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# Well-posedness for generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality and optimization problems

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**Abstract**

The purpose of this paper is to focus on the well-posedness for a generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality and optimization problems with a constraint. We establish a metric characterization of well-posedness in terms of an approximate solution set. Also we prove that well-posedness of optimization problem is closely related to that of generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality problems.

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## 1 Introduction

The theory of variational inequality for multi-valued mappings has been studied by several authors (see [1, 4, 9, 14, 16, 25]). Since variational inequality theory is closely related to mathematical programming problems under mild conditions, consequently the concept of Tykhonov well-posedness has also been generalized to variational inequalities [7–12] and equilibrium problems, fixed point problems, optimization problems, mixed quasi-variational-like inequality with constraints etc. [15, 17, 18, 24, 26].

In 2000, Lignola and Morgan [20] defined the parametric well-posedness for optimization problems with variational inequality constraints by using the approximating sequences. Lignola [19] discussed the well-posedness, $L$-well-posedness and metric characterizations of well-posedness for quasi-variational-inequality problems. Ceng and Yao [3] extended these concepts to derive the conditions under which the generalized mixed variational inequality problems are well-posed. Thereafter, Lin and Chuang [21] established well-posedness for variational inclusion, and optimization problems with variational inclusion and scalar equilibrium constraints in a generalized sense. In 2010, Fang et al. [11] extended the notion of well-posedness by perturbations to a mixed variational inequality problem in a Banach space. Recently, Ceng et al. [2] suggested the conditions of well-posedness for hemivariational inequality problems involving Clarkes generalized directional derivative under different types of monotonicity assumptions.

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Inspired and motivated by recent work [6, 7, 13–16, 23, 25], we consider and study well-posedness for generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality problems and optimization problems with constrained involving a relaxed $\eta$-$\alpha_g$-$P$-monotone operator.

## 2 Preliminaries

Assume that $\mathcal{X}$ and $\mathcal{Y}$ are two real Banach spaces. Let $\mathcal{D} \subset \mathcal{X}$ be a nonempty closed convex subset of $\mathcal{X}$ and $P \subset \mathcal{Y}$ a closed convex and proper cone with nonempty interior. Throughout this paper, we shall use the following inequalities. For all $x, y \in \mathcal{Y}$:

(i) $x \leq_P y \Leftrightarrow y - x \in P$;

(ii) $x \nleq_P y \Leftrightarrow y - x \notin P$;

(iii) $x \leq_{P^0} y \Leftrightarrow y - x \in P^0$;

where $P^0$ denotes the interior of $P$.

If $\leq_P$ is a partial order, then $(\mathcal{Y}, \leq_P)$ is called an ordered Banach space ordered by $P$. Let $T : \mathcal{X} \to 2^{L(\mathcal{X},\mathcal{Y})}$ be a set-valued mapping where $L(\mathcal{X}, \mathcal{Y})$ denotes the space of all continuous linear mappings from $\mathcal{X}$ into $\mathcal{Y}$. Assume that $Q : L(\mathcal{X}, \mathcal{Y}) \times \mathcal{D} \to L(\mathcal{X}, \mathcal{Y})$, $\varphi : \mathcal{D} \times \mathcal{D} \to \mathcal{Y}$, $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are bi-mappings and $g : \mathcal{D} \to \mathcal{D}$ is single-valued mapping. We consider the following generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality problem for finding $x \in \mathcal{D}$ and $u \in T(x)$ such that

$$\bigl\langle Q(u,x), \eta\bigl(y, g(x)\bigr)\bigr\rangle + \varphi\bigl(g(x), y\bigr) \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}. \tag{2.1}$$

Denote by

$$\Omega = \bigl\{x \in \mathcal{D} : \exists u \in T(x) \text{ such that } \bigl\langle Q(u,x), \eta\bigl(y, g(x)\bigr)\bigr\rangle + \varphi\bigl(g(x), y\bigr) \nleq_{P^0} 0, \forall y \in \mathcal{D}\bigr\}$$

the solution set of the problem (2.1).

**Definition 2.1** A mapping $\phi : \mathcal{D} \to \mathcal{Y}$ is said to be

(i) $P$-convex, if

$$\phi\bigl(\mu x + (1-\mu)y\bigr) \leq_P \mu\phi(x) + (1-\mu)\phi(y), \quad \forall x, y \in \mathcal{D}, \mu \in [0,1];$$

(ii) $P$-concave, if

$$\phi\bigl(\mu x + (1-\mu)y\bigr) \geq_P \mu\phi(x) + (1-\mu)\phi(y), \quad \forall x, y \in \mathcal{D}, \mu \in [0,1].$$

**Definition 2.2** ([25]) A set-valued mapping $T : \mathcal{D} \to 2^{L(\mathcal{X},\mathcal{Y})}$ is said to be monotone with respect to the first variable of $Q$, if

$$\bigl\langle Q(u,\cdot) - Q(v,\cdot), x - y\bigr\rangle \geq_P 0, \quad \forall x, y \in \mathcal{D}, u \in T(x), v \in T(y).$$

**Definition 2.3** Let $g : \mathcal{D} \to \mathcal{D}$ be a single-valued mapping. A set-valued mapping $T : \mathcal{D} \to 2^{L(\mathcal{X},\mathcal{Y})}$ is said to be relaxed $\eta$-$\alpha_g$-$P$-monotone with respect to the first variable of $Q$ and $g$, if

$$\bigl\langle Q(u,\cdot) - Q(v,\cdot), \eta\bigl(g(x), y\bigr)\bigr\rangle - \alpha_g(x-y) \geq_P 0, \quad \forall x, y \in \mathcal{D}, u \in T(x), v \in T(y),$$

where $\alpha_g : \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\alpha_g(tz) = t^p \alpha_g(z)$, $\forall t > 0$, $z \in \mathcal{X}$, and $p > 1$ is a constant.

