+1}$, $\mathrm{D}_e = (t\partial_t, \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}, t\partial_{y_{n+1}}, \dots, t\partial_{y_{n+q}})$ in $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q$, and $\mathrm{D}_{\mathrm{cor}} = (r\partial_r, \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}, rt\partial_t)$ in $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$ respectively. Now we present our main conclusions of this paper as follows.

Theorem 1.1 *Let $\gamma \in \mathbb{R}$, $1 \le p_i < \infty$ for $0 \le i \le n$, and $\sum_{i=0}^{n}\frac{1}{p_i} > 1$. Set $\frac{1}{s} = \frac{1}{n+1}(\sum_{i=0}^{n}\frac{1}{p_i} - 1)$. Then we have the following cone type Troisi's inequality for all $u(t,x) \in C_0^\infty(\mathbb{R}_+^{n+1})$:*

$$\|u\|_{L_s^\gamma} \le \big(c_{01}\|u\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}} + c_{02}\|t\partial_t u\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}}\big)\prod_{i=1}^{n} c_i\|\partial_{x_i}u\|_{L_{p_i}^{\gamma_i^*}}^{\frac{1}{n+1}}, \tag{1.6}$$

where $\|u\|_{L_p^\gamma} = (\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} t^{n+1}|t^{-\gamma}u(t,x)|^p \frac{\mathrm{d}t}{t}\,\mathrm{d}x)^{\frac{1}{p}}$, $\gamma_j^* = -(\frac{n+1}{s} - \gamma - \frac{n+1}{p_j})$ for $0 \le j \le n$, $c_{01} = \frac{1}{2}[(1 + \frac{s(p_0-1)}{p_0})|\frac{n+1}{s} - \gamma|]^{\frac{1}{n+1}}$, $c_{02} = \frac{1}{2}(1 + \frac{s(p_0-1)}{p_0})^{\frac{1}{n+1}}$, and $c_i = (1 + \frac{s(p_i-1)}{p_i})^{\frac{1}{n+1}}$ for $1 \le i \le n$.

Moreover, as a special case, we obtain the following cone type Sobolev inequality which was first proved by [7, Theorem 2.1] in studying Dirichlet problem for nonlinear elliptic boundary value problem on a manifold with conical singularities.

Corollary 1.2 *In addition to the conditions included in Theorem 1.1, if $p_i = p \geq 1$ for $0 \leq i \leq n$, then we have the following cone type Sobolev inequality*:

$$\|u\|_{L_s^{\gamma}} \leq \hat{c}_0 \|u\|_{L_p^{\gamma+1}} + \hat{c}_1 \left(\big\|(t\partial_t)u\big\|_{L_p^{\gamma+1}} + \sum_{i=1}^{n} \|\partial_{x_i} u\|_{L_p^{\gamma+1}} \right),$$

where $\hat{c}_0 = \frac{n|n+1-p(\gamma+1)|}{2(n+1)(n+1-p)}$, $\hat{c}_1 = \frac{np}{2(n+1)(n+1-p)}$, $\frac{1}{s} = \frac{1}{p} - \frac{1}{n+1}$, and $p < n+1$.

Secondly, we consider the following edge type Torisi's inequality.

Theorem 1.3 *Given $1 \leq p_i < \infty$ for $0 \leq i \leq n+q$, and $\sum_{i=0}^{n+q} \frac{1}{p_i} > 1$. Let $\frac{1}{s} = \frac{1}{n+q+1}(\sum_{i=0}^{n+q} \frac{1}{p_i} - 1)$. Then we have the following edge type Troisi's inequality for all $u(t,x,y) \in C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q)$ and $\gamma \in \mathbb{R}$*:

$$\begin{aligned} \|u\|_{\mathcal{L}_s^{\gamma}} \leq{}& \big(c_{01} \|u\|_{\mathcal{L}_{p_0}^{\gamma_0^*}}^{\frac{1}{n+q+1}} + c_{02} \|t\partial_t u\|_{\mathcal{L}_{p_0}^{\gamma_0^*}}^{\frac{1}{n+q+1}}\big) \prod_{i=1}^{n} c_i \|\partial_{x_i} u\|_{\mathcal{L}_{p_i}^{\gamma_i^*}}^{\frac{1}{n+q+1}} \\ &\times \prod_{i=n+1}^{n+q} c_i \big\|(t\partial_{y_i})u\big\|_{\mathcal{L}_{p_i}^{\gamma_i^*}}^{\frac{1}{n+q+1}}, \end{aligned} \tag{1.7}$$

where $\|u\|_{\mathcal{L}_p^{\gamma}} = (\int_{\mathbb{R}_+^N} t^{n+q+1} |t^{-\gamma} u(t,x,y)|^p \frac{\mathrm{d}t}{t}\, \mathrm{d}x \frac{\mathrm{d}y}{t})^{\frac{1}{p}}$, $\gamma_j^* = -(\frac{n+q+1}{s} - \gamma - \frac{n+q+1}{p_j})$ for $0 \leq j \leq n+q$, $c_{01} = \frac{1}{2}[(1 + \frac{s(p_0-1)}{p_0})|\frac{n+q+1}{s} - \gamma|]^{\frac{1}{n+q+1}}$, $c_{02} = \frac{1}{2}(1 + \frac{s(p_0-1)}{p_0})^{\frac{1}{n+q+1}}$, and $c_i = (1 + \frac{s(p_i-1)}{p_i})^{\frac{1}{n+q+1}}$ for $1 \leq i \leq n+q$.

In particular, the following edge type Sobolev inequality can be regarded as a special case of the edge type Troisi's inequality above. This kind of edge type Sobolev inequality was first given by [8, Proposition 3.2] in studying Dirichlet problem for semilinear edge-degenerate elliptic equations.

Corollary 1.4 *Under the assumptions in Theorem 1.3, if $p_i = p \geq 1$ for $0 \leq i \leq n+q$, then we have the following edge type Sobolev inequality*:

$$\|u\|_{\mathcal{L}_s^{\gamma}} \leq \hat{c}_0 \|u\|_{\mathcal{L}_p^{\gamma+1}} + \hat{c}_1 \left(\big\|(t\partial_t)u\big\|_{\mathcal{L}_p^{\gamma+1}} + \sum_{i=1}^{n} \|\partial_{x_i} u\|_{\mathcal{L}_p^{\gamma+1}} + \sum_{i=n+1}^{n+q} \big\|(t\partial_{y_i})u\big\|_{\mathcal{L}_p^{\gamma+1}} \right),$$

where $\hat{c}_0 = \frac{(n+q)|n+q+1-p(\gamma+1)|}{2(n+q+1)(n+q+1-p)}$ and $\hat{c}_1 = \frac{(n+q)p}{2(n+q+1)(n+q+1-p)}$, $\frac{1}{s} = \frac{1}{p} - \frac{1}{n+q+1}$ and $p < n+q+1$.

Finally, we give the corner type Troisi's inequality.

Theorem 1.5 *If $1 \leq p_i < \infty$ for $0 \leq i \leq n+1$, $\sum_{i=0}^{n+1} \frac{1}{p_i} > 1$, $\frac{1}{s} = \frac{1}{n+2}(\sum_{i=0}^{n+1} \frac{1}{p_i} - 1)$ and $\bar{\gamma}, \gamma \in \mathbb{R}$, then the following corner type Troisi's inequality holds for all $u(r,x,t) \in C_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$*:

$$\begin{aligned} \|u\|_{\mathfrak{L}_s^{\bar{\gamma},\gamma}} \leq{}& \big(\hat{c}_0 \|u\|_{\mathfrak{L}_{p_0}^{\bar{\gamma}_0^*,\gamma_0^*}}^{\frac{1}{n+2}} + c_0 \big\|(r\partial_r)u\big\|_{\mathfrak{L}_{p_0}^{\bar{\gamma}_0^*,\gamma_0^*}}^{\frac{1}{n+2}}\big) \prod_{i=1}^{n} c_i \|\partial_{x_i} u\|_{\mathfrak{L}_{p_i}^{\bar{\gamma}_i{}^*,\gamma_i^*}}^{\frac{1}{n+2}} \\ &\times \big(\hat{c}_{n+1} \|u\|_{\mathfrak{L}_{p_{n+1}}^{\bar{\gamma}_{n+1}^*-1,\gamma_{n+1}^*}}^{\frac{1}{n+2}} + c_{n+1} \big\|(rt\partial_t)u\big\|_{\mathfrak{L}_{p_{n+1}}^{\bar{\gamma}_{n+1}^*,\gamma_{n+1}^*}}^{\frac{1}{n+2}}\big), \end{aligned} \tag{1.8}$$

