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Global solutions to a two-species chemotaxis system with singular sensitivity and logistic source

Ting Huang[1], Lu Yang[1] and Yongjie Han[1*]

*Correspondence:
yojihan@163.com
[1]School of Science, Xihua University, Chengdu, China

Abstract

This paper is concerned with a chemotaxis system with singular sensitivity and logistic source,

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{w}\nabla w) + \mu_1 u - \mu_1 u^\alpha, & x \in \Omega, t > 0, \\ \upsilon_t = \Delta v - \chi_2 \nabla \cdot (\frac{\upsilon}{w}\nabla w) + \mu_2 \upsilon - \mu_2 \upsilon^\beta, & x \in \Omega, t > 0, \\ w_t = \Delta w - (u + \upsilon)w, & x \in \Omega, t > 0, \end{cases}$$

under the homogeneous Neumann boundary conditions and for widely arbitrary positive initial data in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) with smooth boundary, where χ_i, $\mu_i > 0$ ($i = 1, 2$) and α, $\beta > 1$. It is proved that there exists a global classical solution if $\max\{\chi_1, \chi_2\} < \sqrt{\frac{2}{n}}, \min\{\mu_1, \mu_2\} > \frac{n-2}{n}, \alpha = \beta = 2$ for $n \geq 2$ or any $\chi_i > 0$ ($i = 1, 2$), $\mu_i > 0$ ($i = 1, 2$), α, $\beta > 1$ for $n = 1$.

MSC: 35K55; 35Q92

Keywords: Chemotaxis; Global existence; Singular sensitivity; Logistic source

1 Introduction

The chemotaxis system describes a part of the life cycle of cellular slime molds with chemotaxis. In more detail, slime molds move towards higher concentration of the chemical substance when they plunge into hunger. After the pioneering work of Keller–Segel [8], in a variety of work the classical Keller–Segel model and its variations were investigated.

Particularly, for the chemotaxis system with a consumption mechanism,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\phi(\upsilon)\nabla \upsilon), \\ \upsilon_t = \Delta \upsilon - u\upsilon, \end{cases} \tag{1.1}$$

in the case of $\phi(\upsilon) = 1$, it is well known that for all suitably regular initial data (u_0, v_0) an associated Neumann-type initial-boundary value problem, posed in a smooth n-dimensional domain, admits a global bounded classical solution if $n = 2$ and an asymptotically smooth weak solution for $n = 3$ [20]. When $\phi(\upsilon) = \frac{\chi}{\upsilon}$, it has been shown in [28] that the system possesses a global generalized solution with $\upsilon \to 0$ in $L^p(\Omega)$ as $t \to 0$ in

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the two-dimensional case. A prototypical variant of model (1.1) in which a logistic source function $f(u)$ is considered in the first equation, $u_t = \Delta u - \nabla \cdot (u\phi(\upsilon)\nabla\upsilon) + f(u)$, has also been investigated in the past years. When $\phi(\upsilon)$ is a constant and $f(u) = \kappa u - \mu u^2$, [11] established the existence of a global bounded classical solution for suitably large μ and proved that for any $\mu > 0$ there exists a weak solution in the three-dimensional case. When $\phi(\upsilon) = \frac{\chi}{\upsilon}$ and $f(u) = \kappa u - \mu u^2$, for any n-dimensional domain ($n \geq 2$), there exists a global classical solution to (1.1) provided that $0 < \chi < \sqrt{\frac{2}{n}}$, $\mu > \frac{n-2}{n}$ [10]. When $\phi(s) \in C^1(0,\infty)$ satisfying $\phi(s) \to \infty$ as $s \to 0$, for the more general logistic source $f(u) = ru - \mu u^k$ ($r, \mu > 0, k > 1$), it has been shown in [31] that the problem (1.1) possesses a unique positive global classical solution provided $k > 1$ with $n = 1$ or $k > 1 + \frac{n}{2}$ with $n \geq 2$. Besides the above work, global solutions for the corresponding variant of (1.1) such as coupled chemotaxis–fluid system have also been investigated (see e.g. [17, 18, 24, 25, 27] and the references therein) by many authors.