**Definition 2.4** A mapping $\gamma : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is said to be affine with respect to the first variable if, for any $x_i \in \mathcal{D}$ and $\lambda_i \ge 0$ $(1 \le i \le n)$ with $\sum_{i=1}^n \lambda_i = 1$ and for any $y \in \mathcal{D}$,

$$\gamma\left(\sum_{i=1}^n \lambda_i x_i, y\right) = \sum_{i=1}^n \lambda_i \gamma(x_i, y).$$

**Lemma 2.5** ([5]) *Let $(\mathcal{Y}, P)$ be an ordered Banach space with closed convex pointed cone $P$ and $P^0 \neq \emptyset$. Then, for all $x, y, z \in \mathcal{Y}$, we have*

(i) $z \nleq_{P^0} x,\ x \ge_P y \Rightarrow z \nleq_{P^0} y$;

(ii) $z \ngeq_{P^0} x,\ x \le_P y \Rightarrow z \ngeq_{P^0} y$.

**Lemma 2.6** ([22]) *Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space and $\mathfrak{H}$ be a Hausdorff metric on the collection $CB(\mathcal{X})$ of all nonempty, closed and bounded subsets of $\mathcal{X}$ induced by metric*

$$d(u,v) = \|u - v\|,$$

*which is defined by*

$$\mathfrak{H}(A,B) = \max\left\{\sup_{u\in A}\inf_{v\in B}\|u-v\|, \sup_{v\in B}\inf_{u\in A}\|u-v\|\right\}, \quad \forall A,B \in CB(\mathcal{X}).$$

*If $A$, $B$ are compact sets in $\mathcal{X}$, then for each $u \in A$ there exists $v \in B$ such that*

$$\|u - v\| \le \mathfrak{H}(A,B).$$

**Definition 2.7** A set-valued mapping $T : \mathcal{D} \to 2^{L(\mathcal{X},\mathcal{Y})}$ is said to be $\mathfrak{H}$-hemicontinuous, if

$$\mathfrak{H}\big(T\big(x + \tau(y - x)\big), T(x)\big) \to 0 \quad \text{as } \tau \to 0^+, \forall x, y \in \mathcal{D}, \tau \in (0,1),$$

where $\mathfrak{H}$ is the Hausdorff metric defined on $CB(L(\mathcal{X},\mathcal{Y}))$.

**Lemma 2.8** *Let $\mathcal{D}$ be a closed convex subset of a real Banach space $\mathcal{X}$, $\mathcal{Y}$ be a real Banach space ordered by a nonempty closed convex pointed cone $P$ with apex at the origin and $P^0 \neq \emptyset$. Assume that $Q : L(\mathcal{X},\mathcal{Y}) \to L(\mathcal{X},\mathcal{Y})$ is a continuous mapping and $T : \mathcal{D} \to 2^{L(\mathcal{X},\mathcal{Y})}$ is a nonempty compact set-valued mapping. If the following conditions are satisfied*:

(i) *$\varphi : \mathcal{D} \times \mathcal{D} \to \mathcal{Y}$ is a $P$-convex in the second variable with $\varphi(x,x) = 0$, $\forall x \in \mathcal{D}$*;

(ii) *$\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is an affine mapping in the first variable with $\eta(x,x) = 0$, $\forall x \in \mathcal{D}$*;

(iii) *$T : \mathcal{D} \to 2^{L(\mathcal{X},\mathcal{Y})}$ is $\mathfrak{H}$-hemicontinuous and relaxed $\eta$-$\alpha$-$P$-monotone with respect to $Q$*;

*then the following two problems are equivalent*:

(a) *there exist $x_0 \in \mathcal{D}$ and $u_0 \in T(x_0)$ such that*

$$\big\langle Q(u_0), \eta(y, x_0)\big\rangle + \varphi(x_0, y) \nleq_{P^0} 0, \quad \forall y \in \mathcal{D},$$

(b) *there exists* $x_0 \in \mathcal{D}$ *such that*

$$\langle Q(v), \eta(y, x_0)\rangle + \varphi(x_0, y) - \alpha(y - x_0) \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}, v \in T(y).$$

## 3 Well-posedness for problem (2.1)

In this section, we established the well-posedness for problem (2.1) with relaxed $\eta$-$\alpha_g$-$P$-monotone operator.

**Definition 3.1** A sequence $\{x_n\} \in \mathcal{D}$ is said to be an approximating sequence for problem (2.1) if, there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}, e \in \operatorname{int} P.$$

**Definition 3.2** The generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality problem is said to be well-posed if

(i) there exists a unique solution $x_0$ of problem (2.1);

(ii) every approximating sequence of problem (2.1) converges to $x_0$.

**Corollary 3.3** *From Definition* 3.2, *it follows that if the generalized* $(\eta, g, \varphi)$*-mixed vector variational-type inequality problem is well-posed, then*

(i) *the solution set* $\Omega$ *of problem* (2.1) *is nonempty*;

(ii) *every approximating sequence has a subsequence that converges to some point of* $\Omega$.

To investigate well-posedness of problem (2.1), we denote the approximate solution set of problem (2.1) by

$$\begin{aligned}\Omega_\epsilon = \{&x \in \mathcal{D} : \exists u \in T(x) \text{ such that} \\ &\langle Q(u, x), \eta(y, g(x))\rangle + \varphi(g(x), y) + \epsilon e \nleq_{P^0} 0, \forall y \in \mathcal{D}, \epsilon \geq 0\}.\end{aligned}$$

*Remark* 3.4 We note that, if $\epsilon = 0$ then $\Omega = \Omega_\epsilon$, and if $\epsilon > 0$ then $\Omega \subseteq \Omega_\epsilon$.

