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Relations of a planar domains bounded by hyperbolas with families of holomorphic functions

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Abstract

We consider a family of analytic and normalized functions with the property that $zf'(z)/f(z)$ (or $1 + zf''(z)/f'(z)$) lies in a domain bounded by a right branch of a hyperbola $\rho = \rho(s) = (2\cos\frac{\varphi}{s})^{-s}$, where $0 < s \leq 1$ and $|\varphi| < (\pi s)/2$. A comprehensive characteristic of that families and relations with the well known families of univalent functions are presented. Some relevant examples are indicated.

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1 Introduction and motivation

Let $\mathcal{H}$ denote the class of functions f *holomorphic* in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$, and with the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}).$$

By $\mathcal{S}$ we denote the subclass of $\mathcal{H}$ consisting of univalent functions, and by $\mathcal{ST}, \mathcal{CV}$ the best known subclasses of $\mathcal{S}$ containing functions starlike with respect to the origin and convex, respectively. Their geometric properties and various applications have caused rapid development of research in these directions. Many its subclasses have been introduced and investigated. The main feature of the new definitions is usually fact that either $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is located in a some region contained in a right halfplane. A uniform approach to such definitions was proposed by Ma and Minda [7]. They introduced a special class of functions φ that map the unit disk onto the domains $\varphi(\mathbb{D})$ symmetric with respect to the real axis, starlike with respect to $\varphi(0) = 1$, and with $\varphi'(0) > 0$. A family $\mathcal{P}(\varphi)$ that consists of holomorphic functions p such that $p(0) = 1$ and $p(\mathbb{D}) \subset \varphi(\mathbb{D})$ (or in terms of subordination as $p \prec \varphi$) have also been defined in [7] (in connection with the function φ). The Carathéodory class $\mathcal{P} = \{p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \Re p(z) > 0, z \in \mathbb{D}\}$ is simply the class $\mathcal{P}((1 + z)/(1 - z))$. The class of *starlike* univalent functions $\mathcal{ST}$ (and *convex* functions $\mathcal{CV}$, resp.) can be described as $zf'/f \in \mathcal{P}((1 + z)/(1 - z))$ (or $1 + zf''/f' \in \mathcal{P}((1 + z)/(1 - z))$,

resp.). It is seen that if φ varies, $\mathcal{P}(\varphi)$ generates a number of well known subclasses of the Carathéodory class. In a such way, various similar subclasses of $\mathcal{P}$ were defined and studied through a years. Leading examples of $\mathcal{P}$ are $\mathcal{P}(\alpha) = \mathcal{P}(\varphi_\alpha(z)) = \mathcal{P}((1 + (1 - 2\alpha)z)/(1 - z))$, $\mathcal{P}_\gamma = \mathcal{P}(\varphi^\gamma(z)) = \mathcal{P}(((1 + z)/(1 - z))^\gamma)$, $\mathcal{P}(A, B) = \mathcal{P}(\varphi_{A,B}(z)) = \mathcal{P}((1 + Az)/(1 + Bz))$, etc. Related families of *starlike and convex functions of order α* $(0 \le \alpha < 1)$ (*strongly starlike and convex of order γ* $(0 < \gamma \le 1)$, resp.) were defined as follows:

$$\mathcal{ST}(\alpha) = \left\{ f \in \mathcal{H} : zf'/f \prec \varphi_\alpha \text{ in } \mathbb{D} \right\} = \left\{ f \in \mathcal{H} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{D} \right\},$$

$$\mathcal{ST}_\gamma = \left\{ f \in \mathcal{H} : zf'/f \prec \varphi^\gamma \text{ in } \mathbb{D} \right\} = \left\{ f \in \mathcal{H} : \left| \mathrm{Arg} \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\gamma \pi}{2}, z \in \mathbb{D} \right\},$$

and also the corresponding subclasses $\mathcal{CV}(\alpha)$, $\mathcal{CV}_\gamma$ of convex functions, for which $1 + zf''/f' \prec \varphi_\alpha$ or $1 + zf''/f' \prec \varphi^\gamma(z)$, respectively (see, for example [1, 9, 11]). We note that $\mathcal{ST} = \mathcal{ST}(0) = \mathcal{ST}_1$ and $\mathcal{CV} = \mathcal{CV}(0) = \mathcal{CV}_1$.

An interesting families of the domains that are bounded by a conic sections were introduced and studied in [5, 6]. Such domains are related to *k-starlike functions k-$\mathcal{ST}$*, and *k-uniformly convex function*, denoted *k-$\mathcal{UCV}$*. A univalent function $f \in k\text{-}\mathcal{ST}$ if the expression zf'/f satisfies

$$\Re \frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}),$$

and $f \in k\text{-}\mathcal{UCV}$ if $\Re(1 + zf''(z)/f'(z)) > k|zf''(z)/f'(z)|$ for $z \in \mathbb{D}$. Equivalently, these families may be geometrically described by the fact that the expression $p(z) = zf'(z)/f(z)$ (or $p(z) = 1 + zf''(z)/f'(z)$, resp.) lies in a domain Ω_k for $z \in \mathbb{D}$, where

$$\Omega_k = \left\{ u + iv : u^2 > k^2(u - 1)^2 + k^2 v^2 \right\}.$$

We note that Ω_k is bounded by the conic sections of a shape depending on a parameter k and contained in a right halfplane. $\partial \Omega_k$ reduces to the imaginary axis for $k = 0$, a hyperbola for $0 < k < 1$, parabola for $k = 1$, and finally to an ellipse, when $1 < k < \infty$. It was observed in [5, 6] that each Ω_k is symmetric with respect to the real axis and starlike with respect to the point 1. Additionally the functions p_k, constructed in [6], see also [4], that map the unit disk onto Ω_k satisfy $p_k(0) = 1$, $p'_k(0) > 0$, for $0 \le k < \infty$, therefore fulfill the Ma and Minda assertions.

