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Several new cyclic Jensen type inequalities and their applications

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Abstract

We present some fundamental results and definitions regarding Jensen's inequality with the aim of obtaining new generalizations of cyclic refinements of Jensen's inequality from convex to higher order convex functions using Taylor's formula. We discuss the monotonicity of functionals for n -convex functions at a point. Applications of our work include new bounds for some important inequalities used in information theory.

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1 Introduction

The convexity of functions has been frequently used in various fields of pure and applied mathematics, for instance in function theory, mathematical analysis, functional analysis, probability theory, optimization theory, operational research, information theory. In short, convex functions entail a strong and elegant interaction between analysis and geometry. The simple generalization to a convex function extensively widens our scope for analysis.

Inequalities are frequently used in solving several problems of the applied sciences. Some recent work on the applications of mathematical inequalities can be found in [7, 12] and [24].

The Jensen inequality for convex functions plays a pivotal role in the theory of inequalities because of the fact that various other inequalities, for instance the Holder and Minkowski inequalities and the arithmetic mean–geometric mean inequality can be obtained as particular cases.

Some useful results on Jensen's inequality for strongly convex functions and generalized majorization inequalities can be found in [20] and [13].

For $\psi : [a, b] \rightarrow \mathbb{R}$ where $\psi^{(n-1)}$ is absolutely continuous, the renowned Taylor formula $\forall z \in [a, b]$ at the point $c \in [a, b]$ is

$$\psi(z) = \sum_{w=0}^{n-1} \frac{\psi^{(w)}(c)}{w!} (z-c)^w + \frac{1}{(n-1)!} \int_c^z \psi^{(n)}(\xi) (z-\xi)^{n-1} d\xi. \quad (1)$$

Taking $\nu = 1, \dots, 5$, the Green functions $G_\nu : [a, b] \times [a, b] \rightarrow \mathbb{R}$ are defined as

$$G_1(z, \xi) = \begin{cases} \frac{(b-z)(a-\xi)}{b-a}, & a \leq \xi \leq z, \\ \frac{(b-\xi)(a-z)}{b-a}, & z \leq \xi \leq b, \end{cases} \tag{2}$$

$$G_2(z, \xi) = \begin{cases} a - \xi, & a \leq \xi \leq z, \\ a - z, & z \leq \xi \leq b, \end{cases} \tag{3}$$

$$G_3(z, \xi) = \begin{cases} z - b, & a \leq \xi \leq z, \\ \xi - b, & z \leq \xi \leq b, \end{cases} \tag{4}$$

$$G_4(z, \xi) = \begin{cases} z - a, & a \leq \xi \leq z, \\ \xi - a, & z \leq \xi \leq b, \end{cases} \tag{5}$$

$$G_5(z, \xi) = \begin{cases} b - \xi, & a \leq \xi \leq z, \\ b - z, & z \leq \xi \leq b. \end{cases} \tag{6}$$

Each of these functions is convex and continuous with respect to the two variables z and ξ .

Lemma 1 ([19]) *Suppose $\psi \in C^2[a, b]$, then the following identities are valid:*

$$\psi(z) = \frac{a-z}{b-a} \psi(a) + \frac{z-a}{b-a} \psi(b) + \int_a^b G_1(z, \xi) \psi''(\xi) d\xi, \tag{7}$$

$$\psi(z) = \psi(a) + (z-a) \psi'(a) + \int_a^b G_2(z, \xi) \psi''(\xi) d\xi, \tag{8}$$

$$\psi(z) = \psi(b) + (b-z) \psi'(b) + \int_a^b G_3(z, \xi) \psi''(\xi) d\xi, \tag{9}$$

$$\psi(z) = \psi(b) - (b-a) \psi'(b) + (z-a) \psi'(a) + \int_a^b G_4(z, \xi) \psi''(\xi) d\xi, \tag{10}$$

$$\psi(z) = \psi(a) + (b-a) \psi'(a) - (b-z) \psi'(b) + \int_a^b G_5(z, \xi) \psi''(\xi) d\xi. \tag{11}$$

Proof Consider the integral

$$\int_a^b G_\nu(z, r) \psi''(r) dr = \int_a^z G_\nu(z, r) \psi''(r) dr + \int_z^b G_\nu(z, r) \psi''(r) dr.$$

For fixed $\nu = 1, \dots, 5$, after integrating for a particular value of the Green functions, (7)–(11) are obtained. □

In order to state some recent investigations of Jensen’s inequality, we shall use the following assumptions:

- (H₁) Suppose $2 \leq k \leq m$ are integers and probability distributions $\mathbf{p}, \boldsymbol{\lambda} \in \mathbb{R}_+^m$ so that $\sum_i p_i = 1$ and $\sum_k \lambda_k = 1$.
- (H₂) Suppose (Z, \mathcal{B}, μ) represents probability space.

Suppose for a fixed integer $l \geq 2$ and for $m = 1, \dots, l$, \mathcal{B}^l denotes the σ -algebra in Z^l which is generated by the projection mappings $pr_m : Z^l \rightarrow Z$ given as

$$pr_m(z_1, \dots, z_l) := z_m,$$

where μ^l represents a product measure on \mathcal{B}^l . This measure is uniquely (μ is σ -finite) specified by

$$\mu^l(B_1 \times \dots \times B_l) := \mu(B_1) \cdots \mu(B_l), \quad B_m \in \mathcal{B}, m = 1, \dots, l.$$

(H₃) g is μ -integrable on Z with values in $I \subset \mathbb{R}$.

(H₄) Suppose ψ is convex on I with $\psi \circ g$ being μ -integrable on Z .

Considering the assumptions (H₁)–(H₄), we now define the following.

Theorem 1 ([8]) *Assume (H₁) with $\psi : C \rightarrow \mathbb{R}$ being a convex function where C is a convex subset of real vector space Z . If $z_1, \dots, z_m \in C$, then*

$$\begin{aligned} \psi \left(\sum_{i=1}^m p_i z_i \right) &\leq C_{\text{dis}} = C_{\text{dis}}(\psi, z, \mathbf{p}, \boldsymbol{\lambda}) \\ &:= \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \psi \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} z_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \right) \leq \sum_{i=1}^m p_i \psi(z_i), \end{aligned} \tag{12}$$

while $i + j$ means $i + j \pmod m$.