Denote by $\operatorname{diam} \mathcal{B}$ the diameter of a set $\mathcal{B}$ which is defined as

$$\operatorname{diam} \mathcal{B} = \sup_{a, b \in \mathcal{B}} \|a - b\|.$$

**Theorem 3.5** *Let* $g : \mathcal{D} \to \mathcal{D}$ *and* $Q : L(\mathcal{X}, \mathcal{Y}) \times \mathcal{D} \to L(\mathcal{X}, \mathcal{Y})$ *be two continuous mappings. Let* $\varphi(\cdot, y)$, $\eta(y, \cdot)$ *and* $\alpha_g$ *be continuous functions for all* $y \in \mathcal{D}$. *If the conditions in Lemma* 2.8 *are satisfied, then problem* (2.1) *is well-posed if and only if*

$$\Omega_\epsilon \neq \emptyset, \quad \forall \epsilon > 0$$

*and*

$$\operatorname{diam} \Omega_\epsilon \to 0 \quad \textit{as } \epsilon \to 0.$$

*Proof* Assume that problem (2.1) is well-posed, then it has a unique solution $x_0 \in \Omega$. Since $\Omega \subseteq \Omega_\epsilon$, $\forall \epsilon > 0$, this implies that $\Omega_\epsilon \neq \emptyset$, $\forall \epsilon > 0$. On the contrary, if

$$\operatorname{diam} \Omega_\epsilon \nrightarrow 0 \quad \text{as } \epsilon \to 0,$$

then there exist $r > 0$, $m$ (a positive integer), and a sequence $\{\epsilon_n > 0\}$ with $\epsilon_n \to 0$ and $x_n, x'_n \in \Omega_{\epsilon_n}$ such that

$$\|x_n - x'_n\| > r, \quad \forall n \geq m. \tag{3.1}$$

Since $x_n, x'_n \in \Omega_{\epsilon_n}$, there exist $u_n \in T(x_n)$ and $u'_n \in T(x'_n)$ such that

$$\begin{aligned}
&\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}, \\
&\langle Q(u'_n, x'_n), \eta(y, g(x'_n))\rangle + \varphi(g(x'_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}.
\end{aligned}$$

Since the problem is well-posed, the approximating sequences $\{x_n\}$ and $\{x'_n\}$ of problem (2.1) converge to $x_0$. Therefore we have

$$\|x_n - x'_n\| = \|x_n - x_0 + x_0 - x'_n\| \leq \|x_n - x_0\| + \|x_0 - x'_n\| \leq \epsilon,$$

which contradicts to (3.1), for some $\epsilon = r$.

Conversely, assume that $\{x_n\}$ is an approximating sequence of problem (2.1). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}, \tag{3.2}$$

which implies that $x_n \in \Omega_{\epsilon_n}$. Since $\operatorname{diam} \Omega_{\epsilon_n} \to 0$ as $\epsilon_n \to 0$, $\{x_n\}$ is a Cauchy sequence, which converges to some $x_0 \in \mathcal{D}$ (because $\mathcal{D}$ is closed). Again since $T$ is relaxed $\eta$-$\alpha_g$-$P$-monotone with respect to the first variable of $Q$ and $g$ on $\mathcal{D}$, it follows from Definition 2.3, for any $y \in \mathcal{D}$ and $u \in T(y)$, we have

$$\begin{aligned}
&\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) \\
&\quad \leq_P \langle Q(u, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) - \alpha_g(y - x_n).
\end{aligned} \tag{3.3}$$

From the continuity of $g$, $\varphi$, $\eta$ and $\alpha_g$, we have

$$\begin{aligned}
&\langle Q(u, x_0), \eta(y, g(x_0))\rangle + \varphi(g(x_0), y) - \alpha_g(y - x_0) \\
&\quad = \lim_{n\to\infty} \{\langle Q(u, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) - \alpha_g(y - x_n)\}.
\end{aligned}$$

This together with (3.3) shows that

$$\begin{aligned}
&\langle Q(u, x_0), \eta(y, g(x_0))\rangle + \varphi(g(x_0), y) - \alpha_g(y - x_0) \\
&\quad \geq_P \lim_{n\to\infty} \{\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y)\}.
\end{aligned} \tag{3.4}$$

Taking the limit in (3.2), we have

$$\lim_{n\to\infty} \{\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y)\} \nleq_{P^0} 0. \tag{3.5}$$

Combining (3.4) and (3.5) and using Lemma 2.5(ii), we get

$$\langle Q(u, x_0), \eta(y, g(x_0))\rangle + \varphi(g(x_0), y) - \alpha_g(y - x_0) \nleq_{P^0} 0.$$

Thus, by Lemma 2.8, there exist $x_0 \in \mathcal{D}$ and $u_0 \in T(x_0)$ such that

$$\langle Q(u_0, x_0), \eta(y, g(x_0))\rangle + \varphi(g(x_0), y) \nleq_{P^0} 0, \quad \forall y \in \mathcal{D},$$

which implies that $x_0 \in \Omega$. It remains to prove that $x_0$ is a unique solution of the problem (2.1).