where $\|u\|_{\mathfrak{L}_p^{\gamma_1,\gamma_2}} = (\int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+} |r^{\frac{n+2}{p}-\gamma_1} t^{\frac{n+2}{p}-\gamma_2} u(r,x,t)|^p \frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt})^{\frac{1}{p}}$, $\bar{\gamma}_i^* = -(\frac{n+2}{s} - \bar{\gamma} - \frac{n+2}{p_i})$, $\gamma_i^* = -(\frac{n+2}{s} - \gamma - \frac{n+2}{p_i})$, $c_0 = \frac{1}{2}(1+\frac{s(p_0-1)}{p_0})^{\frac{1}{n+2}}$, $c_i = (1+\frac{s(p_i-1)}{p_i})^{\frac{1}{n+2}}$ *for* $1 \le i \le n+1$, $\hat{c}_0 = \frac{1}{2}[(1+\frac{s(p_0-1)}{p_0})|\frac{n+2}{s} - \bar{\gamma}|]^{\frac{1}{n+2}}$, $\hat{c}_{n+1} = [(1+\frac{s(p_{n+1}-1)}{p_{n+1}})|\frac{n+2}{s} - \gamma|]^{\frac{1}{n+2}}$.

Likewise, it follows from Theorem 1.5 that we can derive the corner type Sobolev inequality as follows, which was first obtained by [9, Proposition 3.1] in studying the existence of multiple solutions for semi-linear corner degenerate elliptic equations.

Corollary 1.6 *Based on Theorem 1.5, further if we choose $p_i = p \ge 1$ for $0 \le i \le n+1$, then we have the following corner type Sobolev inequality*:

$$\begin{aligned}\|u\|_{\mathfrak{L}_s^{\bar{\gamma},\gamma}} &\le \widetilde{\mu}_0\|u\|_{\mathfrak{L}_p^{\bar{\gamma}+1,\gamma+1}} + \mu_0\big\|(r\partial_r)u\big\|_{\mathfrak{L}_p^{\bar{\gamma}+1,\gamma+1}} + \sum_{i=1}^n \mu_i\|\partial_{x_i}u\|_{\mathfrak{L}_p^{\bar{\gamma}+1,\gamma+1}} \\ &\quad + \hat{\mu}_{n+1}\|u\|_{\mathfrak{L}_p^{\bar{\gamma},\gamma+1}} + \mu_{n+1}\big\|(rt\partial_t)u\big\|_{\mathfrak{L}_p^{\bar{\gamma}+1,\gamma+1}},\end{aligned}$$

where $\widetilde{\mu}_0 = \frac{(n+1)|n+2-p(\bar{\gamma}+1)|}{2(n+2)(n+2-p)}$, $\mu_i = \frac{p(n+1)}{2(n+2)(n+2-p)}$ *for* $0 \le i \le n+1$, $\hat{\mu}_{n+1} = \frac{(n+1)|n+2-p(\gamma+1)|}{2(n+2)(n+2-p)}$, $\frac{1}{s} = \frac{1}{p} - \frac{1}{n+2}$ *and* $p < n+2$.

The outline of this paper is as follows. In Sect. 2, we introduce cone type, edge type, and corner type weighted p-Sobolev spaces respectively. Then, in Sect. 3, we give the proof of Theorem 1.1. Finally, the proofs of Theorem 1.3 and Theorem 1.5 will be provided in Sect. 4.

2 Definitions of singular weighted *p*-Sobolev spaces

Let X be a closed compact C^∞ manifold and $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\}\times X)$ be a local model considered as a cone with the base X. In particular, let $X \subset S^n$ be a bounded open set in the unit sphere of $\mathbb{R}^{n+1}$, and the straight cone X^Δ is defined as

$$X^\Delta = \left\{\tilde{x} \in \mathbb{R}^{n+1} : \tilde{x} = 0 \text{ or } \frac{\tilde{x}}{|\tilde{x}|} \in X\right\}.$$

Thus, $X^\wedge = \mathbb{R}_+ \times X$ is called as corresponding open stretched cone with the base X. In local coordinates, $\mathbb{R}_+ \times \mathbb{R}^n$ can be interpreted as an open stretched cone. The typical differential operators, defined on a manifold with conical singularities, are called Fuchs type, i.e.,

$$A = t^{-m}\sum_{k=0}^m a_k(t)(-t\partial_t)^k = t^{-m}A_{\mathbb{X}^\Delta}, \tag{2.1}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $a_k(t) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{m-k}(\mathbb{R}^n))$, $\mathrm{Diff}^j(\mathbb{R}^n)$ refers to the set of differential operators of order j on $\mathbb{R}^n$, and $A_{\mathbb{X}^\Delta}$ are called degenerated cone operators.

Let g be Riemannian metrics on $\mathbb{R}_+ \times \mathbb{R}^n$, then

$$g := \mathrm{d}t^2 + t^2\,\mathrm{d}x^2 = t^2\left[\left(\frac{\mathrm{d}t}{t}\right)^2 + \mathrm{d}x^2\right]. \tag{2.2}$$

Hence the cone type gradient operator here is defined as $D_c := (t\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$. Now we introduce the following cone type weighted L_p-spaces.

Definition 2.1 For $(t,x) \in \mathbb{R}_+^{n+1} (:= \mathbb{R}_+ \times \mathbb{R}^n)$, $1 \le p < +\infty$, and $u(t,x)$ in distribution space $\mathcal{D}'(\mathbb{R}_+^{n+1})$, then we consider that $u(t,x) \in L_p(\mathbb{R}_+^{n+1}, \frac{\mathrm{d}t}{t}\,\mathrm{d}x)$ if

$$\|u\|_{L_p(\mathbb{R}_+^{n+1})} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}_+} t^{n+1} \left| u(t,x) \right|^p \frac{\mathrm{d}t}{t}\,\mathrm{d}x \right)^{\frac{1}{p}} < +\infty. \tag{2.3}$$

Furthermore, the weighted cone type L_p-spaces with weight data $\gamma \in \mathbb{R}$ are denoted by $L_p^\gamma(\mathbb{R}_+^{n+1}, \frac{\mathrm{d}t}{t}\,\mathrm{d}x)$. Namely, if $u(t,x) \in L_p^\gamma(\mathbb{R}_+^{n+1}, \frac{\mathrm{d}t}{t}\,\mathrm{d}x)$, then $t^{-\gamma}u(t,x) \in L_p(\mathbb{R}_+^{n+1}, \frac{\mathrm{d}t}{t}\,\mathrm{d}x)$, and

$$\|u\|_{L_p^\gamma(\mathbb{R}_+^{n+1})} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}_+} t^{n+1} \left| t^{-\gamma} u(t,x) \right|^p \frac{\mathrm{d}t}{t}\,\mathrm{d}x \right)^{\frac{1}{p}} < +\infty. \tag{2.4}$$

Now we give the definition of singular weighted p-Sobolev spaces on stretched cone $\mathbb{R}_+ \times \mathbb{R}^n$ as follows (cf. [7]).

Definition 2.2 For $\gamma \in \mathbb{R}$, $m \in \mathbb{N}$, and $1 \le p < +\infty$, the singular weighted p-Sobolev spaces are defined as

$$H_p^{m,\gamma}(\mathbb{R}_+^{n+1}) := \left\{ u \in \mathcal{D}'(\mathbb{R}_+^{n+1}) : (t\partial_t)^\alpha \partial_x^\beta u(t,x) \in L_p^\gamma\left(\mathbb{R}_+^{n+1}, \frac{\mathrm{d}t}{t}\,\mathrm{d}x \right) \right\} \tag{2.5}$$

for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $|\alpha| + |\beta| \le m$.