Theorem 2 ([8]) *Suppose (H₁)–(H₄) hold. Then*

$$\psi \left(\int_Z g \, d\mu \right) \leq C_{\text{par}}(t) \leq C_{\text{int}} \leq \int_Z \psi \circ g \, d\mu, \quad t \in [0, 1],$$

where

$$\begin{aligned} C_{\text{int}} &= C_{\text{int}}(\psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \\ &:= \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \int_{Z^m} \psi \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} g(z_{i+j})}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \right) d\mu^m(z_1, \dots, z_m), \end{aligned} \tag{13}$$

and for $t \in [0, 1]$

$$\begin{aligned} C_{\text{par}}(t) &= C_{\text{par}}(t, \psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \\ &:= \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \\ &\quad \cdot \int_{Z^m} \psi \left(t \frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} g(z_{i+j})}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} + (1-t) \int_Z g \, d\mu \right) d\mu^m(z_1, \dots, z_m), \end{aligned} \tag{14}$$

while $i + j$ means $i + j \pmod m$.

For the purpose of achieving our targets, we construct non-negative linear functionals from the above theorem.

Remark 1 Under the assumptions of Theorem 1 and Theorem 2 for ψ to be a convex function, we have

$$\begin{aligned}
 J_1(\psi) &= J_1(\mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}; \psi) := \sum_{i=1}^m p_i \psi(z_i) - C_{\text{dis}}(\psi, \mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}) \geq 0, \\
 J_2(\psi) &= J_1(\mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}; \psi) := C_{\text{dis}}(\psi, \mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}) - \psi\left(\sum_{i=1}^m p_i z_i\right) \geq 0, \\
 J_3(\psi) &= J_3(\psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := \int_X \psi \circ g \, d\mu - C_{\text{int}}(\psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \geq 0, \\
 J_4(\psi) &= J_4(t, \psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := \int_X \psi \circ g \, d\mu - C_{\text{par}}(t, \psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \geq 0; \quad t \in [0, 1], \\
 J_5(\psi) &= J_5(t, \psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := C_{\text{int}}(\psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) - C_{\text{par}}(t, \psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) \geq 0; \quad t \in [0, 1], \\
 J_6(\psi) &= J_6(t, \psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) := C_{\text{par}}(t, \psi, g, \mu, \mathbf{p}, \boldsymbol{\lambda}) - \psi\left(\int_X g \, d\mu\right) \geq 0; \quad t \in [0, 1].
 \end{aligned}$$

2 Main results

To start for real weights, we need the following assumptions:

- (A₁) For $J_u(\cdot)$ ($u = 1, 2$), suppose $\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} z_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \in [a, b]$ for $i = 1, \dots, m$.
- (A₂) For the linear functionals $J_u(\cdot)$ ($u = 3, \dots, 6$), suppose (H₂)–(H₄) are satisfied and

$$\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} g(z_{i+j})}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \in [a, b] \quad \text{for } i = 1, \dots, m.$$

We take cyclic refinements of Jensen’s inequality in discrete and continuous versions and form the following identities with real weights by using Taylor’s formula.

Theorem 3 Suppose $m, k \in \mathbb{N}$, p_1, \dots, p_m and $\lambda_1, \dots, \lambda_k$ are real tuples for $2 \leq k \leq m$, such that $\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \neq 0$ for $i = 1, \dots, m$ with $\sum_{i=1}^m p_i = 1$ and $\sum_{j=1}^k \lambda_j = 1$. Also let $z \in [a, b] \subset \mathbb{R}$ and $\mathbf{z} \in [a, b]^m$. Consider the function $\psi : [a, b] \rightarrow \mathbb{R}$ such that $\psi^{(n-1)}$ is absolutely continuous and G_ν ($\nu = 1, \dots, 5$) are same as given in (2)–(6), respectively. Then for ($u = 1, \dots, 6$) along with the assumptions (A₁) and (A₂), we have the following generalized identities:

(a)

$$J_u(\psi) = \sum_{w=1}^{n-1} \frac{\psi^{(w)}(a)}{w!} J_u((z-a)^w) + \frac{1}{(n-1)!} \int_a^b \psi^{(n)}(\xi) J_u((z-\xi)_+^{n-1}) \, d\xi, \quad (15)$$

(b)

$$J_u(\psi) = \sum_{w=1}^{n-1} \frac{(-1)^w \psi^{(w)}(b)}{w!} J_u((b-z)^w) - \frac{(-1)^{n-1}}{(n-1)!} \int_a^b \psi^{(n)}(\xi) J_u((\xi-z)_+^{n-1}) \, d\xi, \quad (16)$$

(c)

$$\begin{aligned}
 J_u(\psi) &= \left(\frac{\psi(b) - \psi(a)}{b - a}\right) J_u(z) + \int_a^b J_u(G_1(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad + \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_\xi^b J_u(G_1(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{17}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= \psi'(b) J_u(z) + \int_a^b J_u(G_2(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad + \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_\xi^b J_u(G_2(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{18}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= -\psi'(a) J_u(z) + \int_a^b J_u(G_3(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad + \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_\xi^b J_u(G_3(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{19}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= \psi'(a) J_u(z) + \int_a^b J_u(G_4(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad + \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_\xi^b J_u(G_4(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{20}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= \psi'(b) J_u(z) + \int_a^b J_u(G_5(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad + \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_\xi^b J_u(G_5(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{21}
 \end{aligned}$$

(d)

$$\begin{aligned}
 J_u(\psi) &= \left(\frac{\psi(b) - \psi(a)}{b - a}\right) J_u(z) + \int_a^b J_u(G_1(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - b)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad - \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_a^\xi J_u(G_1(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{22}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= \psi'(b) J_u(z) + \int_a^b J_u(G_2(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - b)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad - \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_a^\xi J_u(G_2(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= -\psi'(a) J_u(z) + \int_a^b J_u(G_3(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - b)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad - \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_a^\xi J_u(G_3(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{24}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= \psi'(a) J_u(z) + \int_a^b J_u(G_4(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - b)^{w-2}}{(w - 2)!}\right) dr \\
 &\quad - \frac{1}{(n - 3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_a^\xi J_u(G_4(z, r))(r - \xi)^{n-3} dr\right) d\xi, \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 J_u(\psi) &= \psi'(b)J_u(z) + \int_a^b J_u(G_5(z,r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r-b)^{w-2}}{(w-2)!} \right) dr \\
 &\quad - \frac{1}{(n-3)!} \int_a^b \psi^{(n)}(\xi) \left(\int_a^\xi J_u(G_5(z,r))(r-\xi)^{n-3} dr \right) d\xi,
 \end{aligned} \tag{26}$$

where $(z - \xi)_+ = \max(z - \xi, 0)$.