Assume contrary that $x_1$ and $x_2$ are two distinct solutions of (2.1). Then

$$0 < \|x_1 - x_2\| \le \operatorname{diam} \Omega_\epsilon \to 0 \quad \text{as } \epsilon \to 0.$$

This is absurd and the proof is completed. □

**Corollary 3.6** *Assume that all assumptions of Lemma 2.8 hold and $g, \varphi(\cdot, y)$, $\eta(y, \cdot)$ and $\alpha_g$ are continuous functions for all $y \in \mathcal{D}$. Then the problem (2.1) is well-posed if and only if*

$$\Omega \neq \emptyset$$

*and*

$$\operatorname{diam} \Omega_\epsilon \to 0 \quad \text{as } \epsilon \to 0.$$

**Theorem 3.7** *Let $\mathcal{D}$ be a closed convex subset of a real Banach space $\mathcal{X}$. Let $\mathcal{Y}$ be a real Banach space ordered by a nonempty closed convex pointed cone $P$ with the apex at the origin and $P^0 \neq \emptyset$. Assume that $Q : L(\mathcal{X}, \mathcal{Y}) \times \mathcal{D} \to L(\mathcal{X}, \mathcal{Y})$ is a continuous mapping and $T : \mathcal{D} \to 2^{L(\mathcal{X}, \mathcal{Y})}$ is a nonempty compact set-valued mapping. If the following conditions are satisfied*:

(i) *$g : \mathcal{D} \to \mathcal{D}$ is continuous and $P$-convex*;
(ii) *$\varphi : \mathcal{D} \times \mathcal{D} \to \mathcal{Y}$ is $P$-convex in the second variable and $P$-concave in the first argument with $\varphi(g(x), x) = 0$, $\forall x \in \mathcal{D}$*;
(iii) *$\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is an affine mapping in the first and second variables with $\eta(g(x), x) = 0$, $\forall x \in \mathcal{D}$*;
(iv) *$T : \mathcal{D} \to 2^{L(\mathcal{X}, \mathcal{Y})}$ is $\mathfrak{H}$-hemicontinuous and relaxed $\eta$-$\alpha_g$-$P$-monotone with respect to first the variable of $Q$ and $g$*;
(v) *$\varphi(\cdot, y)$, $\eta(y, \cdot)$ and $\alpha_g$ are continuous functions for all $y \in \mathcal{D}$.*

*Then problem (2.1) is well-posed if and only if it has a unique solution.*

*Proof* Assume that problem (2.1) is well-posed, then it has a unique solution.

Conversely, let (2.1) have a unique solution $x_0$. If the problem (2.1) is not well-posed, then there exists an approximating sequence $\{x_n\}$ of (2.1) which does not converge to $x_0$. Since $\{x_n\}$ is an approximating sequence, there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}. \tag{3.6}$$

Now, we prove that $\{x_n\}$ is bounded. Suppose that $\{x_n\}$ is not bounded. Then, without loss of generality, we can suppose that

$$\|x_n\| \to +\infty \quad \text{as } n \to +\infty.$$

Let

$$t_n = \frac{1}{\|x_n - x_0\|}$$

and

$$w_n = x_0 + t_n(x_n - x_0).$$

Without loss of generality, we can assume that $t_n \in (0,1)$ and

$$w_n \to w \neq x_0.$$

By the hypothesis, $T$ is relaxed $\eta - \alpha_g - P$-monotone with respect to the first variable of $Q$ and $g$; therefore, for any $x, y \in \mathcal{D}$, we have

$$\langle Q(u,x_0) - Q(u_0,x_0), \eta(y, g(x_0))\rangle - \alpha_g(y - x_0) \geq_P 0, \quad \forall u_0 \in T(x_0), u \in T(y),$$

which implies that

$$\begin{aligned} &\langle Q(u_0,x_0), \eta(y,g(x_0))\rangle + \varphi(g(x_0),y) \\ &\quad \leq_P \langle Q(u,x_0), \eta(y,g(x_0))\rangle + \varphi(g(x_0),y) - \alpha_g(y - x_0). \end{aligned} \tag{3.7}$$

Since $x_0$ is a solution of (2.1), there exists $u_0 \in T(x_0)$ such that

$$\langle Q(u_0,x_0), \eta(y,g(x_0))\rangle + \varphi(g(x_0),y) \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}. \tag{3.8}$$

Combining (3.7) and (3.8) and, using Lemma 2.5(ii), we get

$$\langle Q(u,x_0), \eta(y,g(x_0))\rangle + \varphi(g(x_0),y) - \alpha_g(y - x_0) \nleq_{P^0} 0. \tag{3.9}$$

From the continuity of $g$, $\varphi$, $\eta$ and $\alpha_g$, we obtain

$$\begin{aligned} &\langle Q(u,w), \eta(y,g(w))\rangle + \varphi(g(w),y) - \alpha_g(y - w) \\ &\quad = \lim_{n\to\infty} \{Q(u,w_n), \eta(y,g(w_n)) + \varphi(g(w_n),y) - \alpha_g(y - w_n)\}. \end{aligned}$$

Since $\eta$ is affine in the second variable, $\varphi$ is $P$-concave in the first variable and using $w_n = x_0 + t_n(x_n - x_0)$, the above equation can be rewritten as

$$\begin{aligned} &\langle Q(u,w), \eta(y,g(w))\rangle + \varphi(g(w),y) - \alpha_g(y - w) \\ &\quad \geq_P \langle Q(u,x_0), \eta(y,g(x_0))\rangle + \varphi(g(x_0),y) - \alpha_g(y - x_0). \end{aligned} \tag{3.10}$$

Using (3.9), (3.10) and Lemma 2.5(ii), we obtain

$$\langle Q(u,w), \eta(y,g(w))\rangle + \varphi(g(w),y) - \alpha_g(y - w) \nleq_{P^0} 0.$$

Therefore, by Lemma 2.8, there exist $w \in \mathcal{D}$ and $w_0 \in T(w)$ such that

$$\langle Q(w_0, w), \eta(y, g(w))\rangle + \varphi(g(w), y) \leq_{P^0} 0, \quad \forall y \in \mathcal{D}.$$

The above inequality implies that $w$ is also a solution of (2.1), which contradicts the uniqueness of $x_0$. Hence, $\{x_n\}$ is a bounded sequence having a convergent subsequence $\{x_{n_\ell}\}$ which converges to $\bar{x}$ (say) as $\ell \to \infty$. Therefore from the definition of relaxed $\eta$-$\alpha_g$-$P$-monotonicity, for any $x_{n_\ell}, y \in \mathcal{D}$, we have

$$\langle Q(u, y) - Q(u_{n_\ell}, y), \eta(y, g(x_{n_\ell}))\rangle - \alpha_g(y - x_{n_\ell})) \geq_P 0, \quad \forall u_{n_\ell} \in T(x_{n_\ell}), u \in T(y).$$