We also recall that the hyperbola considered in [5, 6] is given by

$$\partial \Omega_k = \left\{ u + iv : u^2 = k^2(u - 1)^2 + k^2 v^2, u > 0 \right\},$$

and corresponds to the parameter $0 < k < 1$. Moreover, the points of intersection of $\partial \Omega_k$ with the real axis is $(u, 0) = (k/(k + 1), 0)$ and the slope angle of asymptotes of $\partial \Omega_k$ to the real axis equals $\arctan(1/k)$.

This work is intended as an attempt at defining a new subfamily of $\mathcal{P}$ related to the domains bounded by a right branch of a hyperbolas $H(s)$ for $0 < s \le 1$, where

$$H(s) = \left\{ \rho e^{i\varphi} : \rho = \frac{1}{(2 \cos \frac{\varphi}{s})^s}, -\frac{\pi s}{2} < \varphi < \frac{\pi s}{2} \right\}.$$

It is easy to observe that the intersection point of $H(s)$ and real axis is $(u, 0) = (1/2^s, 0)$ and the slope angle to the real axis is equal $(\pi s)/2$. Taking into account these facts, we want to emphasize then the fact that none hyperbola $\partial\Omega_k$ reduces to $H(s)$ and conversely. Indeed, considering a location of common points of the hyperbola $H(s)$, hyperbola $\partial\Omega_k$ with the real axis and, we obtain the relation $k/(k + 1) = 1/2^s$ that holds for $k = 1/(2^s - 1) > 1$. But when $k > 1$ the $\partial\Omega_k$ is the ellipse.

It is of interest to study the common relations between $H(s)$ and $\partial\Omega_k$ for any $0 \le k < \infty$ and $0 < s \le 1$. The detailed investigation of geometric behavior of $\partial\Omega_k$ and $H(s)$ provides a following relations:

$$H(s) \cap \partial\Omega_k = \emptyset \quad \text{for } 0 \le k \le 1/2^s \text{ and } k > 1/(2^s - 1), 0 < s \le 1, \tag{1.1}$$

$$H(s) \cap \partial\Omega_k = \{M\} \quad \text{for } k = 1/(2^s - 1), 0 < s \le 1, \text{where } M = (1/2^s, 0), \tag{1.2}$$

$$H(s) \cap \partial\Omega_k = \{M_1, M_2\} \quad \text{for } 1/2^s < k < 1/(2^s - 1), 0 < s \le 1, \tag{1.3}$$

where

$$M_{1,2} = \left(\frac{k2^{s-1}}{k2^s - 1}, \pm\frac{\sqrt{(1 - k^2)4^{s-1} + k2^s - 1}}{k2^s - 1} \right).$$

For the case $0 \le k \le 1/2^s$ a domain bounded by $H(s)$ is contained in a domain bounded by $\partial\Omega_k$, and for $k \ge 1/(2^s - 1)$ conversely, the ellipse bounded by $\partial\Omega_k$ is contained in a domain bounded by a hyperbola $H(s)$. Let us denote by $\mathbb{H}(s)$ a domain bounded by hyperbola $H(s)$. Thus, we have the following inclusion:

$$\mathbb{H}(s) \subset \Omega_k \quad \text{for } 0 \le k \le 1/2^s, 0 < s \le 1$$
$$\Omega_k \subset \mathbb{H}(s) \quad \text{for } k \ge 1/(2^s - 1), 0 < s \le 1. \tag{1.4}$$

In the remaining cases of k the considered domains have nonempty intersection set (see Fig. 1). These observation will have significant consequences for the further relations between corresponding classes of analytic functions, but also means that the domains bounded by hyperbolas $H(s)$ $(0 < s \le 1)$ are completely different from the similar domains bounded by the hyperbola $\partial\Omega_k$.

We also note the obvious difference between $\mathbb{H}(s)$ and other domains contained in a right halfplane: disks, halfplanes and angular domains, that have appeared in the investigations of various subclasses of analytic functions.

From the above reasons we wish to find an analytic function that map the unit disk onto a domain bounded by the hyperbola $H(s)$, define and investigate a subfamily of $\mathcal{P}$ related to those domains. As a natural extension of that investigation we will define a subfamily $\mathcal{ST}_{\mathrm{hpl}}(s)$ of the starlike functions for which zf'/f is contained in a domain bounded by the hyperbola $H(s)$ (and related class $\mathcal{CV}_{\mathrm{hpl}}(s)$ of convex functions). It is also natural to try to give a sufficient condition for the function $f \in \mathcal{H}$ to be a member of the class $\mathcal{ST}_{\mathrm{hpl}}(s)$ (or $\mathcal{CV}_{\mathrm{hpl}}(s)$, resp.) and solve several extremal problems.

In order to achieve our aim we recall some definitions and preliminary results from the geometric theory of analytic functions. By $\mathcal{B}$ we denote the class of analytic self-mappings

Figure 1 The mutual location of a hyperbola $H(s)$ (for $s = 1/2$) and $\partial \Omega_k$
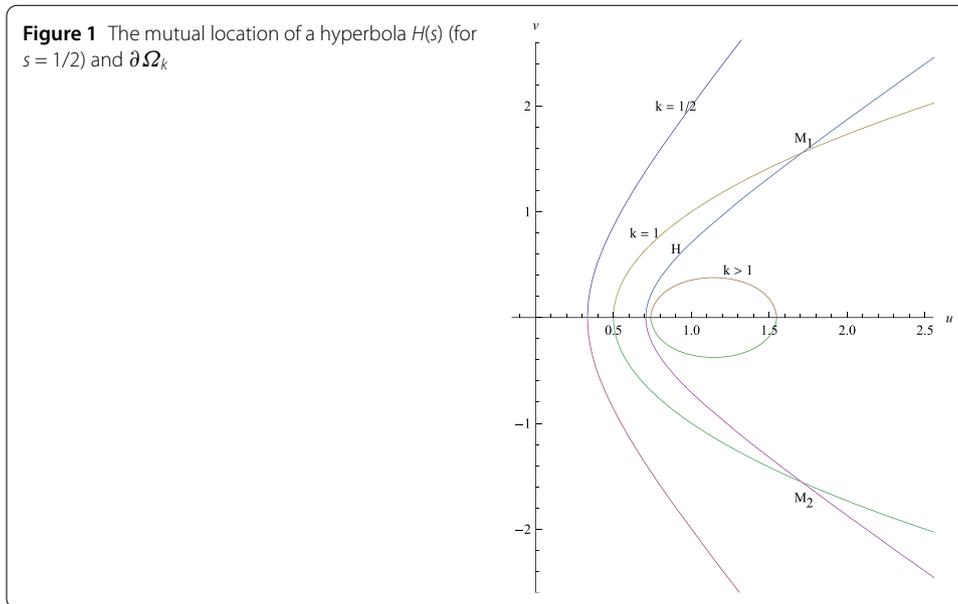
of the unit disk that send the origin onto the origin, i.e.