Moreover, the spaces $H_p^{m,\gamma}(\mathbb{R}_+^{n+1})$ will be Banach spaces endowed with the norm

$$\|u\|_{H_p^{m,\gamma}(\mathbb{R}_+^{n+1})} = \sum_{|\alpha|+|\beta| \le m} \left(\iint_{\mathbb{R}_+^{n+1}} t^{n+1} \left| t^{-\gamma} (t\partial_t)^\alpha \partial_x^\beta u(t,x) \right|^p \frac{\mathrm{d}t}{t}\,\mathrm{d}x \right)^{\frac{1}{p}}. \tag{2.6}$$

Next, we introduce the following edge type p-Sobolev spaces. First, we assume X^Δ is a straight cone, then for a bounded domain Y in $\mathbb{R}^q$, $W := X^\Delta \times Y$ is a corresponding wedge in $\mathbb{R}^{1+n+q}$. Thus the stretched wedge $\mathbb{W}$ to W is $\overline{\mathbb{R}}_+ \times X \times Y$, which is a manifold with smooth boundary $\{0\} \times X \times Y$. In local coordinates, the open stretched wedge will be $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q$.

The typical degenerate differential operator on the open stretched wedge $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q$ has the form of

$$t^{-\nu} \sum_{j+|\alpha| \le \nu} a_{j\alpha}(t,y)(t\partial_t)^j (t\partial_y)^\alpha = t^{-\nu} A_{\mathbb{W}}, \tag{2.7}$$

where $A_{\mathbb{W}}$ is a degenerate edge operator, $a_{j\alpha} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^q, \mathrm{Diff}^{\nu-(j+|\alpha|)}(\mathbb{R}^n))$ for all j, α, and $\mathrm{Diff}^i(\mathbb{R}^n)$ denotes the set of differential operators of order i on $\mathbb{R}^n$.

Furthermore, let g be Riemannian metrics on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q$. Then

$$g := \mathrm{d}t^2 + t^2\,\mathrm{d}x^2 + \mathrm{d}y^2 = t^2 \left[\left(\frac{\mathrm{d}t}{t} \right)^2 + \mathrm{d}x^2 + \left(\frac{\mathrm{d}y}{t} \right)^2 \right]. \tag{2.8}$$

Thus, the edge type gradient operator is defined as $\mathrm{D}_e = (t\partial_t, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, t\partial_{y_{n+1}}, \ldots, t\partial_{y_{n+q}})$. At present, we give the following definition of edge type weighted $\mathcal{L}_p$-spaces.

Definition 2.3 Assume $N = 1 + n + q$, $(t,x,y) \in \mathbb{R}^N_+ (:= \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q)$, and $u(t,x,y) \in \mathcal{D}'(\mathbb{R}^N_+)$. We consider that $u(t,x,y) \in \mathcal{L}_p(\mathbb{R}^N_+, \frac{dt}{t}\,dx\frac{dy}{t})$ if

$$\|u\|_{\mathcal{L}_p} = \left(\int_{\mathbb{R}^N_+} t^N \bigl|u(t,x,y)\bigr|^p \frac{dt}{t}\,dx\frac{dy}{t}\right)^{\frac{1}{p}} < +\infty. \tag{2.9}$$

Moreover, the weighted edge type $\mathcal{L}_p$-spaces with weight $\gamma \in \mathbb{R}$ are denoted by $\mathcal{L}^\gamma_p(\mathbb{R}^N_+, \frac{dt}{t}\,dx\frac{dy}{t})$, which include functions $u(t,x,y)$ such that

$$\|u\|_{\mathcal{L}^\gamma_p} = \left(\int_{\mathbb{R}^N_+} t^N \bigl|t^{-\gamma} u(t,x,y)\bigr|^p \frac{dt}{t}\,dx\frac{dy}{t}\right)^{\frac{1}{p}} < +\infty. \tag{2.10}$$

The edge type weighted p-Sobolev spaces (cf. [8]) can be defined for all $1 \le p < +\infty$ as follows.

Definition 2.4 Taking $\gamma \in \mathbb{R}$, $m \in \mathbb{N}$, and $N = 1 + n + q$, the edge type weighted p-Sobolev spaces are defined as

$$\mathcal{H}^{m,\gamma}_p\bigl(\mathbb{R}^N_+\bigr) := \left\{u \in \mathcal{D}'\bigl(\mathbb{R}^N_+\bigr) : (t\partial_t)^k \partial^\alpha_x (t\partial_y)^\beta u \in \mathcal{L}^\gamma_p\left(\mathbb{R}^N_+, \frac{dt}{t}\,dx\frac{dy}{t}\right)\right\} \tag{2.11}$$

for $k \in \mathbb{N}$, multi-indexes $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^q$, and $k + |\alpha| + |\beta| \le m$.

The edge type p-Sobolev spaces $\mathcal{H}^{m,\gamma}_p(\mathbb{R}^N_+)$ are Banach spaces with the norm

$$\|u\|_{\mathcal{H}^{m,\gamma}_p(\mathbb{R}^N_+)} = \sum_{k+|\alpha|+|\beta|\le m} \left(\int_{\mathbb{R}^N_+} t^N \bigl|t^{-\gamma} (t\partial_t)^k \partial^\alpha_x (t\partial_y)^\beta u(t,x,y)\bigr|^p \frac{dt}{t}\,dx\frac{dy}{t}\right)^{\frac{1}{p}}. \tag{2.12}$$

Finally, a corner can be defined as (cf. [9])

$$E^\Delta = \bigl(\overline{\mathbb{R}}_+ \times X^\Delta\bigr)/\bigl(\{0\} \times X^\Delta\bigr),$$

where X^Δ is a cone. Then the corresponding stretched corner will be $\mathbb{E}^\Delta := [0,r) \times X \times [0,t)$, $t, r \in \mathbb{R}_+$ with the boundary $\{0\} \times X \times \{0\}$. Thus, under the local coordinates, the open stretched corner is $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$. The typical degenerate differential operator on the open stretched corner $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$ will be

$$(rt)^{-\nu} \sum_{j+k\le\nu} a_{jk}(r,t)(r\partial_r)^j (rt\partial_t)^k = (rt)^{-\nu} A_{\mathbb{E}^\Delta} \tag{2.13}$$

with coefficients $a_{jk}(r,t) \in C^\infty(\overline{\mathbb{R}}_+, \operatorname{Diff}^{\nu-j-k}(\mathbb{R}^n))$ and $A_{\mathbb{E}^\Delta}$ is called a degenerate corner operator. Indeed, we have the following Riemannian metric on the corner $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$:

$$g := dt^2 + t^2\bigl(dr^2 + r^2\,dx^2\bigr) = (rt)^2\left[\left(\frac{dt}{rt}\right)^2 + \left(\frac{dr}{r}\right)^2 + dx^2\right]. \tag{2.14}$$

Then the corner type gradient operator will be $\mathrm{D}_{\mathrm{cor}} := (r\partial_r, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, rt\partial_t)$.

Further, we give the definition of corner type weighted $\mathfrak{L}_p$-spaces as follows.

Definition 2.5 Let $(r,x,t) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$, weight data $\gamma_1, \gamma_2 \in \mathbb{R}$, and $1 \le p < +\infty$. Then $\mathfrak{L}_p^{\gamma_1,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt})$ denote the spaces of all $u(r,x,t) \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ such that

$$\|u\|_{\mathfrak{L}_p^{\gamma_1,\gamma_2}} = \left(\int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+} \left|r^{\frac{N}{p}-\gamma_1} t^{\frac{N}{p}-\gamma_2} u(r,x,t)\right|^p \frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{p}} < +\infty. \tag{2.15}$$

From the weighted $\mathfrak{L}_p^{\gamma_1,\gamma_2}$-spaces, we can define the following weighted p-Sobolev spaces over stretched corner $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$ (cf. [9]).

Definition 2.6 Given $\gamma_1, \gamma_2 \in \mathbb{R}$, $m \in \mathbb{N}$, $1 \le p < +\infty$, and $N = n + 2$, the corner type weighted p-Sobolev spaces can be defined by

$$\begin{aligned}
&\mathbb{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+) \\
&\quad= \Big\{ u \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+) : \\
&\qquad (r\partial_r)^l \partial_x^\alpha (rt\partial_t)^k u(r,x,t) \in \mathfrak{L}_p^{\gamma_1,\gamma_2}\left(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt}\right)\Big\}
\end{aligned} \tag{2.16}$$

for $k, l \in \mathbb{N}$, multi-index $\alpha \in \mathbb{N}^n$, and $k + |\alpha| + l \le m$.