Proof Fix $u = 1, \dots, 6$.

(a) Applying Taylor’s formula (1) at point a , we get

$$\psi(z) = \sum_{w=0}^{n-1} \frac{\psi^{(w)}(a)}{w!} (z-a)^w + \frac{1}{(n-1)!} \int_a^b \psi^{(n)}(\xi) (z-\xi)_+^{n-1} d\xi. \tag{27}$$

Since each $J_u(\cdot)$ is a linear functional, applying it to the Taylor expression (27) and using the constancy property of the $J_u(\cdot)$, we get (15).

(b) Applying Taylor’s formula (1) at point b , we get

$$\psi(z) = \sum_{w=0}^{n-1} \frac{(-1)^w \psi^{(w)}(b)}{w!} (z-b)^w - \frac{(-1)^{n-1}}{(n-1)!} \int_a^b \psi^{(n)}(\xi) (\xi-z)_+^{n-1} d\xi, \tag{28}$$

and following similar steps we get (16).

(c) For fixed $v = 1$, applying (7) to $J_u(\cdot)$ and using the linearity of $J_u(\cdot)$, we have

$$\begin{aligned}
 J_u(\psi) &= \psi(a)J_u\left(\frac{b-z}{b-a}\right) + \psi(b)J_u\left(\frac{z-a}{b-a}\right) + \int_a^b J_u(G_1(z,r))\psi''(r) dr \\
 &= \psi(a)\frac{J_u(b-z)}{b-a} + \psi(b)\frac{J_u(z-a)}{b-a} + \int_a^b J_u(G_1(z,r))\psi''(r) dr \\
 &= \frac{1}{b-a} (\psi(a)J_u(b) - \psi(a)J_u(z) + \psi(b)J_u(z) - \psi(b)J_u(a)) \\
 &\quad + \int_a^b J_u(G_1(z,r))\psi''(r) dr \\
 &= \frac{1}{b-a} (\psi(b)J_u(z) - \psi(a)J_u(z)) + \int_a^b J_u(G_1(z,r))\psi''(r) dr.
 \end{aligned} \tag{29}$$

Differentiating (1) twice and putting $c = a$ and replacing n by $(n - 2)$ or utilizing (1) on the function ψ'' at the point a , we get

$$\psi''(r) = \sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)}{(w-2)!} (r-a)^{w-2} + \frac{1}{(n-3)!} \int_a^r \psi^{(n)}(\xi) (r-\xi)^{n-3} d\xi. \tag{30}$$

Now, using (30) in (29), we get

$$\begin{aligned}
 J_u(\psi) &= \left(\frac{\psi(b) - \psi(a)}{b-a} \right) J_u(z) + \int_a^b J_u(G_1(z,r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r-a)^{w-2}}{(w-2)!} \right) dr \\
 &\quad + \frac{1}{(n-3)!} \int_a^b J_u(G_1(z,r)) \left(\int_a^r \psi^{(n)}(\xi) (r-\xi)^{n-3} d\xi \right) dr.
 \end{aligned}$$

Now applying Fubini’s theorem to the last term gives (17), respectively, for $\nu = 1$ and $u = 1, \dots, 6$. The results for $\nu = 2, 3, 4, 5$ can be obtained in the same way.

(d) Differentiating (1) twice and now taking $c = b$, we get

$$\psi''(r) = \sum_{w=2}^{n-1} \frac{\psi^{(w)}(b)}{(w-2)!} (r-b)^{w-2} - \frac{1}{(n-3)!} \int_r^b \psi^{(n)}(\xi) (r-\xi)^{n-3} d\xi. \tag{31}$$

Analogously, putting (31) in (29) and using Fubini’s theorem gives (22), respectively, for $\nu = 1$ and $u = 1, \dots, 6$. The results for $\nu = 2, 3, 4, 5$ can be obtained in the same way. \square

The following theorem gives a key criterion to test the n -convexity of a function ψ (see [23, p. 16]).

Theorem 4 *If $\psi^{(n)}$ exists, then ψ is n -convex if and only if $\psi^{(n)} \geq 0$.*

We now use Theorem 4 to give generalizations of cyclic Jensen type linear functionals for real weights.

Corollary 1 *Under the assumptions of Theorem 3, if ψ is n -convex, we obtain the following results:*

(a) *If, for all $u = 1, \dots, 6$,*

$$J_u((z-\xi)_+^{n-1}) \geq 0, \quad \xi \in [a, b], \tag{32}$$

holds, then we have

$$J_u(\psi) \geq \sum_{w=1}^{n-1} \frac{\psi^{(w)}(a)}{w!} J_u((z-a)^w) \tag{33}$$

for $u = 1, \dots, 6$.

(b) *If, for all $u = 1, \dots, 6$,*

$$(-1)^{n-1} J_u((\xi-z)_+^{n-1}) \leq 0, \quad \xi \in [a, b], \tag{34}$$

holds, then we have

$$J_u(\psi) \geq \sum_{w=1}^{n-1} \frac{(-1)^w \psi^{(w)}(b)}{w!} J_u((b-z)^w) \tag{35}$$

for $u = 1, \dots, 6$.