In recent years, multi-species chemotaxis systems have been studied (see e.g. [1, 4, 12, 13, 15, 16, 21]). For instance, the following two-species chemotaxis model:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u\nabla w), & x \in \Omega, t > 0, \\ \upsilon_t = \Delta \upsilon - \chi_2 \nabla \cdot (\upsilon\nabla w), & x \in \Omega, t > 0, \\ w_t = \Delta w - \beta w + \alpha_1 u + \alpha_2 \upsilon, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \upsilon}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

has been researched by some authors, where χ_1, χ_2, β, α_1, α_2 are positive constants. [12] proved that, for any $m_i > 0$ ($i = 1,2$), there exists radially symmetric initial data $(u_0, \upsilon_0, w_0) \in (C^0(\Omega))^2 \times W^{1,\infty}(\Omega)$ with $m_1 = \int_\Omega u_0$, $m_2 = \int_\Omega \upsilon_0$ such that the corresponding solution blows up in finite time when Ω is a ball in $\mathbb{R}^n$ ($n \geq 3$). In radial symmetric situation, Espejo Arenas et al. [4] proved that there is simultaneous blow-up for both chemotactic species in the ball $B_R(0)$ of $\mathbb{R}^2$. In higher dimensions, blow-up of the parabolic–elliptic counterpart of (1.2) has been studied by Biler et al. [1].

A more general form of two-species chemotaxis model has been studied by Mizukami and Yokota. They considered the two-species chemotaxis model as follows:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\phi_1(w)\nabla w) + \mu_1 u(1-u), & x \in \Omega, t > 0, \\ \upsilon_t = \Delta \upsilon - \nabla \cdot (\upsilon\phi_2(w)\nabla w) + \mu_2 \upsilon(1-\upsilon), & x \in \Omega, t > 0, \\ w_t = d\Delta w + h(u,\upsilon,w), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \upsilon}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $d \geq 0$, $\mu_i > 0$ ($i = 1,2$), $\phi_i \in C^{1+\theta}([0,\infty)) \cap L^1(0,\infty)$ ($i = 1,2$) for some $\theta > 0$. They proved in [14] that there exists an exact pair (u,υ,w) of nonnegative functions which is uniformly bounded.

Inspired by the above-mentioned work, in this paper, we study the initial-boundary value problem of a chemotaxis system with singular sensitivity and logistic source as

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{w} \nabla w) + \mu_1 u - \mu_1 u^{\alpha}, & x \in \Omega, t > 0, \\ \upsilon_t = \Delta \upsilon - \chi_2 \nabla \cdot (\frac{\upsilon}{w} \nabla w) + \mu_2 \upsilon - \mu_2 \upsilon^{\beta}, & x \in \Omega, t > 0, \\ w_t = \Delta w - (u + \upsilon) w, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \upsilon}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad \upsilon(x,0) = \upsilon_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary $\partial\Omega$, where $\chi_1, \chi_2, \mu_1, \mu_2$ are positive constants, $\alpha > 1, \beta > 1$ and ν is the outward normal vector to $\partial\Omega$. The functions $u = u(x,t)$ and $\upsilon = \upsilon(x,t)$ denote, respectively, the unknown population density of the two species, and $w = w(x,t)$ represents the concentration of the chemoattractant. In contrast to the study of [14], we will consider a singular sensitivity $\frac{\chi_i}{w}$, which is suggested by the Weber–Fechner law of stimulus perception (see [9]) and supported by experimental ([7]) and theoretical ([29]) evidence. Moreover, the exponents α and β do not necessarily be 2. We shall establish the global existence of this system.

Throughout this paper, we suppose that the initial data u_0, υ_0, w_0 satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), & u_0 \geq 0 \text{ and } u_0 \not\equiv 0 \text{ in } \overline{\Omega}, \\ v_0 \in C^0(\overline{\Omega}), & v_0 \geq 0 \text{ and } v_0 \not\equiv 0 \text{ in } \overline{\Omega}, \\ w_0 \in W^{1,\infty}(\Omega), & w_0 > 0 \text{ in } \overline{\Omega}. \end{cases} \tag{1.5}$$

Our main results read as follows.