It can be proved that $\mathbb{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ are Banach spaces equipped with the norm

$$\begin{aligned}
&\|u\|_{\mathbb{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \\
&\quad= \sum_{k+|\alpha|+l\le m} \left(\int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+} \left|r^{\frac{N}{p}-\gamma_1} t^{\frac{N}{p}-\gamma_2} (r\partial_r)^l \partial_x^\alpha (rt\partial_t)^k u(r,x,t)\right|^p \frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.17}$$

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

Proof Let $\sigma_i = 1 + s(1 - \frac{1}{p_i}) \ge 1$ and $v_i(t,x) = (t^{\frac{n+1}{s}-\gamma}|u(t,x)|)^{\sigma_i}$ for $0 \le i \le n$. From $\frac{1}{s} = \frac{1}{n+1}(\sum_{i=0}^n \frac{1}{p_i} - 1)$, then it holds that $\sum_{i=0}^n \sigma_i = ns$.

Since $u(t,x) \in C_0^\infty(\mathbb{R}_+^{n+1})$, then we have, for $i = 0$ and $t > 0$,

$$2v_0(t,x) = \int_0^t \tau \frac{\partial v_0(\tau,x)}{\partial \tau}\frac{\mathrm{d}\tau}{\tau} - \int_t^{+\infty} \tau \frac{\partial v_0(\tau,x)}{\partial \tau}\frac{\mathrm{d}\tau}{\tau}.$$

Thus

$$\begin{aligned}
\left|2v_0(t,x)\right| &\le \int_0^t \left|\tau \frac{\partial v_0(\tau,x)}{\partial \tau}\right|\frac{\mathrm{d}\tau}{\tau} + \int_t^{+\infty} \left|\tau \frac{\partial v_0(\tau,x)}{\partial \tau}\right|\frac{\mathrm{d}\tau}{\tau} \\
&=: \int_0^{+\infty} \left|(t\partial_t)v_0(t,x)\right|\frac{\mathrm{d}t}{t}.
\end{aligned}$$

Analogously, for $1 \le i \le n$,

$$\left|2v_i(t,x)\right| \le \int_{-\infty}^{+\infty} \left|\frac{\partial v_i(t,x)}{\partial x_i}\right|\mathrm{d}x_i.$$

After multiplying the $n+1$ inequalities above, we have

$$2^{\frac{n+1}{n}}\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{s}\leq\left(\int_{0}^{+\infty}\left|(t\partial_t)v_0(t,x)\right|\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty}\left|\frac{\partial v_i(t,x)}{\partial x_i}\right|\mathrm{d}x_i\right)^{\frac{1}{n}}.$$

Now integrating the inequality above over the interval $(0,+\infty)$ with respect to $\frac{\mathrm{d}t}{t}$ and using Hölder's inequality, we obtain

$$\begin{aligned}&2^{\frac{n+1}{n}}\int_{0}^{+\infty}\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{s}\frac{\mathrm{d}t}{t}\\&\quad\leq\left(\int_{0}^{+\infty}\left|(t\partial_t)v_0(t,x)\right|\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\int_{0}^{+\infty}\int_{-\infty}^{+\infty}\left|\frac{\partial v_i(t,x)}{\partial x_i}\right|\mathrm{d}x_i\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}.\end{aligned}$$

Then integrating above inequality again over the interval $(-\infty,+\infty)$ with respect to $x_1,x_2,\ldots,x_n$ and using Hölder's inequality respectively, we can deduce that

$$\begin{aligned}&2^{\frac{n+1}{n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{s}\frac{\mathrm{d}t}{t}\,\mathrm{d}x\\&\quad\leq\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left|(t\partial_t)v_0(t,x)\right|\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left|\frac{\partial v_i(t,x)}{\partial x_i}\right|\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right)^{\frac{1}{n}}.\end{aligned}\tag{3.1}$$

For $1\leq i\leq n$, $|\partial_{x_i}|u(t,x)||=|\partial_{x_i}(u\bar{u})^{\frac{1}{2}}|=\frac{1}{2}|(\bar{u}u)^{-\frac{1}{2}}(\bar{u}\partial_{x_i}u+u\partial_{x_i}\bar{u})|\leq\frac{1}{2}|u|^{-1}(|\bar{u}\partial_{x_i}u|+|u\partial_{x_i}\bar{u}|)\leq|\partial_{x_i}u(t,x)|$. Thus we obtain

$$\left|\partial_{x_i}v_i(t,x)\right|\leq\sigma_i\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{\sigma_i-1}t^{\frac{n+1}{s}-\gamma}\left|\partial_{x_i}u(t,x)\right|.\tag{3.2}$$

Similarly, $|(t\partial_t)|u(t,x)|\leq|(t\partial_t)u(t,x)|$, then we have

$$\begin{aligned}&\left|(t\partial_t)v_0(t,x)\right|\\&\quad\leq\sigma_0\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{\sigma_0-1}\left[\left|\frac{n+1}{s}-\gamma\right|t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|+t^{\frac{n+1}{s}-\gamma}\left|(t\partial_t)u(t,x)\right|\right].\end{aligned}\tag{3.3}$$

Replace the corresponding parts of (3.1) by (3.2) and (3.3), we derive that

$$\begin{aligned}&2^{\frac{n+1}{n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{s}\frac{\mathrm{d}t}{t}\,\mathrm{d}x\\&\quad\leq\left[\sigma_0\left|\frac{n+1}{s}-\gamma\right|\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{\sigma_0-1}\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right.\\&\qquad\left.+\sigma_0\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{\sigma_0-1}\left(t^{\frac{n+1}{s}-\gamma}\left|(t\partial_t)u(t,x)\right|\right)\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right]^{\frac{1}{n}}\\&\qquad\times\prod_{i=1}^{n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\sigma_i\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{\sigma_i-1}t^{\frac{n+1}{s}-\gamma}\left|\partial_{x_i}u(t,x)\right|\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right)^{\frac{1}{n}}\\&\quad=:(I_1+I_2)^{\frac{1}{n}}\cdot I_3.\end{aligned}\tag{3.4}$$

Case 1: $p_i > 1$ for $0 \le i \le n$.

If p_i' satisfies $\frac{1}{p_i'} + \frac{1}{p_i} = 1$, then $(\sigma_i - 1)p_i' = s(1 - \frac{1}{p_i})p_i' = s$ for $0 \le i \le n$. By Hölder's inequality, we can acquire that

$$\begin{aligned}
I_1 &= \sigma_0 \left| \frac{n+1}{s} - \gamma \right| \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} \left(t^{\frac{n+1}{s} - \gamma} \left| u(t,x) \right| \right)^{\sigma_0 - 1} \left(t^{\frac{n+1}{s} - \gamma} \left| u(t,x) \right| \right) \frac{\mathrm{d}t}{t} \,\mathrm{d}x \\
&\le \sigma_0 \left| \frac{n+1}{s} - \gamma \right| \|u\|_{L_s^\gamma}^{\frac{s}{p_0'}} \|u\|_{L_{p_0}^{-(\frac{n+1}{s} - \gamma - \frac{n+1}{p_0})}}, \\
I_2 &= \sigma_0 \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} \left(t^{\frac{n+1}{s} - \gamma} \left| u(t,x) \right| \right)^{\sigma_0 - 1} \left(t^{\frac{n+1}{s} - \gamma} \left| (t\partial_t) u(t,x) \right| \right) \frac{\mathrm{d}t}{t} \,\mathrm{d}x \\
&\le \sigma_0 \|u\|_{L_s^\gamma}^{\frac{s}{p_0'}} \left\| (t\partial_t) u \right\|_{L_{p_0}^{-(\frac{n+1}{s} - \gamma - \frac{n+1}{p_0})}}, \\
I_3 &= \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}_+} \sigma_i \left(t^{\frac{n+1}{s} - \gamma} \left| u(t,x) \right| \right)^{\sigma_i - 1} t^{\frac{n+1}{s} - \gamma} \left| \partial_{x_i} u(t,x) \right| \frac{\mathrm{d}t}{t} \,\mathrm{d}x \right)^{\frac{1}{n}} \\
&\le \prod_{i=1}^{n} \sigma_i^{\frac{1}{n}} \|u\|_{L_s^\gamma}^{\frac{s}{np_i'}} \|\partial_{x_i} u\|_{L_{p_i}^{-(\frac{n+1}{s} - \gamma - \frac{n+1}{p_i})}}^{\frac{1}{n}}.
\end{aligned}$$