(c) *If, for all $u = 1, \dots, 6$ and $\nu = 1, \dots, 5$,*

$$\int_{\xi}^b J_u(G_{\nu}(z, r))(r-\xi)^{n-3} dr \geq 0, \quad \xi \in [a, b], \tag{36}$$

holds, then we have

$$J_u(\psi) \geq \left(\frac{\psi(b) - \psi(a)}{b - a}\right) J_u(z) + \int_a^b J_u(G_1(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr, \tag{37}$$

$$J_u(\psi) \geq \psi'(b) J_u(z) + \int_a^b J_u(G_2(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr, \tag{38}$$

$$J_u(\psi) \geq -\psi'(a) J_u(z) + \int_a^b J_u(G_3(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr, \tag{39}$$

$$J_u(\psi) \geq \psi'(a) J_u(z) + \int_a^b J_u(G_4(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr, \tag{40}$$

$$J_u(\psi) \geq \psi'(b) J_u(z) + \int_a^b J_u(G_5(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r - a)^{w-2}}{(w - 2)!}\right) dr. \tag{41}$$

(d) If, for all $u = 1, \dots, 6$ and $v = 1, \dots, 5$,

$$\int_a^\xi J_u(G_v(z, r))(r - \xi)^{n-3} dr \leq 0, \quad \xi \in [a, b], \tag{42}$$

holds, then we have

$$J_u(\psi) \geq \left(\frac{\psi(b) - \psi(a)}{b - a}\right) J_u(z) + \int_a^b J_u(G_1(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(b)(r - b)^{w-2}}{(w - 2)!}\right) dr, \tag{43}$$

$$J_u(\psi) \geq \psi'(b) J_u(z) + \int_a^b J_u(G_2(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(b)(r - b)^{w-2}}{(w - 2)!}\right) dr, \tag{44}$$

$$J_u(\psi) \geq -\psi'(a) J_u(z) + \int_a^b J_u(G_3(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(b)(r - b)^{w-2}}{(w - 2)!}\right) dr, \tag{45}$$

$$J_u(\psi) \geq \psi'(a) J_u(z) + \int_a^b J_u(G_4(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(b)(r - b)^{w-2}}{(w - 2)!}\right) dr, \tag{46}$$

$$J_u(\psi) \geq \psi'(b) J_u(z) + \int_a^b J_u(G_5(z, r)) \left(\sum_{w=2}^{n-1} \frac{\psi^{(w)}(b)(r - b)^{w-2}}{(w - 2)!}\right) dr. \tag{47}$$

Proof We begin with the proof of (a) and its assumed conditions. Fix $u = 1, \dots, 6$.

By our assumption $\psi^{(n-1)}$ is absolutely continuous on $[a, b]$, as a result $\psi^{(n)}$ exists almost everywhere. Moreover, ψ is supposed to be n -convex, so by Theorem 4, $\psi^{(n)}(z) \geq 0$ almost everywhere on $[a, b]$. Therefore by applying Theorem 3, we get (33).

Similarly, the rest of the inequalities can be proved. □

We now give the final results of the present section.

Theorem 5 Under the assumptions of Theorem 3 if p_1, \dots, p_m and $\lambda_1, \dots, \lambda_k$ are non-negative tuples such that $\sum_{i=1}^m p_i = 1$ and $\sum_{j=1}^k \lambda_j = 1$, then for ψ being n -convex, we have:

(a) Inequality (33) is valid when $n \geq 3$. Besides, for the function

$$F_1(z) := \sum_{w=1}^{n-1} \frac{\psi^{(w)}(a)}{w!} (z-a)^w \tag{48}$$

to be convex, the right side of (33) is non-negative, meaning

$$J_u(\psi) \geq 0, \quad u = 1, \dots, 6. \tag{49}$$

(b) For n even, (35) holds. Furthermore, for the function

$$F_2(z) := \sum_{w=1}^{n-1} \frac{(-1)^w \psi^{(w)}(b)}{w!} (b-z)^w \tag{50}$$

to be convex, the right hand side of (35) is non-negative, particularly (49) holds.

(c) Inequalities (37)–(41) hold for all $n \geq 3$. Moreover, let (37)–(41) be valid and

$$\sum_{w=2}^{n-1} \frac{\psi^{(w)}(a)(r-a)^{w-2}}{(w-2)!} \geq 0, \tag{51}$$

then we get (49) for every $u = 1, \dots, 6$ and $v = 1, \dots, 5$.

(d) If n is even, then (43)–(47) hold. Moreover, let (43)–(47) be valid and

$$\sum_{w=2}^{n-1} \frac{\psi^{(w)}(b)(r-a)^{w-2}}{(w-2)!} \geq 0, \tag{52}$$

then we get (49) for $u = 1, \dots, 6$ and for $v = 1, \dots, 5$.

Proof (a) For fixed $u = 1, \dots, 6$.

For $(n \geq 3)$, $x \mapsto ((x-t)_+)^{n-1}$ is a convex function, so (32) holds by virtue of Remark 1 on account of the given weights being positive. Hence (33) is established by taking into account Corollary 1(a). Moreover, we can write the R.H.S. of (33) in the form $J_u(F_1)$ for $(u = 1, \dots, 6)$, after reorganizing this side. By using Remark 1 we can ensure the non-negativity of the R.H.S. of (33), especially (49) is established.

(b) Similar to the proof of (a).

(c) Fix $u = 1, \dots, 6$.

Considering the positive weights and $v = 1, \dots, 5$, $G_v(z, r)$ is convex. Hence using Remark 1, $J_u(G_v(z, r)) \geq 0$. Since ψ is n -convex, by using Corollary 1(c), we get (37)–(41). Moreover, the linear function z is convex (concave), therefore considering the positive weights, Remark 1 shows that $0 \leq J_u(z) \geq 0$ implying $J_u(z) = 0$. Finally using the positivity of $J_u(G_v(z, r))$ and (51), (49) is obtained.

(d) Similar to the proof of (c). □

3 Applications to monotonic functionals

In [22], Pečarić et al. proposed a more generalized class of n -convex functions, that is, n -convex functions at a point (see also [1]).

Definition 1 Consider the function $\psi : I \rightarrow \mathbb{R}$, d is a point in the interior of I . ψ is called $(n + 1)$ -convex at d if there exists a constant C so that the function

$$F(z) = \psi(z) - \frac{C}{n!}z^n \tag{53}$$

is n -concave on the interval $I \cap (-\infty, d]$ and n -convex on the interval $I \cap [d, \infty)$. The function ψ is called $(n + 1)$ -concave at d if $-\psi$ is $(n + 1)$ -convex at d .