Theorem 1.1 *Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary. Assume that*

$$\max\{\chi_1, \chi_2\} < \sqrt{\frac{2}{n}}, \qquad \min\{\mu_1, \mu_2\} > \frac{n-2}{n}, \qquad \alpha = \beta = 2.$$

Then for any initial data (u_0, υ_0, w_0) as in (1.5) there is a global classical solution (u, υ, w) to (1.4).

Theorem 1.2 *Let $\Omega \subseteq \mathbb{R}$ be an open, bounded interval, $\chi_i > 0$ ($i = 1,2$), $\mu_i > 0$ ($i = 1,2$) and α, $\beta > 1$. Then for any initial data (u_0, υ_0, w_0) as in (1.5) there exists a global classical solution (u, υ, w) to (1.4).*

In the present paper, we shall modify the method in [10, 31] to obtain global existence of the solution. Precisely speaking, we first try to derive the lower bound estimate of w, via building the upper bound estimated of $z := -\ln(\frac{w}{\|w_0\|_{L^\infty(\Omega)}})$, and then obtain estimates for $\|u\|_{L^p(\Omega)}$ and $\|\upsilon\|_{L^p(\Omega)}$ for some $p > 1, p > \frac{n}{2}$.

Before we go to the details of our analysis, let us point out that the global existence, boundedness and stabilization of (weak) solutions to the two-species chemotaxis–fluid system have also been established (see e.g. [2, 5, 6, 12, 19]).

2 Some preliminaries

We begin with the local existence of classical solutions to the system (1.4), the proof of which is standard. Refer to, e.g., [30], Lemma 2.1, for the details.

Lemma 2.1 *Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$) be a bounded, smooth domain. Then, for any u_0, υ_0, w_0 satisfying* (1.5), *there exist $T_{\max} \in (0, +\infty]$ and a unique pair of functions (u, υ, w) with*

$$
\begin{aligned}
&u \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big),\\
&\upsilon \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big),\\
&w \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big) \cap L^{\infty}_{\mathrm{loc}}\big([0, T_{\max}); W^{1,\infty}(\Omega)\big),
\end{aligned}
$$

solving (1.4) *in the classical sense with $u, \upsilon, w > 0$ in $\overline{\Omega} \times (0, T_{\max})$ and if $T_{\max} < \infty$, then*

$$
\big\|u(\cdot, t)\big\|_{L^\infty(\Omega)} + \big\|\upsilon(\cdot, t)\big\|_{L^\infty(\Omega)} + \big\|w(\cdot, t)\big\|_{W^{1,\infty}(\Omega)} \to \infty
$$

as $t \to T_{\max}$.

The following mass-preserving property, which is also frequently used in the study of some other chemotaxis systems (see e.g. [22, 23] and the references therein), can be easily obtained.

Lemma 2.2 *If* (1.5) *holds, then the solution of* (1.4) *satisfies*

$$
\big\|u(\cdot, t)\big\|_{L^1(\Omega)} \leq m_1, \qquad \big\|\upsilon(\cdot, t)\big\|_{L^1(\Omega)} \leq m_2, \quad t \in [0, T_{\max}), \tag{2.1}
$$

for some $m_1 > 0, m_2 > 0$.

Proof As $\alpha > 1$, we can easily obtain from the first equation in (1.4) by using Hölder's inequality that

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u\,dx &= \mu_1 \int_\Omega u\,dx - \mu_1 \int_\Omega u^\alpha\,dx\\
&\leq \mu_1 \int_\Omega u\,dx - \frac{\mu_1}{|\Omega|^{\alpha-1}}\left(\int_\Omega u\,dx\right)^\alpha, \quad t \in (0, T_{\max}),
\end{aligned}
$$

which yields the left-hand inequality of (2.1) by the Bernoulli inequality ([3], Lemma 1.2.4). The right-hand inequality of (2.1) can be quickly proved in the same way. □

Also for w the differential equation directly entails some decay properties, as follows.