$$\mathcal{B} = \big\{ \omega(z) \in \mathcal{H}, \omega(0) = 0, \big| \omega(z) \big| < 1, z \in \mathbb{D} \big\}.$$

The class $\mathcal{B}$ is known as the class of *Schwarz functions*.

We say that f is *subordinate* to F in $\mathbb{D}$, written $f(z) \prec F(z)$ (or $f \prec F$), if there exists a function $\omega \in \mathcal{B}$ such that $f(z) = F(\omega(z))$ ($z \in \mathbb{D}$) see, for example [2, 3]. From that definition it is easy to show that if F is univalent in $\mathbb{D}$ then $f \prec F$ holds if and only, if $f(0) = F(0)$, and $f(\mathbb{D}) \subset F(\mathbb{D})$.

Let us denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{D}} \setminus E(f)$, where

$$E(f) = \Big\{ \zeta : \zeta \in \partial \mathbb{D} \text{ and } \lim_{z \to \zeta} f(z) = \infty \Big\},$$

and that are such that

$$f'(\zeta) \neq 0 \quad \text{for } \zeta \in \partial \mathbb{D} \setminus E(f).$$

Lemma 1.1 ([8]) *Let $q \in \mathcal{Q}$ with $q(0) = 1$ and let $p(z) = 1 + p_1 z + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \neq 1$. If $p \not\prec q$ in $\mathbb{D}$ then there exist points $z_0 \in \mathbb{D}$ and $\zeta \in \partial \mathbb{D} \setminus E(q)$ and there exist a real number $m \geq 1$ for which*

$$p\big(|z| < |z_0|\big) \subset q(\mathbb{D}), \qquad p(z_0) = q(\zeta), \qquad z_0 p'(z_0) = m \zeta q'(\zeta).$$

2 The class $\mathcal{P}(\mathsf{q}_s)$ and its properties

This section provides a detailed exposition of an analytic function that maps the unit disk onto a domain bounded by a hyperbola $H(s)$ and contained in a right halfplane. Let

$$\mathsf{q}_s(z) := \frac{1}{(1-z)^s} = e^{-s \log(1-z)} \quad (0 < s \leq 1, z \in \mathbb{D}),$$

where the branch of the logarithm is determined by $q_s(0) = 1$. It is easy to check that

$$q_s(z) = 1 + \sum_{n=1}^{\infty} q_n z^n = 1 + \sum_{n=1}^{\infty} \frac{s(s+1)\cdots(s+n-1)}{n!} z^n$$

$$= 1 + sz + \frac{s(s+1)}{2} z^2 + \frac{s(s+1)(s+2)}{6} z^3 + \cdots \quad (z \in \mathbb{D}).$$

The function q_s maps the unit disk onto a domain bounded by a right branch of a hyperbola

$$H(s) = \left\{ \rho e^{i\varphi} : \rho = \frac{1}{(2\cos\frac{\varphi}{2})^s}, -\frac{\pi s}{2} < \varphi < \frac{\pi s}{2} \right\}.$$

Indeed, setting $z = e^{i\theta}$ $(\theta \in (0, 2\pi))$ we obtain

$$\rho e^{i\varphi} = q_s\left(e^{i\theta}\right) = \frac{1}{(1 - e^{i\theta})^s} = e^{-s\log|1-e^{i\theta}|-s\operatorname{Arg}(1-e^{i\theta})} = \frac{e^{i\frac{\pi-\theta}{2}s}}{(2\sin\frac{\theta}{2})^s}.$$

Thus, we have

$$\Re q_s\left(e^{i\theta}\right) = \frac{\cos(\frac{\pi-\theta}{2}s)}{(2\sin\frac{\theta}{2})^s}, \qquad \Im q_s\left(e^{i\theta}\right) = \frac{\sin(\frac{\pi-\theta}{2}s)}{(2\sin\frac{\theta}{2})^s},$$

$$\rho = \frac{1}{(2\sin\frac{\theta}{2})^s}, \quad \text{with } \varphi = \frac{\pi-\theta}{2}s.$$

Setting $\varphi = (\theta - \pi)s/2$, after simple computation, we obtain

$$\rho = \frac{1}{(2\cos\frac{\varphi}{s})^s} \quad \text{with} \left(-\frac{\pi s}{2} < \varphi < \frac{\pi s}{2} \right). \tag{2.1}$$