This implies that

$$\begin{aligned} &\langle Q(u_{n_\ell}, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y) \\ &\quad \leq_P \langle Q(u, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y) - \alpha_g(y - x_{n_\ell}). \end{aligned} \tag{3.11}$$

Again from the continuity of $g$, $\varphi$, $\eta$ and $\alpha_g$, we have

$$\begin{aligned} &\langle Q(u, \bar{x}), \eta(y, g(\bar{x}))\rangle + \varphi(g(\bar{x}), y) - \alpha_g(y - \bar{x}) \\ &\quad = \lim_{\ell \to \infty} \{\langle Q(u, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y) - \alpha_g(y - x_{n_\ell})\}. \end{aligned}$$

This together with (3.11) shows that

$$\begin{aligned} &\langle Q(u, \bar{x}), \eta(y, g(\bar{x}))\rangle + \varphi(g(\bar{x}), y) - \alpha_g(y - \bar{x}) \\ &\quad \geq_P \lim_{\ell \to \infty} \{\langle Q(u_{n_\ell}, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y)\}. \end{aligned} \tag{3.12}$$

By virtue of (3.6), we can obtain

$$\lim_{\ell \to \infty} \{\langle Q(u_{n_\ell}, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y)\} \nleq_{P^0} 0. \tag{3.13}$$

From (3.12), (3.13) and Lemma 2.5(ii), we get

$$\langle Q(u, \bar{x}), \eta(y, g(\bar{x}))\rangle + \varphi(g(\bar{x}), y) - \alpha_g(y - \bar{x}) \nleq_{P^0} 0.$$

Thus, by Lemma 2.8, there exist $\bar{x} \in \mathcal{D}$ and $\bar{u} \in T(\bar{x})$ such that

$$\langle Q(\bar{u}, \bar{x}), \eta(y, g(\bar{x}))\rangle + \varphi(g(\bar{x}), y) \nleq_{P^0} 0,$$

which shows that $\bar{x}$ is a solution to (2.1). Hence,

$$x_{n_\ell} \to \bar{x}, \quad i.e., \quad x_{n_\ell} \to x_0.$$

Since $\{x_n\}$ is an approximating sequence, we have

$$x_n \to x_0.$$

The proof of Theorem 3.7 is completed. □

*Example* 3.8 Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{D} = [0,1]$ and $P = [0,\infty)$. Let us define the mappings $T : \mathcal{D} \to 2^{L(\mathcal{X},\mathcal{Y})}$, $\varphi : \mathcal{D} \times \mathcal{D} \to \mathcal{Y}$, $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, and $Q : L(\mathcal{X},\mathcal{Y}) \times \mathcal{D} \to L(\mathcal{X},\mathcal{Y})$ as follows:

$$\begin{cases} T(x) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is a continuous linear mapping such that } u(x) = -x\}; \\ g(x) = x; \\ \varphi(g(x), y) = y - x; \\ \eta(y, g(x)) = \frac{1}{2}(y - x); \\ Q(v, y) = v; \\ \alpha_g = -x^2. \end{cases}$$

In this case, the generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality problem (2.1) is to find $x \in \mathcal{D}$ and $u \in T(x)$ such that

$$\left\langle u, \frac{1}{2}(x - y)\right\rangle + y - x \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}. \tag{$*$}$$

It easy to see that $\Omega = \{0\}$ and $T$ is relaxed $\eta$-$\alpha_g$-$P$-monotone with respect to the first variable of $Q$ and $g$, and all conditions in Theorem 3.7 are satisfied. Therefore the problem ($*$) is well-posed.

**Theorem 3.9** *Suppose that all the conditions in Lemma* 2.8 *are satisfies. Further, assume that $\mathcal{D}$ is a compact set and $g, \varphi(\cdot, y), \eta(y, \cdot), \alpha_g$ are continuous functions for all $y \in \mathcal{D}$. Then problem* (2.1) *is well-posed if and only if the solution set $\Omega$ is nonempty.*

*Proof* Suppose that problem (2.1) is well-posed. Then its solution set $\Omega$ is nonempty. Conversely, let $\{x_n\}$ be an approximating sequence of problem (2.1). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}. \tag{3.14}$$

By the hypothesis, $\Omega$ is compact; hence, $\{x_n\}$ has a subsequence $\{x_{n_\ell}\}$ converging to some point $x_0 \in \mathcal{D}$. Since $T$ is relaxed $\eta$-$\alpha_g$-$P$-monotone with respect to the first variable of $Q$ and $g$, by Definition 2.3, for any $y \in \mathcal{D}$, we have

$$\begin{aligned} &\langle Q(u, x_{n_\ell}) - Q(u_{n_\ell}, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle - \alpha_g(y - x_{n_\ell}) \\ &\quad \geq_P 0, \quad \forall x_{n_\ell} \in \mathcal{D}, u_{n_\ell} \in T(x_{n_\ell}), u \in T(y), \end{aligned}$$

which implies

$$\begin{aligned} &\lim_{\ell\to\infty} \{\langle Q(u, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y) - \alpha_g(y - x_{n_\ell})\} \\ &\quad \geq_P \lim_{\ell\to\infty} \{\langle Q(u_{n_\ell}, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y)\}. \end{aligned}$$

Since $g$, $\eta$, $\varphi$, $\alpha_g$ are continuous,

$$\begin{aligned} &\langle Q(u, x_0), \eta(y, g(x_0))\rangle + \varphi(g(x_0), y) - \alpha_g(y - x_0) \\ &\quad = \lim_{\ell\to\infty} \{\langle Q(u, x_{n_\ell}), \eta(y, g(x_{n_\ell}))\rangle + \varphi(g(x_{n_\ell}), y) - \alpha_g(y - x_{n_\ell})\}. \end{aligned}$$