Lemma 2.3 *For every $p \in [1, \infty)$, the map $(0, T_{\max}) \ni t \mapsto \|w(\cdot, t)\|^p_{L^p(\Omega)}$ is monotone decreasing. In particular, $\|w(\cdot, t)\|_{L^p(\Omega)} \leq \|w_0\|_{L^p(\Omega)}$ for all $t \in [0, T_{\max})$.*

Proof It is easy to see from the third equation in (1.4) that

$$
\frac{d}{dt}\int_\Omega w^p = p\int_\Omega w^{p-1} w_t
$$

$$\begin{aligned}&= p\int_\Omega w^{p-1}\Delta w - p\int_\Omega w^p u - p\int_\Omega w^p \upsilon \\ &= -p(p-1)\int_\Omega w^{p-2}|\nabla w|^2 - p\int_\Omega w^p u - p\int_\Omega w^p\upsilon \le 0,\end{aligned}$$

due to $u, \upsilon, w > 0$. □

Lemma 2.4 $\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)}$ *for every* $t \in [0, T_{\max})$.

Proof The consequence can be obtained by the comparison theorem (Theorem B.1 of [10]). □

Lemma 2.5 *Let* $z := -\ln(\frac{w}{\|w_0\|_{L^\infty(\Omega)}})$. *Then we have* $z \ge 0$ *and* $z_t = \Delta z - |\nabla z|^2 + u + \upsilon$ *on* $\Omega \times (0, T_{\max})$.

Proof According to Lemma 2.4, we know $\frac{w}{\|w_0\|_{L^\infty(\Omega)}} \le 1$ for $(x,t) \in (\Omega \times (0, T_{\max}))$ and thus $z \ge 0$. Moreover, on $\Omega \times (0, T_{\max})$,

$$\begin{aligned}z_t &= -\frac{\|w_0\|_{L^\infty(\Omega)} w_t}{w\|w_0\|_{L^\infty(\Omega)}} = -\frac{w_t}{w}, \\ \nabla z &= -\frac{\|w_0\|_{L^\infty(\Omega)}\nabla w}{w\|w_0\|_{L^\infty(\Omega)}} = -\frac{\nabla w}{w},\end{aligned}$$

which entails

$$\Delta z = \nabla\cdot\nabla z = \nabla\cdot\left(-\frac{\nabla w}{w}\right) = -\frac{\Delta w}{w} + \frac{|\nabla w|^2}{w^2} = -\frac{\Delta w}{w} + |\nabla z|^2.$$

Together with $w_t = \Delta w - (u+v)w$ this proves

$$z_t = -\frac{w_t}{w} = -\frac{\Delta w}{w} + u + \upsilon = \Delta z - |\nabla z|^2 + u + \upsilon \tag{2.2}$$

on $\Omega \times (0, T_{\max})$. □

Accordingly, the pair (u, υ, z) solves the PDE system

$$\begin{cases} u_t = \Delta u + \chi_1\nabla\cdot(u\nabla z) + \mu_1 u - \mu_1 u^\alpha, & x\in\Omega, t>0, \\ \upsilon_t = \Delta\upsilon + \chi_2\nabla\cdot(\upsilon\nabla z) + \mu_2\upsilon - \mu_2\upsilon^\beta, & x\in\Omega, t>0, \\ z_t = \Delta z - |\nabla z|^2 + u + \upsilon, & x\in\Omega, t>0, \\ \frac{\partial u}{\partial\nu} = \frac{\partial\upsilon}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega, t>0, \\ u(x,0) = u_0(x), \qquad \upsilon(x,0) = \upsilon_0(x), & \\ z(x,0) = z_0(x) := -\ln\frac{w_0}{\|w_0\|_{L^\infty(\Omega)}}, & x\in\Omega. \end{cases} \tag{2.3}$$

For solution (u, υ, z) of the system (2.3), we present the following proposition.