The function

$$g(\theta) = \Re q_s\left(e^{i\theta}\right) = \frac{\cos(\frac{\pi-\theta}{2}s)}{(2\sin\frac{\theta}{2})^s}$$
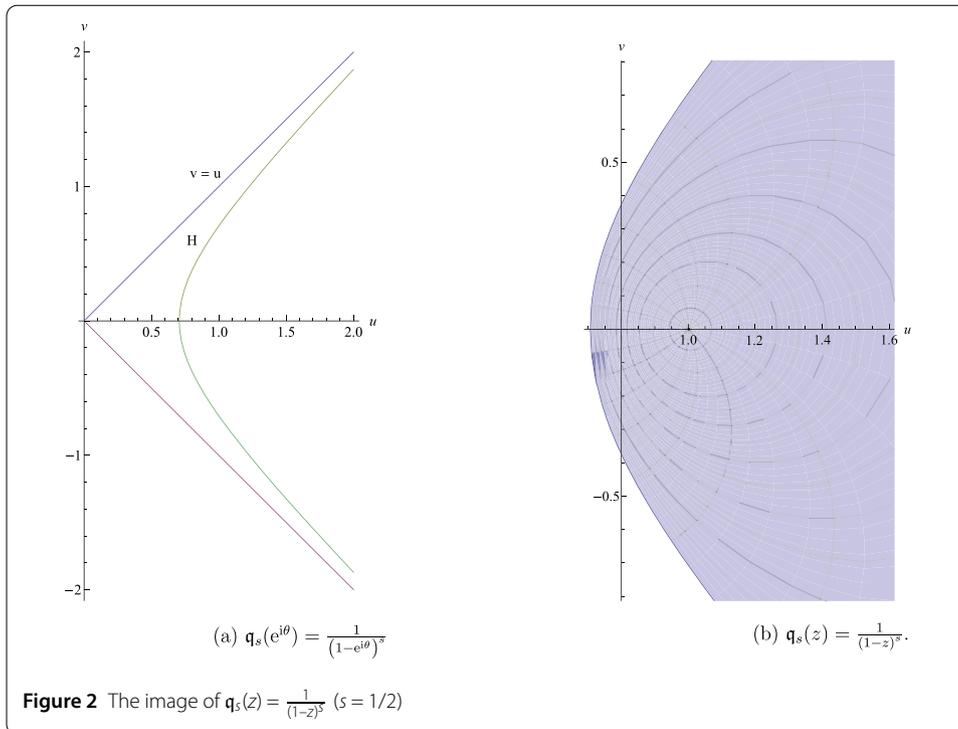
attains its minimal value when $\theta = \pi$, i.e., we have

$$\Re q_s\left(e^{i\theta}\right) \geq \frac{1}{2^s} \quad (0 < \theta < 2\pi).$$

Also, we note that $\Re q_s(e^{i\theta}) \to \infty$ for $\theta \to 0$. From (2.1) we obtain that the boundary $q_s(e^{i\theta})$ satisfies the equality $1/\rho = (2\sin\frac{\theta}{2})^s = (2\cos\frac{\varphi}{s})^s$. Additionally, we have $q_s(0) = 1$, therefore a right branch of a hyperbola

$$H(s) = \left\{ \rho e^{i\varphi} \in \mathbb{C} : \rho = \frac{1}{(2\cos\frac{\theta}{s})^s}, |\varphi| < \frac{\pi s}{2} \right\}$$

is a boundary of a domain $q_s(\mathbb{D})$, which we denote by $\mathbb{H}(s)$. The domain $\mathbb{H}(s)$ is symmetric about the real axis, starlike with respect to the point $q_s(0) = 1$, and it satisfies $q_s'(0) = s > 0$. Also, the hyperbola $H(s) = \partial\mathbb{H}(s)$ has a vertex at $(2^{-s}, 0)$ and the slant asymptotes $v = \pm\tan(\pi s/2)u$ (see Fig. 2).

(a) $\mathfrak{q}_s(e^{i\theta}) = \frac{1}{(1-e^{i\theta})^s}$

(b) $\mathfrak{q}_s(z) = \frac{1}{(1-z)^s}.$

Figure 2 The image of $\mathfrak{q}_s(z) = \frac{1}{(1-z)^s}$ $(s = 1/2)$

A special and interesting case of $H(s)$ is that for $s = 1/2$. We then obtain $\mathfrak{q}_{1/2}(z) = \frac{1}{\sqrt{1-z}}$, and the hyperbola reduces to that given by the relation

$$H(1/2) = \left\{u + iv : u^2 - v^2 = 1/2, u > 0\right\}.$$

Lemma 2.1 *The function* $\mathfrak{q}_s$ *is convex univalent in* $\mathbb{D}$, *moreover* $g(z) = (\mathfrak{q}_s(z) - 1)/s \in \mathcal{CV}((1 - s)/2)$. *Also, if* $|z| = r < 1$, *then*

$$\min_{|z|=r}\left|\mathfrak{q}_s(z)\right| = \mathfrak{q}_s(-r) = \frac{1}{(1 + r)^s} \quad and \quad \max_{|z|=r}\left|\mathfrak{q}_s(z)\right| = \mathfrak{q}_s(r) = \frac{1}{(1 - r)^s}. \tag{2.2}$$

Proof We first observe that $\mathfrak{q}_s$ is convex by a convexity property of a set $\mathfrak{q}_s(\mathbb{D})$. To deduce the univalence, we use the results on univalence of some integrals from [10]. In order to prove (2.2) we consider the function $Q(\theta)$ for $\theta \in [0, 2\pi)$, where

$$Q(\theta) = \left|\mathfrak{q}_s\left(re^{i\theta}\right)\right| = \frac{1}{|1 - re^{i\theta}|^s} = \frac{1}{(1 + r^2 - 2r\cos\theta)^{s/2}} \quad (0 < r < 1).$$

Since $Q(\theta)$ attains its minimum at $\theta = \pi$ and maximum at $\theta = 0$, (2.2) follows.