A property that explains the name of this class is that a function is n -convex on an interval if and only if it is n -convex at every point of the interval (see [22]). Witkowski et al. in [22] deduced the conditions which are necessary and sufficient on two linear functionals,

$$\Gamma : C([a, d]) \rightarrow \mathbb{R}$$

and

$$\Upsilon : C([d, b]) \rightarrow \mathbb{R}$$

so that $\Gamma(\psi) \leq \Upsilon(\psi)$ is valid for each ψ which is n -convex at d . We shall apply this extended class of n -convex functions to obtaining monotonic linear functionals of Jensen type. Suppose ζ^u represents the monomials $\zeta^u(x) = x^u$, $u \in \mathbb{N}_0$. For the remaining part of this section, $\Gamma_u(\psi_{[a,d]})$ and $\Upsilon_u(\psi_{[d,b]})$ will represent the linear functionals obtained as the difference of the L.H.S. and R.H.S. of inequality (33), where $u = 1, \dots, 6$, applied to the intervals $[a, d]$ and $[d, b]$, respectively, i.e., for $\mathbf{z} \in [a, d]^m$ and $\mathbf{y} \in [d, b]^m$. Let

$$\Gamma_u(\psi_{[a,d]}) := \Gamma_u(\mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}; \psi_{[a,d]}) = J_u(\psi) - \sum_{w=1}^{n-1} \frac{\psi^{(w)}(a)}{w!} J_u((z-a)^w), \tag{54}$$

$$\Upsilon_u(\psi_{[d,b]}) := \Upsilon_u(\mathbf{y}, \mathbf{q}, \tilde{\boldsymbol{\lambda}}; \psi_{[d,b]}) = J_u(\psi) - \sum_{w=1}^{n-1} \frac{\psi^{(w)}(d)}{w!} J_u((y-d)^w). \tag{55}$$

In a similar vein, by using the inequality (35), for $u = 1, \dots, 6$, we define linear functionals as

$$\hat{\Gamma}_u(\psi_{[a,d]}) := \hat{\Gamma}_u(\mathbf{z}, \mathbf{p}, \boldsymbol{\lambda}; \psi_{[a,d]}) = J_u(\psi) - \sum_{w=1}^{n-1} \frac{(-1)^w \psi^{(w)}(d)}{w!} J_u((d-z)^w), \tag{56}$$

$$\hat{\Upsilon}_u(\psi_{[d,b]}) := \hat{\Upsilon}_u(\mathbf{y}, \mathbf{q}, \tilde{\boldsymbol{\lambda}}; \psi_{[d,b]}) = J_u(\psi) - \sum_{w=1}^{n-1} \frac{(-1)^w \psi^{(w)}(b)}{w!} J_u((b-y)^w). \tag{57}$$

By constructing new linear functionals $\Gamma_u(\psi_{[a,d]})$ and $\Upsilon_u(\psi_{[d,b]})$ using identity (15) for $u = 1, \dots, 6$, on $[a, d]$ and $[d, b]$ we obtain

$$\Gamma_u(\psi_{[a,d]}) = \frac{1}{(n-1)!} \int_a^d \psi^{(n)}(\xi) J_u((z-\xi)_+^{n-1}) d\xi, \tag{58}$$

$$\Upsilon_u(\psi_{[d,b]}) = \frac{1}{(n-1)!} \int_d^b \psi^{(n)}(\xi) J_u((y-\xi)_+^{n-1}) d\xi. \tag{59}$$

By applying identity (16) for $u = 1, \dots, 6$ to the intervals $[a, d]$ and $[d, b]$ we get

$$\hat{\Gamma}_u(\psi_{[a,d]}) = -\frac{(-1)^{n-1}}{(n-1)!} \int_a^d \psi^{(n)}(\xi) J_u((\xi-z)_+^{n-1}) d\xi, \tag{60}$$

$$\hat{\Upsilon}_u(\psi_{[d,b]}) = -\frac{(-1)^{n-1}}{(n-1)!} \int_d^b \psi^{(n)}(\xi) J_u((\xi-y)_+^{n-1}) d\xi. \tag{61}$$

Immediately, we can state the first theorem of the present section.

Theorem 6 Consider $\mathbf{z} \in [a, d]^m$ and $\mathbf{y} \in [d, b]^m$ where we have the following.

(i) For $u = 1, \dots, 6$, consider

$$J_u((z-\xi)_+^{n-1}) \geq 0, \quad \xi \in [a, d], \tag{62}$$

$$J_u((y-\xi)_+^{n-1}) \geq 0, \quad \xi \in [d, b], \tag{63}$$

$$\int_a^d J_u((z-\xi)_+^{n-1}) d\xi = \int_d^b J_u((y-\xi)_+^{n-1}) d\xi, \tag{64}$$

and let $\Gamma_u(\psi_{[a,d]})$ and $\Upsilon_u(\psi_{[d,b]})$ be the linear functionals introduced in (54) and (55). If $\psi : [a, b] \rightarrow \mathbb{R}$ is $(n+1)$ -convex at the point d , then

$$\Gamma_u(\psi_{[a,d]}) \leq \Upsilon_u(\psi_{[d,b]}) \quad \text{for } u = 1, \dots, 6. \tag{65}$$

(ii) Similarly, for $u = 1, \dots, 6$, suppose

$$(-1)^{n-1} J_u((\xi-z)_+^{n-1}) \leq 0, \quad \xi \in [a, d], \tag{66}$$

$$(-1)^{n-1} J_u((\xi-y)_+^{n-1}) \leq 0, \quad \xi \in [d, b], \tag{67}$$

$$\int_a^d J_u((\xi-z)_+^{n-1}) d\xi = \int_d^b J_u((\xi-y)_+^{n-1}) d\xi, \tag{68}$$

and let $\hat{\Gamma}_u(\psi_{[a,d]})$ and $\hat{\Upsilon}_u(\psi_{[d,b]})$ be the linear functionals which are given by (56) and (57). If $\psi : [a, b] \rightarrow \mathbb{R}$ is $(n+1)$ -convex at the point d , then

$$\hat{\Gamma}_u(\psi_{[a,d]}) \leq \hat{\Upsilon}_u(\psi_{[d,b]}) \quad \text{for } u = 1, \dots, 6. \tag{69}$$

Proof (i) Fix $u = 1, \dots, 6$. Using Definition 1, construct the function $F = \psi - \frac{C}{m} \zeta^n$ such that F is n -concave on $[a, d]$ and is n -convex on $[d, b]$. Now by applying Corollary 1 to F in the interval $[a, d]$, we get

$$0 \geq \Gamma_u(\Psi) = \Gamma_u(\psi_{[a,d]}) - \frac{C}{(n)!} \Gamma_u(\zeta_{[a,d]}^n). \tag{70}$$