Returning to (3.4) and setting $\gamma_i^* = -(\frac{n+1}{s} - \gamma - \frac{n+1}{p_i})$ for $0 \le i \le n$, we get

$$\begin{aligned}
2^{\frac{n+1}{n}} \|u\|_{L_s^\gamma}^s \le{}& \left(\sigma_0 \left| \frac{n+1}{s} - \gamma \right| \|u\|_{L_{p_0}^{\gamma_0^*}} + \sigma_0 \left\| (t\partial_t) u \right\|_{L_{p_0}^{\gamma_0^*}} \right)^{\frac{1}{n}} \prod_{i=0}^{n} \|u\|_{L_s^\gamma}^{\frac{s}{np_i'}} \\
&\times \prod_{i=1}^{n} \sigma_i^{\frac{1}{n}} \|\partial_{x_i} u\|_{L_{p_i}^{\gamma_i^*}}^{\frac{1}{n}}.
\end{aligned} \tag{3.5}$$

In view of $\frac{n+1}{s} + 1 = \sum_{i=0}^{n} \frac{1}{p_i}$, we deduce $\sum_{i=0}^{n} \frac{s}{np_i'} = \frac{s}{n}(n + 1 - \sum_{i=0}^{n} \frac{1}{p_i}) = s - \frac{n+1}{n}$. According to (3.5), we find that

$$2^{\frac{n+1}{n}} \|u\|_{L_s^\gamma}^{\frac{n+1}{n}} \le \left(\sigma_0 \left| \frac{n+1}{s} - \gamma \right| \|u\|_{L_{p_0}^{\gamma_0^*}} + \sigma_0 \left\| (t\partial_t) u \right\|_{L_{p_0}^{\gamma_0^*}} \right)^{\frac{1}{n}} \left(\prod_{i=1}^{n} \sigma_i \|\partial_{x_i} u\|_{L_{p_i}^{\gamma_i^*}} \right)^{\frac{1}{n}}.$$

That means

$$\begin{aligned}
\|u\|_{L_s^\gamma} &\le \frac{1}{2} \left(\sigma_0 \left| \frac{n+1}{s} - \gamma \right| \|u\|_{L_{p_0}^{\gamma_0^*}} + \sigma_0 \left\| (t\partial_t) u \right\|_{L_{p_0}^{\gamma_0^*}} \right)^{\frac{1}{n+1}} \prod_{i=1}^{n} \sigma_i^{\frac{1}{n+1}} \|\partial_{x_i} u\|_{L_{p_i}^{\gamma_i^*}}^{\frac{1}{n+1}} \\
&\le \frac{1}{2} \left[\left(\sigma_0 \left| \frac{n+1}{s} - \gamma \right| \right)^{\frac{1}{n+1}} \|u\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}} + \sigma_0^{\frac{1}{n+1}} \left\| (t\partial_t) u \right\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}} \right] \prod_{i=1}^{n} \sigma_i^{\frac{1}{n+1}} \|\partial_{x_i} u\|_{L_{p_i}^{\gamma_i^*}}^{\frac{1}{n+1}}.
\end{aligned} \tag{3.6}$$

Set $c_{01} = \frac{1}{2}(\sigma_0 |\frac{n+1}{s} - \gamma|)^{\frac{1}{n+1}} = \frac{1}{2}[(1 + \frac{s(p_0-1)}{p_0})|\frac{n+1}{s} - \gamma|]^{\frac{1}{n+1}}$, $c_{02} = \frac{1}{2}\sigma_0^{\frac{1}{n+1}} = \frac{1}{2}(1 + \frac{s(p_0-1)}{p_0})^{\frac{1}{n+1}}$, and $c_i = \sigma_i^{\frac{1}{n+1}} = (1 + \frac{s(p_i-1)}{p_i})^{\frac{1}{n+1}}$ for $1 \le i \le n$. As a consequence,

$$\|u\|_{L_s^\gamma} \le \left(c_{01} \|u\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}} + c_{02} \left\| (t\partial_t) u \right\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}} \right) \prod_{i=1}^{n} c_i \|\partial_{x_i} u\|_{L_{p_i}^{\gamma_i^*}}^{\frac{1}{n+1}}. \tag{3.7}$$

Case 2: There exists at least one $p_i \in \{p_0, p_1, \dots, p_n\}$ such that $p_i = 1$.

Without loss of generality, set $p_0, p_1, p_2, \dots, p_{i_0} = 1$ and $p_{i_0+1}, \dots, p_n > 1$. We deduce that $\sigma_i = 1 (0 \le i \le i_0)$, $\sigma_i > 1 (i_0 + 1 \le i \le n)$, and $\frac{1}{s} = \frac{1}{n+1}(i_0 + \sum_{i=i_0+1}^{n} \frac{1}{p_i})$. Thus inequality (3.4) becomes

$$\begin{aligned}
&2^{\frac{n+1}{n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} \left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^s \frac{\mathrm{d}t}{t}\,\mathrm{d}x \\
&\quad\le \left[\left|\frac{n+1}{s}-\gamma\right| \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\frac{\mathrm{d}t}{t}\,\mathrm{d}x + \int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left(t^{\frac{n+1}{s}-\gamma}\left|(t\partial_t)u(t,x)\right|\right)\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right]^{\frac{1}{n}} \\
&\qquad \times \left(\prod_{i=1}^{i_0}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} t^{\frac{n+1}{s}-\gamma}\left|\partial_{x_i}u(t,x)\right|\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right)^{\frac{1}{n}} \\
&\qquad \times \left(\prod_{i=i_0+1}^{n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} \sigma_i\left(t^{\frac{n+1}{s}-\gamma}\left|u(t,x)\right|\right)^{\sigma_i-1} t^{\frac{n+1}{s}-\gamma}\left|\partial_{x_i}u(t,x)\right|\frac{\mathrm{d}t}{t}\,\mathrm{d}x\right)^{\frac{1}{n}} \\
&\quad =: (\hat{I}_1 + \hat{I}_2)^{\frac{1}{n}}\hat{I}_3\hat{I}_4.
\end{aligned}$$

For $\hat{I}_4$, setting $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ $(i_0 + 1 \le i \le n)$ and using Hölder's inequality again, we have

$$\hat{I}_4 \le \prod_{i=i_0+1}^{n} \|u\|_{L_s^{\gamma}}^{\frac{s}{np_i'}} \prod_{i=i_0+1}^{n} \sigma_i^{\frac{1}{n}} \|\partial_{x_i}u\|_{L_{p_i}^{-(\frac{n+1}{s}-\gamma-\frac{n+1}{p_i})}}^{\frac{1}{n}}.$$

Further, it follows from $\sum_{i=i_0+1}^{n} \frac{s}{np_i'} = \frac{s}{n}(n - i_0 - \sum_{i=i_0+1}^{n}\frac{1}{p_i}) = s - \frac{n+1}{n}$ that

$$\begin{aligned}
2^{\frac{n+1}{n}}\|u\|_{L_s^{\gamma}}^{\frac{n+1}{n}} \le{}& \left(\left|\frac{n+1}{s}-\gamma\right|\|u\|_{L_1^{-(\frac{n+1}{s}-\gamma-n-1)}} + \left\|(t\partial_t)u\right\|_{L_1^{-(\frac{n+1}{s}-\gamma-n-1)}}\right)^{\frac{1}{n}} \\
&\times \left(\prod_{i=1}^{i_0}\|\partial_{x_i}u\|_{L_1^{-(\frac{n+1}{s}-\gamma-n-1)}}\right)^{\frac{1}{n}} \left(\prod_{i=i_0+1}^{n}\sigma_i\|\partial_{x_i}u\|_{L_{p_i}^{-(\frac{n+1}{s}-\gamma-\frac{n+1}{p_i})}}\right)^{\frac{1}{n}}.
\end{aligned}$$