Let us consider $g(z) = (\mathfrak{q}_s(z) - 1)/s$ $(z \in \mathbb{D})$. Then we have

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) = \Re\left(\frac{1 + sz}{1 - z}\right) > \frac{1 - s}{2},$$

so that $g \in \mathcal{CV}((1 - s)/2)$, as asserted. $\square$

Definition 2.2 By $\mathcal{P}(\mathfrak{q}_s)$ we denote a class of all analytic functions p such that $p(0) = 1$ and $p(z) \prec \mathfrak{q}_s(z)$, $z \in \mathbb{D}$, that is,

$$\mathcal{P}(\mathfrak{q}_s) = \left\{ p(z) = 1 + p_1 z + p_2 z^2 + \cdots, p(z) \prec \mathfrak{q}_s(z), z \in \mathbb{D} \right\} \subset \mathcal{P}.$$

Theorem 2.3 *If $p \in \mathcal{P}(\mathfrak{q}_s)$, then*

$$\Re p(z) > \frac{1}{2^s}, \qquad \left| \mathrm{Arg}\{p(z)\} \right| < \frac{\pi}{2} s \quad (z \in \mathbb{D}), \tag{2.3}$$

and

$$\left| p^{1/s}(z) - 1 \right| < \left| p(z) \right|^{1/s} \quad (z \in \mathbb{D}), \tag{2.4}$$

or

$$\left| p^{1/(2s)}(z) + 1 \right| - \left| p^{1/(2s)}(z) - 1 \right| > \sqrt{2} \quad \textit{and} \quad \Re p^{1/s}(z) > \frac{1}{2} \quad (z \in \mathbb{D}). \tag{2.5}$$

Conversely, if p is analytic with $p(0) = 1$, $| \mathrm{Arg}\{p(z)\} | < (\pi s)/2$ and p satisfies (2.4) *(or one of the conditions of* (2.5)*), then $p \prec \mathfrak{q}_s$ in $\mathbb{D}$.*

Proof The subordination $p \prec \mathfrak{q}_s$ with $p(0) = \mathfrak{q}_s(0)$, and the geometric properties of $\mathfrak{q}_s(\mathbb{D})$ from Sect. 1, yield (2.3). We next claim that (2.4) is satisfied. Repeated application of the properties of subordination $p \prec \mathfrak{q}_s$ shows that there exists $\omega \in \mathcal{B}$, such that

$$p(z) = \frac{1}{(1 - \omega(z))^s} \quad \text{or} \quad p^{1/s}(z) = \frac{1}{1 - \omega(z)} \quad (z \in \mathbb{D}),$$

which is equivalent to

$$\omega(z) = \frac{p^{1/s}(z) - 1}{p^{1/s}(z)}, \qquad \left| \omega(z) \right| < 1 \quad (z \in \mathbb{D}),$$

and the assertion (2.4) now follows. Squaring (2.4) we may rewrite it as

$$\left(p^{1/s}(z) - 1 \right) \left(\overline{p^{1/s}(z)} - 1 \right) < \left| p^{1/s}(z) \right|^2,$$

or

$$\left| p^{1/s}(z) \right|^2 - p^{1/s}(z) - \overline{p^{1/s}(z)} + 1 < \left| p^{1/s}(z) \right|^2,$$

which is nothing but the second relation of (2.5). For the proof of the first relation of (2.5) we rewrite (2.4) as

$$2 + 2 \left| p^{1/s}(z) \right| - 2 \left| p^{1/s}(z) - 1 \right| > 2,$$

or equivalently

$$\left| p^{1/(2s)}(z) + 1 \right|^2 + \left| p^{1/(2s)}(z) - 1 \right|^2 - 2 \left| p^{\frac{1}{s}}(z) - 1 \right| > 2,$$

which immediately becomes

$$\left(\left|p^{1/(2s)}(z) + 1\right| - \left|p^{1/(2s)}(z) - 1\right|\right)^2 > 2,$$

and, from this, the desired conclusion follows.

Conversely, if for p is analytic in $\mathbb{D}$ with $p(0) = 1$ and the condition (2.4) (or one of the conditions of (2.5)) is satisfied, then we easily show that $p = \rho e^{i\theta}$ lies in a domain $\mathbb{H}(s) = \mathfrak{q}_s(\mathbb{D})$, which completes the proof. □

Theorem 2.4 *If $p \in \mathcal{P}$ satisfies*

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\} > -\frac{s}{2} \quad (z \in \mathbb{D}), \tag{2.6}$$

then

$$p(z) \prec \mathfrak{q}_s(z) \quad (z \in \mathbb{D}).$$

Proof Suppose, on the contrary, that $p(z) \not\prec \mathfrak{q}_s(z)$ on $\mathbb{D}$. Then, by $p(0) = \mathfrak{q}_s(0) = 1$, and by Lemma 1.1, there exist $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}$ ($\zeta_0 \neq 1$) such that

$$p(z_0) = \mathfrak{q}_s(\zeta_0), \qquad z_0 p'(z_0) = m\zeta_0 \mathfrak{q}'_s(\zeta_0) \quad \text{with } m \geq 1.$$

Hence

$$\Re\left\{\frac{z_0 p'(z_0)}{p(z_0)}\right\} = \Re\left\{\frac{m\zeta_0 \mathfrak{q}'_s(\zeta_0)}{\mathfrak{q}_s(\zeta_0)}\right\} = ms\Re\left\{\frac{\zeta_0}{1 - \zeta_0}\right\} = -\frac{ms}{2} \leq -\frac{s}{2},$$

which contradicts the assumption (2.6), and therefore $p \prec \mathfrak{q}_s$ on $\mathbb{D}$. □

3 The classes $\mathcal{ST}_{\mathrm{hpl}}(s)$, $\mathcal{CV}_{\mathrm{hpl}}(s)$

In this section we introduce and examine the notion of starlikeness and convexity related to the domains bounded by the hyperbolas $H(s)$. We indicate how the properties of the domains $\mathbb{H}(s)$ and class $\mathcal{P}(\mathfrak{q}_s)$ influence on the properties of a newly introduced classes.