Similarly, applying Corollary 1 to F in $[d, b]$, we have

$$0 \leq \Upsilon_u(\Psi) = \Upsilon_u(\psi_{[d,b]}) - \frac{C}{(n)!} \Upsilon_u(\zeta_{[d,b]}^n). \tag{71}$$

Moreover, utilizing the monomials $\zeta^n(\cdot)$ in (58) and (59) we get

$$\Gamma_u(\zeta^n_{[a,d]}) = n \int_a^d J_u((z - \xi)_+^{n-1}) d\xi, \tag{72}$$

$$\Upsilon_u(\zeta^n_{[d,b]}) = n \int_d^b J_u((y - \xi)_+^{n-1}) d\xi. \tag{73}$$

Therefore assumption (64) is equivalent to

$$\Gamma_u(\zeta^n_{[a,d]}) = \Upsilon_u(\zeta^n_{[d,b]}).$$

So from (70) and (71), one can get

$$\Gamma_u(\psi_{[a,d]}) \leq \frac{C}{(n)!} \Gamma_u(\zeta^n_{[a,d]}) = \frac{C}{(n)!} \Upsilon_u(\zeta^n_{[d,b]}) \leq \Upsilon_u(\psi_{[d,b]}). \tag{74}$$

So (65) is obtained for $u = 1, \dots, 6$.

(ii) A method similar to the above can be employed by using the identities (60) and (61). Hence by deducing supposition (68), we have (69) for $u = 1, \dots, 6$. □

We conclude with the following remarks.

Remark 2 Note that inequality (65) and (69) are also valid on replacing assumptions (64) and (68) with the weaker assumptions that

$$\begin{aligned} C(\Upsilon_u(\zeta^n_{[d,b]}) - \Gamma_u(\zeta^n_{[a,d]})) &\geq 0 \quad \text{and} \\ C(\hat{\Upsilon}_u(\zeta^n_{[d,b]}) - \hat{\Gamma}_u(\zeta^n_{[a,d]})) &\geq 0 \end{aligned}$$

for $u = 1, \dots, 6$, respectively.

Remark 3 Similar results can also be formulated by employing Theorem 6 for the inequalities (37) to (41) and (43) to (47) in Corollary 1.

4 New entropic bounds

In fields like probability theory, mathematical statistics and information theory, measures of dissimilarity between probability distributions play a pivotal role. Various divergence measures have been introduced for this purpose. For instance, we have the f -divergence, some particular cases of which are the Kullback–Leibler divergence and the Jensen–Shannon divergence. Entropies are used to quantify the uncertainty, diversity and the randomness of a system. The idea is frequently used in several scientific disciplines. Some new estimations for the Shannon and Zipf–Mandelbrot entropies can be found in [14].

In the current section we will work in discrete space, i.e., with discrete probability distributions.

We first introduce some important definitions and results used for the rest of this section. Csiszár in [5] introduced the following notion of a divergence functional.

Definition 2 Let $\psi : (0, \infty) \rightarrow (0, \infty)$ be a convex function and let $\mathbf{p} := (p_1, \dots, p_m)$ and $\mathbf{q} := (q_1, \dots, q_m)$ be positive probability distributions. The ψ -divergence functional is

$$I_\psi(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^n q_i \psi\left(\frac{p_i}{q_i}\right).$$

It is possible to use non-negative probability distributions in the ψ -divergence functional, by defining

$$\psi(0) := \lim_{t \rightarrow 0^+} \psi(t); \quad 0\psi\left(\frac{0}{0}\right) := 0; \quad 0\psi\left(\frac{a}{0}\right) := \lim_{t \rightarrow 0^+} t\psi\left(\frac{a}{t}\right), \quad a > 0.$$

Based on the previous definition, we introduce a new functional.

Definition 3 Suppose $\psi : I \rightarrow \mathbb{R}$ with I being an interval in \mathbb{R} . Suppose $\mathbf{p} := (p_1, \dots, p_m) \in \mathbb{R}^m$ and $\mathbf{q} := (q_1, \dots, q_m) \in (0, \infty)^m$ such that

$$\frac{p_i}{q_i} \in I, \quad i = 1, \dots, m.$$

Then let

$$\tilde{I}_\psi(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^m q_i \psi\left(\frac{p_i}{q_i}\right). \tag{75}$$

Remark 4 As a consequences of Corollary 1 for the case $u = 1$, we consider the explicit form of our generalized Jensen inequalities for n -convex functions, from (33) and (35):

$$\begin{aligned} & \sum_{i=1}^m p_i \psi(z_i) - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \psi\left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} z_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \right) \\ & \geq \sum_{w=1}^{n-1} \frac{\psi^{(w)}(a)}{w!} \left(\sum_{i=1}^m p_i (z_i - a)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} z_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} - a \right)^w \right), \tag{76} \\ & \sum_{i=1}^m p_i \psi(z_i) - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \psi\left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} z_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \right) \\ & \geq \sum_{w=1}^{n-1} \frac{(-1)^w \psi^{(w)}(b)}{w!} \left(\sum_{i=1}^m p_i (b - z_i)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right) \left(b - \frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} z_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}} \right)^w \right). \tag{77} \end{aligned}$$

For the rest of this section, we will use the following assumptions:

(A₃) Suppose for $m, k \in \mathbb{N}$ ($2 \leq k \leq m$), $\lambda_1, \dots, \lambda_k$ are positive probability distributions.

(A₄) Let $\mathbf{q} := (q_1, \dots, q_m) \in (0, \infty)^m$.