Hence we can acquire that

$$\begin{aligned}
\|u\|_{L_s^{\gamma}} \le{}& \frac{1}{2}\left(\left|\frac{n+1}{s}-\gamma\right|^{\frac{1}{n+1}}\|u\|_{L_1^{-(\frac{n+1}{s}-\gamma-n-1)}}^{\frac{1}{n+1}} + \left\|(t\partial_t)u\right\|_{L_1^{-(\frac{n+1}{s}-\gamma-n-1)}}^{\frac{1}{n+1}}\right) \\
&\times \left(\prod_{i=1}^{i_0}\|\partial_{x_i}u\|_{L_1^{-(\frac{n+1}{s}-\gamma-n-1)}}\right)^{\frac{1}{n+1}} \prod_{i=i_0+1}^{n}\sigma_i^{\frac{1}{n+1}}\|\partial_{x_i}u\|_{L_{p_i}^{-(\frac{n+1}{s}-\gamma-\frac{n+1}{p_i})}}^{\frac{1}{n+1}} \\
={}& \left(c_{01}\|u\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}} + c_{02}\left\|(t\partial_t)u\right\|_{L_{p_0}^{\gamma_0^*}}^{\frac{1}{n+1}}\right)\prod_{i=1}^{n}c_i\|\partial_{x_i}u\|_{L_{p_i}^{\gamma_i^*}}^{\frac{1}{n+1}},
\end{aligned}$$

where c_{01}, c_{02}, $c_i (1 \le i \le n)$ and $\gamma_i^* (0 \le i \le n)$ are the same as those in (3.7). Theorem 1.1 is proved. □

4 Proofs of Theorem 1.3 and Theorem 1.5

4.1 Proof of Theorem 1.3

Proof Let $\sigma_i = 1 + s(1 - \frac{1}{p_i}) \geq 1$ and $v_i(t,x) = (t^{\frac{n+q+1}{s}-\gamma}|u(t,x)|)^{\sigma_i}$ for $0 \leq i \leq n+q$. Since $\frac{1}{s} = \frac{1}{n+q+1}(\sum_{i=0}^{n+q}\frac{1}{p_i} - 1)$, then we have $\sum_{i=0}^{n+q}\sigma_i = (n+q)s$.

For $u(t,x,y) \in C_0^\infty(\mathbb{R}_+^{n+q+1})$, it holds that

$$2v_0(t,x,y) = \int_0^t \tau\frac{\partial v_0(\tau,x,y)}{\partial\tau}\frac{\mathrm{d}\tau}{\tau} - \int_t^{+\infty}\tau\frac{\partial v_0(\tau,x,y)}{\partial\tau}\frac{\mathrm{d}\tau}{\tau}.$$

Thus

$$\begin{aligned}
\bigl|2v_0(t,x,y)\bigr| &\leq \int_0^t\left|\tau\frac{\partial v_0(\tau,x,y)}{\partial\tau}\right|\frac{\mathrm{d}\tau}{\tau} + \int_t^{+\infty}\left|\tau\frac{\partial v_0(\tau,x,y)}{\partial\tau}\right|\frac{\mathrm{d}\tau}{\tau}\\
&\leq \int_0^{+\infty}\bigl|(t\partial_t)v_0(t,x,y)\bigr|\frac{\mathrm{d}t}{t}.
\end{aligned}$$

Also, for $1 \leq i \leq n$, one has

$$\begin{aligned}
\bigl|2v_i(t,x,y)\bigr| &\leq \int_{-\infty}^{x_i}\left|\frac{\partial v_i(t,x_1,x_2,\ldots,x_{i-1},\hat{x}_i,x_{i+1},\ldots,x_n,y)}{\partial\hat{x}_i}\right|\mathrm{d}\hat{x}_i\\
&\quad+\int_{x_i}^{+\infty}\left|\frac{\partial v_i(t,x_1,x_2,\ldots,x_{i-1},\hat{x}_i,x_{i+1},\ldots,x_n,y)}{\partial\hat{x}_i}\right|\mathrm{d}\hat{x}_i\\
&\leq \int_{-\infty}^{+\infty}\bigl|\partial_{x_i}v_i(t,x,y)\bigr|\,\mathrm{d}x_i,
\end{aligned}$$

where $y = (y_{n+1}, y_{n+2}, \ldots, y_{n+q})$. Similarly, for $n+1 \leq i \leq n+q$, we derive

$$\begin{aligned}
\bigl|2v_i(t,x,y)\bigr| &\leq \int_{-\infty}^{y_i}\left|\frac{\partial v_i(t,x,y_{n+1},\ldots,y_{i-1},\hat{y}_i,y_{i+1},\ldots,y_{n+q})}{\partial\hat{y}_i}\right|\mathrm{d}\hat{y}_i\\
&\quad+\int_{y_i}^{+\infty}\left|\frac{\partial v_i(t,x,y_{n+1},\ldots,y_{i-1},\hat{y}_i,y_{i+1},\ldots,y_{n+q})}{\partial\hat{y}_i}\right|\mathrm{d}\hat{y}_i\\
&= \int_{-\infty}^{+\infty}\bigl|(t\partial_{\hat{y}_i})v_i(t,x,y_{n+1},\ldots,y_{i-1},\hat{y}_i,y_{i+1},\ldots,y_{n+q})\bigr|\frac{\mathrm{d}\hat{y}_i}{t}\\
&=: \int_{-\infty}^{+\infty}\bigl|(t\partial_{y_i})v_i(t,x,y)\bigr|\frac{\mathrm{d}y_i}{t},
\end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. By multiplying the $n+q+1$ inequalities above, we have

$$\begin{aligned}
\prod_{i=0}^{n+q}\bigl|2v_i(t,x)\bigr| &= 2^{n+q+1}\bigl(t^{\frac{n+q+1}{s}-\gamma}\bigl|u(t,x,y)\bigr|\bigr)^{\sum_{i=0}^{n+q}\sigma_i}\\
&= 2^{n+q+1}\bigl(t^{\frac{n+q+1}{s}-\gamma}\bigl|u(t,x,y)\bigr|\bigr)^{s(n+q)}\\
&\leq \int_0^{+\infty}\bigl|(t\partial_t)v_0(t,x,y)\bigr|\frac{\mathrm{d}t}{t}\prod_{i=1}^{n}\int_{-\infty}^{+\infty}\bigl|\partial_{x_i}v_i(t,x,y)\bigr|\,\mathrm{d}x_i\\
&\quad\times\prod_{i=n+1}^{n+q}\int_{-\infty}^{+\infty}\bigl|(t\partial_{y_i})v_i(t,x,y)\bigr|\frac{\mathrm{d}y_i}{t}.
\end{aligned}$$

That implies

$$
\begin{aligned}
2^{\frac{n+q+1}{n+q}}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{s} \le{}& \left(\int_{0}^{+\infty}\left|(t\partial_t)v_0(t,x,y)\right|\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n+q}} \\
&\times\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty}\left|\partial_{x_i}v_i(t,x,y)\right|\mathrm{d}x_i\right)^{\frac{1}{n+q}} \\
&\times\prod_{i=n+1}^{n+q}\left(\int_{-\infty}^{+\infty}\left|(t\partial_{y_i})v_i(t,x,y)\right|\frac{\mathrm{d}y_i}{t}\right)^{\frac{1}{n+q}}.
\end{aligned}
$$

Now integrating these inequalities over the interval $(0,+\infty)$ with respect to $\frac{\mathrm{d}t}{t}$ and using Hölder inequality will lead to

$$
\begin{aligned}
&2^{\frac{n+q+1}{n+q}}\int_{0}^{+\infty}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{s}\frac{\mathrm{d}t}{t} \\
&\le\left(\int_{0}^{+\infty}\left|(t\partial_t)v_0(t,x,y)\right|\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n+q}}\prod_{i=1}^{n}\left(\int_{0}^{+\infty}\int_{-\infty}^{+\infty}\left|\partial_{x_i}v_i(t,x,y)\right|\mathrm{d}x_i\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n+q}} \\
&\quad\times\prod_{i=n+1}^{n+q}\left(\int_{0}^{+\infty}\int_{-\infty}^{+\infty}\left|(t\partial_{y_i})v_i(t,x,y)\right|\frac{\mathrm{d}y_i}{t}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n+q}}.
\end{aligned}
$$