Definition 3.1 Let $\mathcal{ST}_{\mathrm{hpl}}(s)$ denote the subfamily of $\mathcal{S}$ consisting of the functions f satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \mathfrak{q}_s(z) \quad (z \in \mathbb{D}), \tag{3.1}$$

and let $\mathcal{CV}_{\mathrm{hpl}}(s)$ be a class of analytic functions f such that

$$1 + \frac{zf''(z)}{f'(z)} \prec \mathfrak{q}_s(z) \quad (z \in \mathbb{D}). \tag{3.2}$$

Geometrically, the condition (3.1) and (3.2) means that the expression $zf'(z)/f(z)$ (or $1 + zf''(z)/f'(z)$, resp.) lies in a domain $\mathbb{H}(s)$. Since $\mathbb{H}(s) = \mathfrak{q}_s(\mathbb{D})$ is contained in a right half-plane, we deduce that $\mathcal{ST}_{\mathrm{hpl}}(s)$ is a proper subset of a class of a starlike functions $\mathcal{ST}$ (and

$\mathcal{CV}_{hpl}(s) \subset \mathcal{CV}$, resp.). Furthermore, the properties of $\mathbb{H}(s)$ considered in Sect. 1, yield

$$\mathcal{ST}_{hpl}(s) \subset \mathcal{ST}(\alpha) \quad \text{and} \quad \mathcal{CV}_{hpl}(s) \subset \mathcal{CV}(\alpha) \quad \text{for } 0 \leq \alpha \leq 2^{-s}, 0 < s \leq 1,$$

$$\mathcal{ST}_{hpl}(s) \subset \mathcal{ST}_{\gamma} \quad \text{and} \quad \mathcal{CV}_{hpl}(s) \subset \mathcal{CV}_{\gamma} \quad \text{for } 0 < s \leq \gamma \leq 1.$$

The above inclusions additionally give

$$\mathcal{ST}_{hpl}(s) \subset \mathcal{ST}_s \cap \mathcal{ST}(1/2^s), \qquad \mathcal{CV}_{hpl}(s) \subset \mathcal{CV}_s \cap \mathcal{CV}(1/2^s), \quad 0 < s \leq 1.$$

Also, geometric properties of Ω_k and $\mathbb{H}(s)$ given by (1.1), (1.2), (1.3) and also (1.4) imply

$$\mathcal{ST}_{hpl}(s) \subset k\text{-}\mathcal{ST} \quad \text{and} \quad \mathcal{CV}_{hpl}(s) \subset k\text{-}\mathcal{CV} \quad \text{for } 0 \leq k \leq 2^{-s}, 0 < s \leq 1,$$

$$k\text{-}\mathcal{ST} \subset \mathcal{ST}_{hpl}(s) \quad \text{and} \quad k\text{-}\mathcal{CV} \subset \mathcal{CV}_{hpl}(s) \quad \text{for } k \geq 2^s - 1, 0 < s \leq 1,$$

$$k\text{-}\mathcal{ST} \cap \mathcal{ST}_{hpl}(s) \neq \emptyset \quad \text{and} \quad k\text{-}\mathcal{CV} \cap \mathcal{CV}_{hpl}(s) \neq \emptyset \quad \text{for } 2^{-s} < k < 2^s - 1, 0 < s \leq 1.$$

Moreover

$$k\text{-}\mathcal{ST} \not\subset \mathcal{ST}_{hpl}(s) \quad \text{and} \quad \mathcal{ST}_{hpl}(s) \not\subset k\text{-}\mathcal{ST} \quad \text{for } 2^{-s} < k < 2^s - 1, 0 < s \leq 1,$$

and

$$k\text{-}\mathcal{CV} \not\subset \mathcal{CV}_{hpl}(s) \quad \text{and} \quad \mathcal{CV}_{hpl}(s) \not\subset k\text{-}\mathcal{CV} \quad \text{for } 2^{-s} < k < 2^s - 1, 0 < s \leq 1.$$

Since q_s is the extremal function in $\mathcal{P}(\mathsf{q}_s)$, the obvious integral representation of $\mathcal{ST}_{hpl}(s)$ and $\mathcal{CV}_{hpl}(s)$ immediately follows.

Theorem 3.2 *A function f is in the class $\mathcal{ST}_{hpl}(s)$ if, and only if there exists p such that $p \prec \mathsf{q}_s$, and*

$$f(z) = z \exp\left(\int_0^z \frac{p(t) - 1}{t}\, dt \right).$$

Theorem 3.3 *A function f is in the class $\mathcal{CV}_{hpl}(s)$ if, and only if there exists p such that $p \prec \mathsf{q}_s$, and*

$$f(z) = \int_0^z \exp\left(\int_0^x \frac{p(t) - 1}{t}\, dt \right) dx.$$

Suppose that $\Phi_{s,n} \in \mathcal{ST}_{hpl}(s)$ is such that

$$\frac{z\Phi_{s,n}'(z)}{\Phi_{s,n}(z)} = \frac{1}{(1 - z^n)^s} \quad (z \in \mathbb{D}, n = 1, 2, \ldots). \tag{3.3}$$

Then the functions $\Phi_{s,n}(z)$ are of the form

$$\Phi_{s,n}(z) = z \exp\left(\int_0^z \frac{\mathsf{q}_s(t^n) - 1}{t}\, dt \right) = z + \frac{s}{n} z^{n+1} + \frac{(n+2)s^2 + ns}{4n^2} z^{2n+1}$$

$$+ \frac{4n^2 s + (9n + 6n^2)s^2 + (2n^2 + 9n + 6)s^3}{36n^3} z^{3n+1} \cdots, \tag{3.4}$$

and these are extremal functions for different problems in the class $\mathcal{ST}_{\mathrm{hpl}}(s)$. For instance

$$
\Phi_s(z) = \Phi_{s,1}(z) = z \exp\left(\int_0^z \frac{\mathfrak{q}_s(t) - 1}{t} \, dt \right)
$$

$$
= z + sz^2 + \frac{3s^2 + s}{4} z^3 + \frac{17s^3 + 15s^2 + 4s}{36} z^4 \cdots \quad (z \in \mathbb{D}). \tag{3.5}
$$