Using the above inequality, we obtain

$$\begin{aligned} &\big\langle Q(u,x_0),\eta\big(y,g(x_0)\big)\big\rangle+\varphi\big(g(x_0),y\big)-\alpha_g(y-x_0)\\ &\quad\geq_P \lim_{\ell\to\infty}\big\{\big\langle Q(u_{n_\ell},x_{n_\ell}),\eta\big(y,g(x_{n_\ell})\big)\big\rangle+\varphi\big(g(x_{n_\ell}),y\big)\big\}. \end{aligned} \tag{3.15}$$

By virtue of (3.14), it can be written as

$$\lim_{\ell\to\infty}\big\{\big\langle Q(u_{n_\ell},x_{n_\ell}),\eta\big(y,g(x_{n_\ell})\big)\big\rangle+\varphi\big(g(x_{n_\ell}),y\big)\big\}\nleq_{P^0} 0. \tag{3.16}$$

It follows from (3.15), (3.16) and Lemma 2.5(ii) that

$$\big\langle Q(u,x_0),\eta\big(y,g(x_0)\big)\big\rangle+\varphi\big(g(x_0),y\big)-\alpha_g(y-x_0)\nleq_{P^0} 0.$$

Thus, by Lemma 2.8, there exist $x_0\in\mathcal{D}$ and $u_0\in T(x_0)$ such that

$$\big\langle Q(u_0,x_0),\eta\big(y,g(x_0)\big)\big\rangle+\varphi\big(g(x_0),y\big)\nleq_{P^0} 0.$$

This implies that $x_0\in\Omega$.

The proof is completed. □

*Example* 3.10 Let $\mathcal{X}=\mathcal{Y}=\mathbb{R}^2$, $\mathcal{D}=[0,1]\times[0,1]$ and $P=[0,\infty)\times[0,\infty)$. Let us define the mappings $T:\mathcal{D}\to 2^{L(\mathcal{X},\mathcal{Y})}$, $\varphi:\mathcal{D}\times\mathcal{D}\to\mathcal{Y}$, $\eta:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$, and $Q:L(\mathcal{X},\mathcal{Y})\times\mathcal{D}\to L(\mathcal{X},\mathcal{Y})$ as follows:

$$\begin{cases} T(x)=\{w,z:\mathbb{R}^2\to\mathbb{R}\mid w,z\text{ are continuous linear mappings}\\ \qquad\qquad\text{such that } w(x_1,x_2)=x_1, z(x_1,x_2)=x_2\};\\ g(x)=x;\\ \varphi(g(x),y)=y-x;\\ \eta(y,g(x))=y-x;\\ Q(u,x)=-u;\\ \alpha_g=0. \end{cases}$$

In this case, the generalized $(\eta,g,\varphi)$-mixed vector variational-type inequality problem (2.1) is to find $x\in\mathcal{D}$ and $u\in T(x)$ such that

$$\langle -u,x-y\rangle+y-x\nleq_{P^0} 0,\quad\forall y\in\mathcal{D}. \tag{$**$}$$

Clearly, $\Omega=[0,1]\times[0,1]$. It can be easily verified that $T$ is relaxed $\eta$-$\alpha_g$-$P$-monotone with respect to the first variable of $Q$ and $g$, and all conditions in Theorem 3.9 are satisfies. Hence, problem ($**$) is well-posed.

**Theorem 3.11** *Assume that all conditions in Lemma 2.8 are satisfied and assume that $g,\varphi(\cdot,y)$, $\eta(y,\cdot)$, $\alpha_g$ are continuous functions for all $y\in\mathcal{D}$. If there exists some $\epsilon>0$ such that $\Omega_\epsilon\neq\emptyset$ and is bounded. Then problem (2.1) is well-posed.*

*Proof* Let $\epsilon > 0$ such that

$$\Omega_\epsilon \neq \emptyset$$

and suppose $\{x_n\}$ is an approximating sequence of problem (2.1). Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \not\leq_{P^0} 0, \quad \forall y \in \mathcal{D},$$

which implies that

$$x_n \in \Omega_\epsilon, \quad \forall n > m.$$

Therefore, $\{x_n\}$ is a bounded sequence which has a convergent subsequence $\{x_{n_\ell}\}$ converging to $x_0$ as $\ell \to \infty$. Following lines similar to the proof of Theorem 3.9, we get $x_0 \in \Omega$. The proof is completed. □

## 4 Well-posedness of optimization problems with constraints

This section is devoted to a study of the well-posedness of optimization problems with generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality constraints:

$$\begin{aligned} &P\text{-minimize } \Psi(x) \\ &\quad \text{subject to } x \in \Omega, \end{aligned} \tag{4.1}$$

where $\Psi : \mathcal{D} \to \mathbf{R}$ is a function, and $\Omega$ is the solution set of problem (2.1).

Denote by $\zeta$ the solution set of (4.1), *i.e.*,

$$\begin{aligned} \zeta = \Big\{ & x \in \mathcal{D} \,\Big|\, \exists u \in T(x) \text{ such that } \Psi(x) \leq_P \inf_{y \in \Omega} \Psi(y) \text{ and} \\ & \langle Q(u, x), \eta(y, g(x))\rangle + \varphi(g(x), y) \not\leq_{P^0} 0, \forall y \in \mathcal{D} \Big\}. \end{aligned}$$

**Definition 4.1** A sequence $\{x_n\} \in \mathcal{D}$ is said to be an approximating sequence for problem (4.1), if

(i) $\lim_{n\to\infty} \sup \Psi(x_n) \leq_P \inf_{y\in\Omega} \Psi(y)$,

(ii) there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \not\leq_{P^0} 0, \quad \forall y \in \mathcal{D}.$$