Then integrating over the interval $(-\infty,+\infty)$ with respect to $x_1,x_2,\ldots,x_n$ and $\frac{\mathrm{d}y_{n+1}}{t},\ldots,\frac{\mathrm{d}y_{n+q}}{t}$ respectively and using Hölder's inequality again, we obtain by setting $\mathrm{d}\eta := \frac{\mathrm{d}t}{t}\,\mathrm{d}x\frac{\mathrm{d}y}{t}$ and $N = n+q+1$ that

$$
\begin{aligned}
&2^{\frac{n+q+1}{n+1}}\int_{\mathbb{R}_+^N}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{s}\mathrm{d}\eta \\
&\le\left(\int_{\mathbb{R}_+^N}\left|(t\partial_t)v_0(t,x,y)\right|\mathrm{d}\eta\right)^{\frac{1}{n+q}}\prod_{i=1}^{n}\left(\int_{\mathbb{R}_+^N}\left|\partial_{x_i}v_i(t,x,y)\right|\mathrm{d}\eta\right)^{\frac{1}{n+q}} \\
&\quad\times\prod_{i=n+1}^{n+q}\left(\int_{\mathbb{R}_+^N}\left|(t\partial_{y_i})v_i(t,x,y)\right|\mathrm{d}\eta\right)^{\frac{1}{n+q}}.
\end{aligned}
\tag{4.1}
$$

For $1 \le i \le n$, we acquire $|\partial_{x_i}|u(t,x,y)|| = |\partial_{x_i}(u\bar{u})^{\frac{1}{2}}| \le \frac{1}{2}|u|^{-1}(|\bar{u}\partial_{x_i}u| + |u\partial_{x_i}\bar{u}|) \le |\partial_{x_i}u(t,x,y)|$. Similar to this deduction, it holds that $|(t\partial_t)|u(t,x,y)|| \le |(t\partial_t)u(t,x,y)|$ and $|(t\partial_{y_i})|u(t,x,y)|| \le |(t\partial_{y_i})u(t,x,y)|$ for $n+1 \le i \le n+q$.

Consequently, for $1 \le i \le n$, we have

$$
\begin{aligned}
\left|\partial_{x_i}v_i(t,x,y)\right| &= \left|\partial_{x_i}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_i}\right| \\
&\le\sigma_i\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_i-1}t^{\frac{n+q+1}{s}-\gamma}\left|\partial_{x_i}u(t,x,y)\right|,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
\left|(t\partial_t)v_0(t,x,y)\right| \le{}& \sigma_0\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_0-1}\left[\left|\frac{n+q+1}{s}-\gamma\right|t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right. \\
&\left.+\,t^{\frac{n+q+1}{s}-\gamma}\left|(t\partial_t)u(t,x,y)\right|\right].
\end{aligned}
\tag{4.3}
$$

Also, for $n+1 \le i \le n+q$, we have

$$\left|(t\partial_{y_i})v_i(t,x,y)\right| \le \sigma_i\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_i-1}t^{\frac{n+q+1}{s}-\gamma}\left|(t\partial_{y_i})u(t,x,y)\right|. \tag{4.4}$$

After rewriting the corresponding parts of (4.1) by (4.2), (4.3), and (4.4), we get

$$\begin{aligned}
&2^{\frac{n+q+1}{n+q}}\int_{\mathbb{R}^N_+}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^s \,\mathrm{d}\eta \\
&\quad\le\left[\sigma_0\left|\frac{n+q+1}{s}-\gamma\right|\int_{\mathbb{R}^N_+}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_0-1}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)\mathrm{d}\eta\right. \\
&\qquad\left.+\sigma_0\int_{\mathbb{R}^N_+}\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_0-1}\left(t^{\frac{n+q+1}{s}-\gamma}\left|(t\partial_t)u(t,x,y)\right|\right)\mathrm{d}\eta\right]^{\frac{1}{n+q}} \\
&\qquad\times\prod_{i=1}^{n}\left[\int_{\mathbb{R}^N_+}\sigma_i\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_i-1}\left(t^{\frac{n+q+1}{s}-\gamma}\left|\partial_{x_i}u(t,x,y)\right|\right)\mathrm{d}\eta\right]^{\frac{1}{n+q}} \\
&\qquad\times\prod_{i=n+1}^{n+q}\left[\int_{\mathbb{R}^N_+}\sigma_i\left(t^{\frac{n+q+1}{s}-\gamma}\left|u(t,x,y)\right|\right)^{\sigma_i-1}\left(t^{\frac{n+q+1}{s}-\gamma}\left|(t\partial_{y_i})u(t,x,y)\right|\right)\mathrm{d}\eta\right]^{\frac{1}{n+q}} \\
&\quad=:(I_1+I_2)^{\frac{1}{n}}\cdot I_3\cdot I_4.
\end{aligned}$$

There are still two cases similar to the proof of Theorem 1.1, i.e., the case of $p_i > 1$ for $0 \le i \le n+q$ and the case that there exists at least one $p_i \in \{p_0, p_1, \dots, p_{n+q}\}$ such that $p_i = 1$. Since the proof process here is also analogous to the corresponding part in the proof of Theorem 1.1, then we omit it here, Theorem 1.3 is proved. □

4.2 Proof of Theorem 1.5

Proof Let $\sigma_i = 1 + s(1-\frac{1}{p_i}) \ge 1$ and $v_i(r,x,t) = (r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}|u(r,x,t)|)^{\sigma_i}$ for $0 \le i \le n+1$. Due to $\frac{1}{s} = \frac{1}{n+2}(\sum_{i=0}^{n+1}\frac{1}{p_i}-1)$, we have $\sum_{i=0}^{n+1}\sigma_i = (n+1)s$.

Since $u(r,x,t) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$, then we obtain, for $i = 0$ and $r > 0$,

$$2v_0(r,x,t) = \int_0^r (\tau\partial_\tau)v_0(\tau,x,t)\frac{\mathrm{d}\tau}{\tau} - \int_r^{+\infty}(\tau\partial_\tau)v_0(\tau,x,t)\frac{\mathrm{d}\tau}{\tau}.$$

Thus

$$\begin{aligned}
\left|2v_0(r,x,t)\right| &\le \int_0^r\left|(\tau\partial_\tau)v_0(\tau,x,y)\right|\frac{\mathrm{d}\tau}{\tau} + \int_r^{+\infty}\left|(\tau\partial_\tau)v_0(\tau,x,t)\right|\frac{\mathrm{d}\tau}{\tau} \\
&= \int_0^{+\infty}\left|(r\partial_r)v_0(r,x,t)\right|\frac{\mathrm{d}r}{r}.
\end{aligned}$$

For $1 \le i \le n$, we obtain, for $r, t \in \mathbb{R}_+$,

$$\begin{aligned}
\left|2v_i(r,x,t)\right| &\le \int_{-\infty}^{x_i}\left|\partial_{\hat{x}_i}v_i(r,x_1,x_2,\dots,x_{i-1},\hat{x}_i,x_{i+1},\dots,x_n,t)\right|\mathrm{d}\hat{x}_i \\
&\quad+\int_{x_i}^{+\infty}\left|\partial_{\hat{x}_i}v_i(r,x_1,x_2,\dots,x_{i-1},\hat{x}_i,x_{i+1},\dots,x_n,t)\right|\mathrm{d}\hat{x}_i \\
&= \int_{-\infty}^{+\infty}\left|\partial_{x_i}v_i(r,x,t)\right|\mathrm{d}x_i.
\end{aligned}$$