The special case $s = 1/2$ gives

$$
\Phi_{1/2,n}(z) = z \exp\left(\int_0^z \frac{\mathfrak{q}_{1/2}(t^n) - 1}{t} \, dt \right) = z \left(\frac{4}{(1 + \sqrt{1 - z^n})^2} \right)^{1/n},
$$

$$
= z + \frac{1}{2n} z^{n+1} + \frac{7 - 2n}{16n^2} z^{2n+1} + \cdots
$$

and

$$
\Phi_{1/2}(z) = \Phi_{1/2,1}(z) = z \exp\left(\int_0^z \frac{\mathfrak{q}_{1/2}(t) - 1}{t} \, dt \right) = \frac{4z}{(1 + \sqrt{1 - z})^2}
$$

$$
= z + \frac{1}{2} z^2 + \frac{5}{16} z^3 + \cdots \quad (z \in \mathbb{D}).
$$

Also, suppose that $K_{s,n} \in \mathcal{CV}_{\mathrm{hpl}}(s)$ is such that

$$
1 + \frac{zK_{s,n}''(z)}{K_{s,n}'(z)} = \frac{1}{(1 - z^n)^s} \quad (z \in \mathbb{D}, n = 1, 2, \ldots). \tag{3.6}
$$

Then the functions $K_{s,n}(z)$ are of the form

$$
K_{s,n}(z) = \int_0^z \exp\left(\int_0^w \frac{\mathfrak{q}_s(t^n) - 1}{t} \, dt \right) dw
$$

$$
= z + \frac{s}{n(n + 1)} z^{n+1} + \frac{(n + 2)s^2 + ns}{4n^2(2n + 1)} z^{2n+1}
$$

$$
+ \frac{4n^2 s + (9n + 6n^2)s^2 + (2n^2 + 9n + 6)s^3}{36n^3(3n + 1)} z^{3n+1} \cdots,
$$

and the $K_{s,n}$ are extremal functions for various problems in the class $\mathcal{CV}_{\mathrm{hpl}}(s)$. The most interesting special cases are for $n = 1$

$$
K_s(z) = K_{s,1}(z) = \int_0^z \exp\left(\int_0^w \frac{\mathfrak{q}_s(t) - 1}{t} \, dt \right) dw = \int_0^z \frac{\Phi_s(w)}{w} \, dw
$$

$$
= z + \frac{s}{2} z^2 + \frac{3s^2 + s}{12} z^3 + \frac{17s^3 + 15s^2 + 4s}{144} z^4 \cdots \quad (z \in \mathbb{D}) \tag{3.7}
$$

and for $s = 1/2, n = 1$

$$
K_{1/2}(z) = K_{1/2,1}(z) = \int_0^z \frac{\Phi_{1/2}(w)}{w} \, dw
$$

$$
= 8 \log\left(\frac{2}{1 + \sqrt{1 - z}} \right) + 4 \left(\frac{\sqrt{1 - z} - 1}{\sqrt{1 - z} + 1} \right)
$$

$$
= z + \frac{1}{4} z^2 + \frac{5}{48} z^3 + \frac{7}{128} z^4 + \frac{21}{640} z^5 + \cdots (z \in \mathbb{D}).
$$

From Theorems 3.2 and 3.3 we obtain the following corollary.

Corollary 3.4 *If $f \in \mathcal{CV}(1 - s/2)$, then $f' \prec q_s$ in $\mathbb{D}$. Thus f is univalent in $\mathbb{D}$. The functions H and G defined by*

$$H(z) = z \exp\left(\int_0^z \frac{f'(t) - 1}{t}\, dt\right), \qquad G(z) = \int_0^z \exp\left(\int_0^x \frac{f'(t) - 1}{t}\, dt\right) dx,$$

belong to $\mathcal{ST}_{\mathrm{hpl}}(s)$ and $\mathcal{CV}_{\mathrm{hpl}}(s)$, respectively.

Examples Defined classes $\mathcal{ST}_{\mathrm{hpl}}(s)$ and $\mathcal{CV}_{\mathrm{hpl}}(s)$ are nonempty. The integral representation given in Theorems 3.2 and 3.3 provides various examples of functions of those classes. For example, if $q_1(z) = 1 + B_1 z$ with $0 < B_1 < 1 - 2^{-s}$, then $q_1 \prec q_s$. Hence $f_1(z) = z \exp(B_1 z) \in \mathcal{ST}_{\mathrm{hpl}}(s)$, and $g_1(z) = (e^{B_1 z} - 1)/B_1 \in \mathcal{CV}_{\mathrm{hpl}}(s)$ (for $0 < B_1 < 1 - 2^{-s}$), respectively. Since

$$f_2(z) = \frac{z}{1 - B_2 z}, \qquad f_3(z) = \frac{z}{(1 - B_3 z)^2},$$

for $0 < |B_2| \leq s/(2 - s)$, $0 < |B_3| \leq s/(4 - s)$ are starlike of order $(1 - s/2)$, applying Corollary 3.4 we see that the appropriate functions

$$H_2(z) = \frac{z}{1 - B_2 z}, \qquad H_3(z) = \frac{z}{1 - B_3 z} \exp\left(\frac{B_3 z}{1 - B_3 z}\right)$$

belong to the class $\mathcal{ST}_{\mathrm{hpl}}(s)$, and the corresponding functions will be elements of $\mathcal{CV}_{\mathrm{hpl}}(s)$.

Theorem 3.5 *The function $\Phi_{s,n}(z)$, defined by (3.4), is normalized, univalent and convex in one direction in $\mathbb{D}$ for $n = 1, 2, \ldots, \lfloor (2^{1-s} + 1)/s \rfloor$, and convex for $n = 1, 2, \ldots, \lfloor 2^{1-s}/s \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function.*

Proof It is a simple matter to check that $\Phi_{s,n}(z)$ is normalized by $\Phi_{s,n}(0) = \Phi'_{s,n}(0) - 1 = 0$. By a definition (3.3) of $\Phi_{s,n}(z)$, we have

$$\frac{\Phi'_{s,n}(z)}{\Phi_{s,n}(z)} = \frac{1}{z(1 - z^n)^s}.$$

Hence

$$\Re\left(1 + \frac{z\Phi''_{s,n}(z)}{\Phi'_{s,n}(z)}\right) = \Re\left(ns\frac{z^n}{1 - z^n} + \frac{1}{(1 - z^n)^s}\right),$$

and from the above we see that

$$\Re\left(1 + \frac{z\Phi''_{s,n}(z)}{\Phi'_{s,n}(z)}\right) > 2^{-s} - \frac{ns}{2}.$$