For $\delta, \epsilon \geq 0$, we denote the approximating solution set of (4.1) by $\zeta(\delta, \epsilon)$, *i.e.*,

$$\begin{aligned} \zeta(\delta, \epsilon) = \Big\{ & x \in \mathcal{D} \,\Big|\, \exists u \in T(x) \text{ such that } \Psi(x) \leq_P \inf_{y \in \Omega} \Psi(y) + \delta \text{ and} \\ & \langle Q(u, x), \eta(y, g(x))\rangle + \varphi(g(x), y) + \epsilon e \not\leq_{P^0} 0, \forall y \in \mathcal{D} \Big\}. \end{aligned}$$

*Remark* 4.2 It is obvious that $\zeta = \zeta(\delta, \epsilon)$ when $(\delta, \epsilon) = (0, 0)$ and

$$\zeta \subseteq \zeta(\delta, \epsilon), \quad \forall \delta, \epsilon > 0.$$

**Theorem 4.3** *Assume that all assumptions of Theorem 3.5 are satisfies and $\Psi$ is lower semicontinuous. Then (4.1) is well-posed if and only if*

$$\zeta(\delta,\epsilon)\neq\emptyset,\quad \forall\delta,\epsilon>0$$

*and*

$$\operatorname{diam}\zeta(\delta,\epsilon)\to 0\quad \text{as } (\delta,\epsilon)\to(0,0).$$

*Proof* The necessary part directly follows from the proof of Theorem 3.5, so it is omitted. Conversely, suppose that $\{x_n\}$ is an approximating sequence of (4.1). Then there exist $u_n\in T(x_n)$ and a sequence of positive real number $\epsilon_n\to 0$ such that

$$\lim_{n\to\infty}\sup\Psi(x_n)\leq_P\inf_{y\in\Omega}\Psi(y), \tag{4.2}$$

$$\bigl\langle Q(u_n,x_n),\eta\bigl(y,g(x_n)\bigr)\bigr\rangle+\varphi\bigl(g(x_n),y\bigr)+\epsilon_n e\nleq_{P^0}0,\quad \forall y\in\mathcal{D}, \tag{4.3}$$

which implies that

$$x_n\in\zeta(\delta_n,\epsilon_n),\quad \text{for } \delta_n\to 0.$$

Since

$$\operatorname{diam}\zeta(\delta,\epsilon)\to 0\quad \text{as } (\delta,\epsilon)\to(0,0),$$

and $\{x_n\}$ is a Cauchy sequence converging to $x_0\in\mathcal{D}$ (because $\mathcal{D}$ is closed). By the same argument as in Theorem 3.5, we get

$$\bigl\langle Q(u_0,x_0),\eta\bigl(y,g(x_0)\bigr)\bigr\rangle+\varphi\bigl(g(x_0),y\bigr)\nleq_{P^0}0,\quad \forall u_0\in T(x_0),y\in\mathcal{D}. \tag{4.4}$$

Since $\Psi$ is lower semicontinuous,

$$\Psi(x_0)\leq_P\lim_{n\to\infty}\inf\Psi(x_n)\leq_P\lim_{n\to\infty}\sup\Psi(x_n).$$

By using (4.1), the above inequality reduces to

$$\Psi(x_0)\leq_P\inf_{y\in\Omega}\Psi(y). \tag{4.5}$$

Thus, from (4.3) and (4.4), we conclude that $x_0$ solve (4.1). The uniqueness of $x_0$ directly follows from the assumption

$$\operatorname{diam}\zeta(\delta,\epsilon)\to 0\quad \text{as } (\delta,\epsilon)\to(0,0).$$

This completes the proof. □

*Example* 4.4 Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{D} = [0,1]$ and $P = [0,\infty)$. Let us define the mappings $\Psi : \mathcal{D} \to \mathbf{R}$, $T : \mathcal{D} \to 2^{L(\mathcal{X},\mathcal{Y})}$, $\varphi : \mathcal{D} \times \mathcal{D} \to \mathcal{Y}$, $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, and $Q : L(\mathcal{X},\mathcal{Y}) \times \mathcal{D} \to L(\mathcal{X},\mathcal{Y})$ as follows:

$$\begin{cases} \Psi(x) = |x^3|; \\ T(x) = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is a continuous linear mapping such that } u(x) = -x\}; \\ g(x) = x; \\ \varphi(g(x), y) = y - x; \\ \eta(g(x), y) = \frac{1}{2}(y - x); \\ Q(v, x) = v; \\ \alpha_g = -x^2. \end{cases}$$

Consider the optimization problem with generalized $(\eta, g, \varphi)$-mixed vector variational-type inequality constraints:

$$\begin{aligned} &P\text{-minimize } \left|x^3\right| \\ &\quad \text{subject to } x \in \Omega, \end{aligned} \tag{4.6}$$

where

$$\Omega = \left\{ x \in \mathcal{D} \;\middle|\; \exists u \in T(x) \text{ such that } \left\langle u, \frac{1}{2}(x - y) \right\rangle + y - x \nleq_{P^0} 0, \forall y \in \mathcal{D} \right\}.$$

We see that $\Omega = \{0\}$. Since

$$\zeta(\delta, \epsilon) = \left\{ x \in \mathcal{D} \;\middle|\; \exists u \in T(x) \text{ such that } \left|x^3\right| \leq_P \delta \text{ and } (y - x)\left(1 + \frac{x}{2}\right) + \epsilon \nleq_{P^0} 0, \forall y \in \mathcal{D} \right\},$$

we have

$$\operatorname{diam} \zeta(\delta, \epsilon) \to 0 \quad \text{as } (\delta, \epsilon) \to (0, 0).$$

It is easily verified that $T$ is relaxed $\eta$-$\alpha_g$-$P$-monotone with respect to the first variable of $Q$ and $g$, and all assumptions of Theorem 4.3 are satisfied. Hence (4.6) is well-posed.