Similarly, for $r, t > 0$,

$$\begin{aligned} \left|2v_{n+1}(r,x,t)\right| &\le \int_0^t \left|(r\mu\partial_\mu)v_{n+1}(r,x,\mu)\right|\frac{\mathrm{d}\mu}{r\mu} + \int_t^{+\infty}\left|(r\mu\partial_\mu)v_{n+1}(r,x,\mu)\right|\frac{\mathrm{d}\mu}{r\mu} \\ &= \int_0^{+\infty}\left|(r\mu\partial_\mu)v_{n+1}(r,x,\mu)\right|\frac{\mathrm{d}\mu}{r\mu} \\ &=: \int_0^{+\infty}\left|(rt\partial_t)v_{n+1}(r,x,t)\right|\frac{\mathrm{d}t}{rt}, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. By multiplying the $n + 2$ inequalities above, we have

$$\begin{aligned} \prod_{i=0}^{n+1}\left|2v_i(r,x,t)\right| &= 2^{n+2}\left(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\left|u(r,x,t)\right|\right)^{s(n+1)} \\ &\le \int_0^{+\infty}\left|(r\partial_r)v_0(r,x,t)\right|\frac{\mathrm{d}r}{r}\prod_{i=1}^{n}\int_{-\infty}^{+\infty}\left|\partial_{x_i}v_i(r,x,t)\right|\mathrm{d}x_i \\ &\quad\times\int_0^{+\infty}\left|(rt\partial_t)v_{n+1}(r,x,t)\right|\frac{\mathrm{d}t}{rt}. \end{aligned}$$

That means

$$\begin{aligned} &2^{\frac{n+2}{n+1}}\left(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\left|u(r,x,t)\right|\right)^{s} \\ &\quad\le\left(\int_0^{+\infty}\left|(r\partial_r)v_0(r,x,t)\right|\frac{\mathrm{d}r}{r}\right)^{\frac{1}{n+1}}\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty}\left|\partial_{x_i}v_i(r,x,t)\right|\mathrm{d}x_i\right)^{\frac{1}{n+1}} \\ &\qquad\times\left(\int_0^{+\infty}\left|(rt\partial_t)v_{n+1}(r,x,t)\right|\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{n+1}}. \end{aligned}$$

Now integrating over the interval $(0, +\infty)$ with respect to $\frac{\mathrm{d}r}{r}$ and $\frac{\mathrm{d}t}{rt}$ and using Hölder's inequality respectively, we obtain

$$\begin{aligned} &2^{\frac{n+2}{n+1}}\int_0^{+\infty}\int_0^{+\infty}\left(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\left|u(r,x,t)\right|\right)^{s}\frac{\mathrm{d}r}{r}\frac{\mathrm{d}t}{rt} \\ &\quad\le\left(\int_0^{+\infty}\int_0^{+\infty}\left|(r\partial_r)v_0(r,x,t)\right|\frac{\mathrm{d}r}{r}\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{n+1}}\left(\int_0^{+\infty}\int_0^{+\infty}\left|(rt\partial_t)v_{n+1}(r,x,t)\right|\frac{\mathrm{d}t}{rt}\frac{\mathrm{d}r}{r}\right)^{\frac{1}{n+1}} \\ &\qquad\times\prod_{i=1}^{n}\left(\int_0^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty}\left|\partial_{x_i}v_i(r,x,t)\right|\mathrm{d}x_i\frac{\mathrm{d}r}{r}\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{n+1}}. \end{aligned}$$

Then from integrating over the interval $(-\infty, +\infty)$ with $x_1, x_2, \ldots, x_n$ respectively and using Hölder's inequality again, we derive that

$$\begin{aligned} &2^{\frac{n+2}{n+1}}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\left|u(r,x,t)\right|\right)^{s}\frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt} \\ &\quad\le\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\left|(r\partial_r)v_0(r,x,t)\right|\frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{n+1}} \end{aligned}$$

$$\times \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\bigl|(rt\partial_t)v_{n+1}(r,x,t)\bigr|\frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{n+1}}$$
$$\times \prod_{i=1}^{n}\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\bigl|\partial_{x_i}v_i(r,x,t)\bigr|\frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt}\right)^{\frac{1}{n+1}}. \tag{4.5}$$

Set $\mathrm{d}\eta := \frac{\mathrm{d}r}{r}\,\mathrm{d}x\frac{\mathrm{d}t}{rt}$, and $N = n + 2$. Similar to the estimation in Theorem 1.3, we acquire that $|\partial_{x_i}|u(r,x,t)|| \le |\partial_{x_i}u(r,x,t)|$ for $1 \le i \le n$, $|(r\partial_r)|u(r,x,t)|| \le |(r\partial_r)u(r,x,t)|$ and $|(rt\partial_t)|u(r,x,t)|| \le |(rt\partial_t)u(r,x,t)|$. As a result,

$$\begin{aligned}\bigl|\partial_{x_i}v_i(r,x,t)\bigr| &= \Bigl|\partial_{x_i}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_i}\Bigr| \\ &\le \sigma_i\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_i-1}r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|\partial_{x_i}u(r,x,t)\bigr|\end{aligned} \tag{4.6}$$

for $1 \le i \le n$,

$$\begin{aligned}&\bigl|(r\partial_r)v_0(r,x,t)\bigr| \\ &\quad\le \sigma_0\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_0-1}\biggl[\biggl|\frac{n+2}{s}-\bar{\gamma}\biggr|r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr| \\ &\qquad + r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|(r\partial_r)u(r,x,t)\bigr|\biggr]\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\bigl|(rt\partial_t)v_{n+1}(r,x,t)\bigr| \\ &\quad\le \sigma_{n+1}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_{n+1}-1}\biggl[\biggl|\frac{n+2}{s}-\gamma\biggr|r^{\frac{n+2}{s}-\bar{\gamma}+1}t^{\frac{n+2}{s}-\gamma} \\ &\qquad \times \bigl|u(r,x,t)\bigr| + r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|(rt\partial_t)u(r,x,t)\bigr|\biggr].\end{aligned} \tag{4.8}$$

Substituting (4.6), (4.7), and (4.8) into (4.5), it is easy to see

$$\begin{aligned}&2^{\frac{n+2}{n+1}}\int_{\mathbb{R}_+^N}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^s\,\mathrm{d}\eta \\ &\quad\le \biggl[\sigma_0\biggl|\frac{n+2}{s}-\bar{\gamma}\biggr|\int_{\mathbb{R}_+^N}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_0-1}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)\,\mathrm{d}\eta \\ &\qquad + \sigma_0\int_{\mathbb{R}_+^N}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_0-1}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|(r\partial_r)u(r,x,t)\bigr|\bigr)\,\mathrm{d}\eta\biggr]^{\frac{1}{n+1}} \\ &\qquad \times \prod_{i=1}^{n}\biggl(\int_{\mathbb{R}_+^N}\sigma_i\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_i-1}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|\partial_{x_i}u(r,x,t)\bigr|\bigr)\,\mathrm{d}\eta\biggr)^{\frac{1}{n+1}} \\ &\qquad \times \biggl[\sigma_{n+1}\biggl|\frac{n+2}{s}-\gamma\biggr|\int_{\mathbb{R}_+^N}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_{n+1}-1} \\ &\qquad \times \bigl(r^{\frac{n+2}{s}-\bar{\gamma}+1}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)\,\mathrm{d}\eta + \sigma_{n+1}\int_{\mathbb{R}_+^N}\bigl(r^{\frac{n+2}{s}-\bar{\gamma}}t^{\frac{n+2}{s}-\gamma}\bigl|u(r,x,t)\bigr|\bigr)^{\sigma_{n+1}-1}\end{aligned}$$

$$\times \left(r^{\frac{n+2}{s}-\bar{\gamma}} t^{\frac{n+2}{s}-\gamma} \left|(rt\partial_t)u(r,x,t)\right|\right) \mathrm{d}\eta \Bigg]^{\frac{1}{n+1}}$$
$$=: (I_1 + I_2)^{\frac{1}{n+1}} \cdot I_3 (I_4 + I_5)^{\frac{1}{n+1}}.$$

Considering that the remaining proofs will be the same as those in both Theorem 1.1 and Theorem 1.3, then Theorem 1.5 is proved. □

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Authors' contributions
HC first raised the core problem of the current paper. Under his supervision and suggestion, YL and JW finished this manuscript together. Then HC read carefully this manuscript for several times and gave some valuable revisions on it. All authors read and approved the final manuscript.

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