Theorem 7 Considering (A₃) and (A₄), let $\mathbf{p} := (p_1, \dots, p_m) \in \mathbb{R}^m$ such that

$$\frac{p_i}{q_i} \in [a, b], \quad i = 1, \dots, m.$$

Let $\psi : [a, b] \rightarrow \mathbb{R}$ where $\psi^{(n-1)}$ is absolutely continuous and ψ is n -convex, then the following inequalities hold:

(a)

$$\begin{aligned} \tilde{I}_\psi(\mathbf{p}, \mathbf{q}) \geq & \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) \psi \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} \right) + \sum_{w=1}^{n-1} \frac{\psi^{(w)}(a)}{w!} \\ & \cdot \left(\sum_{i=1}^m q_i \left(\frac{p_i}{q_i} - a \right)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{i+1} q_{i+j} \right) \left(\frac{\sum_{j=0}^{k-1} \lambda_{i+1} p_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} - a \right)^w \right), \end{aligned} \tag{78}$$

(b)

$$\begin{aligned} \tilde{I}_\psi(\mathbf{p}, \mathbf{q}) \geq & \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) \psi \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} \right) + \sum_{w=1}^{n-1} \frac{(-1)^w \psi^{(w)}(b)}{w!} \\ & \cdot \left(\sum_{i=1}^m q_i \left(b - \frac{p_i}{q_i} \right)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) \left(b - \frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} \right)^w \right). \end{aligned} \tag{79}$$

Proof Replacing p_i with q_i and z_i with $\frac{p_i}{q_i}$ for $(i = 1, \dots, m)$ in (76) and (77), we get (78) and (79), respectively. \square

We now present two significant applications of the previous results.

Shannon’s entropy, also known as a measure of uncertainty, plays a pivotal role in information theory. It is also frequently applied in fields like population genetics, molecular ecology and dynamical systems (see [18]). Some recent bounds for Shannon entropy can be found in [9, 15]. For a positive m -tuple $\mathbf{p} = (p_1, \dots, p_m)$ such that $\sum_{i=1}^m p_i = 1$, the *Shannon entropy* is defined by

$$S(\mathbf{p}) = - \sum_{i=1}^m p_i \ln p_i. \tag{80}$$

Corollary 2 *Suppose (A₃) holds.*

(a) *Considering (A₄), if n is even, we get*

$$\begin{aligned} \sum_{i=1}^m q_i \ln q_i \geq & \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) \ln \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) + \sum_{w=1}^{n-1} \frac{(-1)^w}{w \cdot (a)^w} \\ & \cdot \left(\sum_{i=1}^m q_i \left(\frac{1}{q_i} - a \right)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) \left(\frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} - a \right)^w \right). \end{aligned} \tag{81}$$

(b) *If $\mathbf{q} := (q_1, \dots, q_m)$ is a positive probability distribution, while n is even, we obtain the bounds for the Shannon entropy of \mathbf{q} ,*

$$\begin{aligned} S(\mathbf{q}) \leq & - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) \ln \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) - \sum_{w=1}^{n-1} \frac{(-1)^w}{w \cdot (a)^w} \\ & \cdot \left(\sum_{i=1}^m q_i \left(\frac{1}{q_i} - a \right)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j} \right) \left(\frac{1}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} - a \right)^w \right). \end{aligned} \tag{82}$$

For n odd, (81) and (82) are satisfied in the reverse directions.

Proof (a) Substituting $\psi(t) := -\ln t$, and $\mathbf{p} := (1, 1, \dots, 1)$ in Theorem 7(a), we obtain the desired results.

(b) This part is a special case of (a). □

The Kullback–Leibler measure, also known as the relative entropy, is a famous non-parametric measure. It is also one of the most famous distance functions used in signal processing, information theory and mathematical statistics. Some recent bounds for the relative entropy can be found in [9, 15]. The *Kullback–Leibler* distance [17] between the positive probability distributions $\mathbf{p} = (p_1, \dots, p_m)$ and $\mathbf{q} = (q_1, \dots, q_m)$ is given as

$$D(\mathbf{q} \parallel \mathbf{p}) = \sum_{i=1}^m q_i \ln\left(\frac{q_i}{p_i}\right). \tag{83}$$

Corollary 3 *Suppose (A₃) holds.*

(a) *If $\mathbf{q} := (q_1, \dots, q_m)$, $\mathbf{p} := (p_1, \dots, p_m) \in (0, \infty)^m$. Then, for even n , we get*

$$\begin{aligned} \sum_{i=1}^m q_i \ln\left(\frac{q_i}{p_i}\right) &\geq \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}\right) \ln\left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}\right) + \sum_{w=1}^{n-1} \frac{(-1)^w}{w.(a)^w} \\ &\cdot \left(\sum_{i=1}^m q_i \left(\frac{p_i}{q_i} - a\right)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}\right) \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} - a\right)^w\right). \end{aligned} \tag{84}$$

(b) *If $\mathbf{q} := (q_1, \dots, q_m)$, $\mathbf{p} := (p_1, \dots, p_m)$ are positive probability distributions with n even, we obtain*

$$\begin{aligned} D(\mathbf{q} \parallel \mathbf{p}) &\geq \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}\right) \ln\left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}\right) + \sum_{w=1}^{n-1} \frac{(-1)^w}{w.(a)^w} \\ &\cdot \left(\sum_{i=1}^m q_i \left(\frac{p_i}{q_i} - a\right)^w - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}\right) \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} q_{i+j}} - a\right)^w\right). \end{aligned} \tag{85}$$

In the case n is odd, (84) and (85) are satisfied in the reverse directions.

Proof (a) Setting $\psi(t) := -\ln t$ in Theorem 7(a) gives the required results.

(b) This part is a special case of (a). □

Philologist George Kingsley Zipf (1902–1950) studied statistical occurrences of words in different languages. It was one of the first academic studies of word frequency and was originally prescribed only for linguistics. It was only later that many other disciplines took credit of it: the Pareto law in economy reveals another aspect of it and the “Zipfian distribution” is present in other fields as well: information science, bibliometrics, social sciences etc.

Benoit Mandelbrot (1924–2010), in 1966, generalized the Zipf law and improved for counting the low-rank words. It has various applications in art, in ecological field studies and is also used in information sciences [6, p. 294]. Some of the recent studies regarding the Zipf–Mandelbrot law can be found in the listed references (see [9, 10, 15, 16]).