Lemma 2.6 *For any* $T \le T_{\max}$, $T < \infty$, *if there are a constant* $C = C(T) > 0$ *and some* $p \ge 1$, *satisfying* $p > \frac{n}{2}$, *with* $\|u(\cdot,t)\|_{L^p} + \|\upsilon(\cdot,t)\|_{L^p} \le C$ *on* $(0,T)$, *then* z *is bounded on* $\Omega \times (0,T)$.

Proof By the variation-of-constants formula, z can be represented as

$$
\begin{aligned}
z(\cdot,t) &= e^{t\Delta}z_0 + \int_0^t e^{(t-s)\Delta}\bigl(u(\cdot,s) + \upsilon(\cdot,s) - |\nabla z|^2\bigr)\,ds \\
&\le e^{t\Delta}z_0 + \int_0^t e^{(t-s)\Delta}u(\cdot,s)\,ds + \int_0^t e^{(t-s)\Delta}\upsilon(\cdot,s)\,ds, \quad t\in(0,T),
\end{aligned}
$$

for $-|\nabla z(\cdot,s)|^2 \le 0$ immediately implies $e^{(t-s)\Delta}(-|\nabla z(\cdot,s)|^2) \le 0$. With this representation and semigroup estimates as in [26], Lemma 1.3, we obtain $c_1 > 0$ such that for all $t \in (0,T)$

$$
\begin{aligned}
&\bigl\|z(\cdot,t)\bigr\|_{L^\infty(\Omega)} \\
&\quad\le \bigl\|e^{t\Delta}z_0\bigr\|_{L^\infty(\Omega)} + \int_0^t \bigl\|e^{(t-s)\Delta}u(\cdot,s)\bigr\|_{L^\infty(\Omega)}\,ds + \int_0^t \bigl\|e^{(t-s)\Delta}\upsilon(\cdot,s)\bigr\|_{L^\infty(\Omega)}\,ds \\
&\quad\le \|z_0\|_{L^\infty(\Omega)} + \int_0^t c_1\bigl(1+(t-s)^{-\frac{n}{2p}}\bigr)\bigl\|u(\cdot,s)\bigr\|_{L^p(\Omega)}\,ds \\
&\qquad + \int_0^t c_1\bigl(1+(t-s)^{-\frac{n}{2p}}\bigr)\bigl\|\upsilon(\cdot,s)\bigr\|_{L^p(\Omega)}\,ds \\
&\quad\le \|z_0\|_{L^\infty(\Omega)} + 2c_1C\int_0^T\bigl(1+(t-s)^{-\frac{n}{2p}}\bigr)\,ds < \infty,
\end{aligned}
$$

since $p > \frac{n}{2}$ implies that $-\frac{n}{2p} > -1$ and thus finiteness of the integral. □

From the above lemma, we can obtain the following estimate for w.

Lemma 2.7 *For any $T \le T_{\max}$, $T < \infty$, if there are a constant $C = C(T) > 0$ and some $p \ge 1$, $p > \frac{n}{2}$, with $\|u(\cdot,t)\|_{L^p} + \|\upsilon(\cdot,t)\|_{L^p} \le C$ on $(0,T)$, then there are $d = d(T) > 0$, such that $w \ge d$ and in particular $\frac{1}{w} \le \frac{1}{d}$ on $\Omega \times (0,T)$.*

Proof By Lemma 2.6 there are $C = C(T) > 0$ with $z \le C$ on $\Omega \times (0,T)$. From the definition $z := -\ln(\frac{w}{\|w_0\|_{L^\infty(\Omega)}})$, we directly obtain $w \ge \|w_0\|_{L^\infty(\Omega)}e^{-C} =: d > 0$ on $\Omega \times (0,T)$. □

3 Global existence for *n*-dimensional case ($n \ge 2$)

Now we deal with the global solutions of (1.4) when $n \ge 2, \alpha = \beta = 2$.