In order to get univalence, it is convenient to use the result by Umezawa [12], with the requirement $\Re(1 + zh''(z)/h'(z)) > -1/2$ for univalence of h in $\mathbb{D}$. That condition holds if

$$2^{-s} - \frac{ns}{2} \geq -\frac{1}{2},$$

which is satisfied for $n = 1, 2, \ldots, \lfloor (2^{1-s} + 1)/s \rfloor$. We recall that the same condition gives the convexity in one direction. Furthermore, $\Phi_{s,n}(z)$ is convex in $\mathbb{D}$ if $\Re(1 + z\Phi_{s,n}''(z)/\Phi_{s,n}'(z)) > 0$ holds for $n \leq 2^{1-s}/s$, i.e. for $n = 1, 2, \ldots, \lfloor 2^{1-s}/s \rfloor$. □

We have the following from the results in [7], and Lemma 2.1.

Corollary 3.6 *If $f \in \mathcal{ST}_{\text{hpl}}(s)$ and $|z| = r < 1$, then*

$$-\Phi_s(-r) \leq |f(z)| \leq \Phi_s(r),$$

$$\Phi_s'(-r) \leq |f'(z)| \leq \Phi_s'(r),$$

$$|\text{Arg}\{f(z)/z\}| \leq \max_{|z|=r} \text{Arg}\{\Phi_s(z)/z\},$$

$$f(z)/z \prec \Phi_s(z)/z \quad (z \in \mathbb{D}).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of Φ_s, where Φ_s is given by (3.5). Also, if $f \in \mathcal{ST}_{\text{hpl}}(s)$, then either f is a rotation of Φ_s or

$$\{w \in \mathbb{C} : |w| \leq -\Phi_s(-1)\} \subset f(\mathbb{D}).$$

Here $-\Phi_s(-1)$ is understood to be the limit of $-\Phi_s(-r)$ as r tends to 1.

Similar results hold for functions of $\mathcal{CV}_{\text{hpl}}(s)$.

Corollary 3.7 *If $f \in \mathcal{CV}_{\text{hpl}}(s)$ and $|z| = r < 1$, then*

$$-K_s(-r) \leq |f(z)| \leq K_s(r),$$

$$K_s'(-r) \leq |f'(z)| \leq K_s'(r),$$

$$|\text{Arg}\{f'(z)\}| \leq \max_{|z|=r} \text{Arg}\{K_s'(z)\},$$

$$f'(z) \prec K_s'(z) \quad (z \in \mathbb{D}).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of K_s, where K_s is given by (3.7). Also, if $f \in \mathcal{CV}_{\text{hpl}}(s)$, then either f is a rotation of K_s or

$$\{w \in \mathbb{C} : |w| \leq -K_s(-1)\} \subset f(\mathbb{D}).$$

Here $-K_s(-1)$ is understood to be the limit of $-K_s(-r)$ as r tends to 1.

Theorem 3.8 *Let r_0 denote the positive root of the equation*

$$\frac{1}{(1+r)^s} = \frac{rs}{1-r} \quad (0 \leq r < 1). \tag{3.8}$$

If $f \in \mathcal{ST}_{\text{hpl}}(s)$, then f is convex in the disk $|z| < r_0$.

Proof Let $f \in \mathcal{ST}_{\mathrm{hpl}}(s)$. Then from Definition 3.1 we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1}{(1 - w(z))^s} \quad (z \in \mathbb{D}),$$

where $w \in \mathcal{B}$. Application of Lemma 2.1 and the well-known inequality for Schwarz functions $|w'(z)| \le (1 - |w(z)|^2)/(1 - |z|^2)$ gives

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \ge \Re \frac{1}{(1 - w(z))^s} - s|z| \frac{1 - |w(z)|^2}{[1 - |w(z)|][1 - |z|^2]}$$

$$\ge \frac{1}{(1 + |z|)^s} - \frac{s|z|}{1 - |z|}.$$

The function $\psi(r) = \frac{1}{(1+r)^s} - \frac{rs}{1-r}$, where $r = |z| \in [0, 1)$ is decreasing in $[0, 1)$ with $\psi(0) = 1$ and $\psi(1^-) = -\infty$. Thus, there exists a unique $r_0 \in (0, 1)$ such that $\psi(r_0) = 0$, and for $0 \le r < r_0$ we have $0 < \psi(r) \le 1$. This inequality is equivalent to

$$\frac{1}{(1 + r)^s} > \frac{rs}{1 - r} \quad \bigl(r \in [0, 1)\bigr),$$

which is satisfied for $|z| < r_0$, where r_0 is the only real positive root of (3.8). □

Theorem 3.9 *If $f \in \mathcal{ST}_{\mathrm{hpl}}(s)$, then there exists $\alpha > 1$ such that*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \quad \bigl(|z| = r < 1\bigr).$$

Proof Let $f \in \mathcal{ST}_{\mathrm{hpl}}(s)$. Then, from Definition 3.1, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1}{(1 - w(z))^s} \quad (z \in \mathbb{D}),$$

where $w \in \mathcal{B}$. By Lemma 2.1 and the inequality for Schwarz functions it follows that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \le \Re \frac{1}{(1 - w(z))^s} + s|z| \frac{1 - |w(z)|^2}{[1 - |w(z)|][1 - |z|^2]}$$

$$\le \frac{1}{(1 - |z|)^s} + \frac{s|z|}{1 - |z|}.$$

The function $g(r) = \frac{1}{(1-r)^s} + \frac{rs}{1-r}$, where $r = |z| \in (0, 1)$, is increasing in $(0, 1)$ and $g(0) = 1$. Thus for $z \in \mathbb{D}$ we have $g(|z|) > 1$. □

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