Hence if we rank (r) word in accordance with the frequency of the word occurrence (ψ), the product is a constant (C):

$$C = r\psi.$$

For $m \in \{1, 2, \dots\}$, $t \geq 0$ and $s > 0$ the Zipf–Mandelbrot law is given as

$$\psi(i; m, t, s) = \frac{1}{((i + t)^s H_{t,s}^m)}, \quad i = (1, 2, \dots, m), \tag{86}$$

where

$$H_{t,s}^m = \sum_{k=1}^m \frac{1}{(k + t)^s}.$$

In the formula, i represents the rank of the data, t and s are parameters of the distribution. Suppose $m \in \{1, 2, \dots\}$, $t \geq 0$, $s > 0$, then the Zipf–Mandelbrot entropy is given as

$$Z(H, t, s) = \frac{s}{H_{t,s}^m} \sum_{i=1}^m \frac{\ln(i + t)}{(i + t)^s} + \ln(H_{t,s}^m). \tag{87}$$

Consider

$$q_i = \psi(i; m, t, s) = \frac{1}{((i + t)^s H_{t,s}^m)}. \tag{88}$$

The following results give bounds for the Zipf–Mandelbrot entropy.

Theorem 8 Suppose (A_3) holds and \mathbf{q} is the same as given in (88) by the Zipf–Mandelbrot law with parameters $m \in \{1, 2, \dots\}$, $t \geq 0$, $s > 0$. For n even, we have

$$\begin{aligned} S(\mathbf{q}) &= Z(H, t, s) \\ &\leq - \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{((i + j + t)^s H_{t,s}^m)} \right) \ln \left(\frac{1}{H_{t,s}^m} \sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{((i + j + t)^s)} \right) \\ &\quad - \sum_{w=1}^{n-1} \frac{(-1)^w}{w.(a)^w} \left(\sum_{i=1}^m \frac{1}{((i + t)^s H_{t,s}^m)} \left(((i + t)^s H_{t,s}^m) - a \right)^w \right) \\ &\quad + \sum_{w=1}^{n-1} \frac{(-1)^w}{w.(a)^w} \left(\sum_{i=1}^m \left(\sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{((i + j + t)^s H_{t,s}^m)} \right) \left(\frac{1}{\sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{((i+j+t)^s H_{t,s}^m)}} - a \right)^w \right). \end{aligned} \tag{89}$$

Equation (89) holds in the opposite direction for n odd.

Proof Substituting this $q_i = \frac{1}{((i+t)^s H_{t,s}^m)}$ in Corollary 2(b) gives the required result. Also note that $\sum_{i=1}^m q_i = 1$. Moreover, using the above q_i in the Shannon entropy (80), we get the

Mandelbrot entropy (87),

$$\begin{aligned}
 S(q) &= -q_i \ln q_i \\
 &= -\sum_{i=1}^m \frac{1}{((i+t)^s H_{t,s}^m)} \ln \frac{1}{((i+t)^s H_{t,s}^m)} \\
 &= \frac{-1}{H_{t,s}^m} \sum_{i=1}^m \frac{1}{(i+t)^s} \ln \frac{1}{(i+t)^s H_{t,s}^m} \\
 &= \frac{-1}{H_{t,s}^m} \sum_{i=1}^m \frac{1}{(i+t)^s} (\ln(1) - s \ln(i+t) - \ln(H_{t,s}^m)) \\
 &= \frac{1}{H_{t,s}^m} \sum_{i=1}^m \frac{1}{(i+t)^s} (s \ln(i+t) + \ln(H_{t,s}^m)) \\
 &= \frac{s}{H_{t,s}^m} \sum_{i=1}^m \frac{\ln(i+t)}{(i+t)^s} + \ln(H_{t,s}^m). \tag{90}
 \end{aligned}$$

□

Corollary 4 Suppose (A₃) holds and for $t_1, t_2 \in [0, \infty)$, $s_1, s_2 > 0$, let $H_{t_1, s_1}^m = \sum_{k=1}^m \frac{1}{(k+t_1)^{s_1}}$ and $H_{t_2, s_2}^m = \sum_{k=1}^m \frac{1}{(k+t_2)^{s_2}}$. Now using $q_i = \frac{1}{(i+t_1)^{s_1} H_{t_1, s_1}^m}$ and $p_i = \frac{1}{(i+t_2)^{s_2} H_{t_2, s_2}^m}$ in Corollary 3(b), with n even, we have

$$\begin{aligned}
 D(\mathbf{q} \parallel \mathbf{p}) &= \sum_{i=1}^m \frac{1}{(i+t_1)^{s_1} H_{t_1, s_1}^m} \ln \left(\frac{(i+t_2)^{s_2} H_{t_2, s_2}^m}{(i+t_1)^{s_1} H_{t_1, s_1}^m} \right) \\
 &\geq \sum_{i=1}^m \left(\sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{(i+j+t_1)^{s_1} H_{t_1, s_1}^m} \right) \ln \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} \frac{1}{(i+j+t_1)^{s_1} H_{t_1, s_1}^m}}{\sum_{j=0}^{k-1} \lambda_{j+1} \frac{1}{(i+j+t_2)^{s_2} H_{t_2, s_2}^m}} \right) \\
 &\quad + \sum_{w=1}^{n-1} \frac{(-1)^w}{w.(a)^w} \left(\sum_{i=1}^m \frac{1}{(i+t_1)^{s_1} H_{t_1, s_1}^m} \left(\frac{(i+t_1)^{s_1} H_{t_1, s_1}^m}{(i+t_2)^{s_2} H_{t_2, s_2}^m} - a \right)^w \right) \\
 &\quad - \sum_{w=1}^{n-1} \frac{(-1)^w}{w.(a)^w} \\
 &\quad \cdot \left(\sum_{i=1}^m \left(\sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{(i+j+t_1)^{s_1} H_{t_1, s_1}^m} \right) \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} \frac{1}{(i+j+t_2)^{s_2} H_{t_2, s_2}^m}}{\sum_{j=0}^{k-1} \lambda_{j+1} \frac{1}{(i+j+t_1)^{s_1} H_{t_1, s_1}^m}} - a \right)^w \right). \tag{91}
 \end{aligned}$$

If n is odd, (91) holds in the reverse direction.

5 Concluding remarks

New upper bounds for Grüss and Ostrowski type inequalities can be formulated (see [2]) as a direct application of the results obtained by Dragomir et al. in [4] from new generalized inequalities for n -convex functions obtained in the first section. We can also construct several functionals from the inequalities introduced in Theorem 1 and give mean value theorems of Lagrange and Cauchy type for n -convex functions. Moreover, considering the n -exponentially method introduced by Pečarić et al. in [11] and [21] (see also [3]), we can establish a novel collection of non-trivial examples of functions which are n -exponentially and exponentially convex. Finally we can also construct monotonic Cauchy means.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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