**Theorem 4.5** *Let all conditions in Theorem* 3.7 *hold and let* $\Psi$ *be lower semicontinuous. Then the problem* (4.1) *is well-posed if and only if it has a unique solution.*

*Proof* The necessary condition is obvious. Conversely, let (4.1) have a unique solution $x_0$. Then

$$\begin{aligned} &\Psi(x_0) = \inf_{y \in \Omega} \Psi(y), \\ &\left\langle Q(u_0, x_0), \eta\bigl(y, g(x_0)\bigr)\right\rangle + \varphi\bigl(g(x_0), y\bigr) \nleq_{P^0} 0, \quad \forall u_0 \in T(x_0), y \in \mathcal{D}. \end{aligned}$$

Let $\{x_n\}$ be an approximating sequence. Then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\lim_{n\to\infty} \sup \Psi(x_n) \leq_P \inf_{y\in\Omega} \Psi(y),$$
$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}.$$

Now, following lines similar to the proof of Theorem 3.7, we find that the sequence $\{x_n\}$ has a subsequence $\{x_{n_\ell}\}$ converging to $\bar{x}$, for any $\bar{x} \in \mathcal{D}$ and

$$\langle Q(\bar{u}, \bar{x}), \eta(y, g(\bar{x}))\rangle + \varphi(g(\bar{x}), y) \nleq_{P^0} 0, \quad \forall \bar{u} \in T(\bar{x}), y \in \mathcal{D}. \tag{4.7}$$

Since $\Psi$ is lower semicontinuous, therefore,

$$\Psi(\bar{x}) \leq_P \lim_{\ell\to\infty} \inf \Psi(x_{n_\ell}) \leq_P \lim_{\ell\to\infty} \sup \Psi(x_{n_\ell}) \leq_P \inf_{y\in\Omega} \Psi(y). \tag{4.8}$$

Thus, from (4.7) and (4.8), we conclude that $\bar{x} \in \zeta$, and the proof is completed. □

**Theorem 4.6** *Assume that all assumptions of Theorem 4.5 are satisfies and $\Psi$ is lower semicontinuous, and there exists some $\epsilon > 0$ such that $\zeta(\epsilon, \epsilon) \neq \emptyset$, and it is bounded. Then (4.1) is well-posed.*

*Proof* Let $\epsilon > 0$ such that

$$\zeta(\epsilon, \epsilon) \neq \emptyset$$

and suppose $\{x_n\}$ is an approximating sequence of problem (2.1). Then

(i) $\lim_{n\to\infty} \sup \Psi(x_n) \leq_P \inf_{y\in\Omega} \Psi(y)$,

(ii) there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\langle Q(u_n, x_n), \eta(y, g(x_n))\rangle + \varphi(g(x_n), y) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}, n \in \mathbb{N},$$

which implies that for some positive integer $m$

$$x_n \in \zeta(\epsilon, \epsilon), \quad \forall n > m.$$

Therefore, $\{x_n\}$ is a bounded sequence and there exists a subsequence $\{x_{n_\ell}\}$ such that $\{x_{n_\ell}\}$ converges to $x_0$ as $\ell \to \infty$. Following the lines similar to the proof of Theorem 4.5, we conclude that $x_0 \in \zeta$. Hence, (4.1) is well-posed and the proof is completed. □

## 5 Well-posedness of optimization problems by using well-posedness of constraints

In this section, we derive the well-posedness of problem (4.1) by using the well-posedness of problem (2.1).

**Theorem 5.1** *Let $\mathcal{D}$ be a nonempty compact set and $\Psi$ be lower semicontinuous. Suppose problem (4.1) has a unique solution. If problem (2.1) is well-posed, then problem (4.1) is also well-posed.*

*Proof* If problem (4.1) has a unique solution $x_0$, and $\{x_n\}$ is an approximating sequence for problem (4.1), then there exist $u_n \in T(x_n)$ and a sequence of positive real numbers $\epsilon_n \to 0$ such that

$$\begin{aligned} &\lim_{n\to\infty} \sup \Psi(x_n) \leq_P \inf_{y\in\Omega} \Psi(y), \\ &\bigl\langle Q(u_n,x_n),\eta\bigl(y,g(x_n)\bigr)\bigr\rangle + \varphi\bigl(g(x_n),y\bigr) + \epsilon_n e \nleq_{P^0} 0, \quad \forall y \in \mathcal{D}. \end{aligned}$$

Since $\mathcal{D}$ is compact, there exists a subsequence $\{x_{n_\ell}\}$ of $\{x_n\}$ such that $\{x_{n_\ell}\}$ converges to a $\bar{x}$ (*say*) as $\ell \to \infty$. Since problem (2.1) is well-posed, $\bar{x}$ solves (2.1), *i.e.*,

$$\bigl\langle Q(\bar{u},\bar{x}),\eta\bigl(y,g(\bar{x})\bigr)\bigr\rangle + \varphi\bigl(g(\bar{x}),y\bigr) \nleq_{P^0} 0, \quad \forall \bar{u} \in T(\bar{x}), y \in \mathcal{D}. \tag{5.1}$$

Since $\Psi$ is lower semicontinuous, we have

$$\Psi(\bar{x}) \leq_P \lim_{\ell\to\infty} \inf \Psi(x_{n_\ell}) \leq_P \lim_{\ell\to\infty} \sup \Psi(x_{n_\ell}) \leq_P \inf_{y\in\Omega} \Psi(y). \tag{5.2}$$

Thus, from (5.1) and (5.2) we conclude that $\bar{x}$ solves problem (4.1). But (4.1) has a unique solution $x_0$; therefore,

$$\bar{x} = x_0 \quad \text{and} \quad x_n \to x_0.$$

Hence, (4.1) is well-posed. The proof is completed. □

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