Lemma 3.1 *Let $T \in (0,T_{\max}]$, $T < \infty$, $r,p \in [1,\infty]$ and suppose*

$$
\frac{1}{2} + \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right) < 1.
$$

Then there is $C > 0$ such that for all $t \in (0,T)$ we have

$$
\bigl\|\nabla w(\cdot,t)\bigr\|_{L^r(\Omega)} \le C\Bigl(1 + \sup_{s\in(0,t)}\bigl\|u(\cdot,s)\bigr\|_{L^p(\Omega)} + \sup_{s\in(0,t)}\bigl\|\upsilon(\cdot,s)\bigr\|_{L^p(\Omega)}\Bigr).
$$

Proof First let $p \le r$.

Due to the variation-of-constants formula, for all $t \in (0,T)$ we have

$$
\bigl\|\nabla w(\cdot,t)\bigr\|_{L^r(\Omega)} \le \bigl\|\nabla e^{t\Delta}w_0\bigr\|_{L^r(\Omega)} + \int_0^t \bigl\|\nabla e^{(t-s)\Delta}\bigl((u+\upsilon)w\bigr)\bigr\|_{L^r(\Omega)}\,ds
$$

and the semigroup estimates of [26], Lemma 1.3, entail the existence of $c_1 > 0$ and $\lambda > 0$ such that

$$\|\nabla e^{t\triangle} w_0\|_{L^r(\Omega)} \leq c_1$$

and

$$\begin{aligned}
&\|\nabla e^{(t-s)\Delta}((u+\upsilon)w)\|_{L^r(\Omega)} \\
&\quad \leq c_1\bigl(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\bigr)e^{-\lambda(t-s)}\|(u+\upsilon)w\|_{L^p(\Omega)} \\
&\quad \leq c_1\bigl(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\bigr)e^{-\lambda(t-s)}\|u+\upsilon\|_{L^p(\Omega)}\|w\|_{L^\infty(\Omega)} \\
&\quad \leq c_1\bigl(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\bigr)e^{-\lambda(t-s)}\bigl(\|u\|_{L^p(\Omega)}+\|\upsilon\|_{L^p(\Omega)}\bigr)\|w_0\|_{L^\infty(\Omega)}
\end{aligned}$$

hold for all $t \in (0,T)$, $s \leq t$, where in the last step we have employed Lemma 2.4. Together this results in

$$\begin{aligned}
&\|\nabla w(\cdot,t)\|_{L^r(\Omega)} \\
&\quad \leq c_1 + c_1\|w_0\|_{L^\infty(\Omega)}\Bigl(\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^p(\Omega)} + \sup_{s\in(0,t)}\|\upsilon(\cdot,s)\|_{L^p(\Omega)}\Bigr) \\
&\qquad \times \int_0^t \bigl(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\bigr)e^{-\lambda(t-s)}\,ds
\end{aligned}$$

for all $t \in (0,T)$ and hence in the claim, because the integral $\int_0^\infty (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}) \times e^{-\lambda(t-s)}\,ds$ is finite.

For $p > r$ the claim follows from the previous considerations together with Hölder's inequality. For some c_2, c_3 we have

$$\begin{aligned}
&\|\nabla w(\cdot,t)\|_{L^r(\Omega)} \\
&\quad \leq c_2\|\nabla w(\cdot,t)\|_{L^p(\Omega)} \\
&\quad \leq c_3\Bigl(1+\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^r(\Omega)} + \sup_{s\in(0,t)}\|\upsilon(\cdot,s)\|_{L^r(\Omega)}\Bigr) \\
&\quad \leq c_3\Bigl(1+\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^p(\Omega)}|\Omega|^{\frac{p-r}{rp}} + \sup_{s\in(0,t)}\|\upsilon(\cdot,s)\|_{L^p(\Omega)}|\Omega|^{\frac{p-r}{rp}}\Bigr) \\
&\quad \leq C\Bigl(1+\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^p(\Omega)} + \sup_{s\in(0,t)}\|\upsilon(\cdot,s)\|_{L^p(\Omega)}\Bigr)
\end{aligned}$$

for all $t \in (0,T)$. □

The following lemma asserts that boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ and $\|\upsilon(\cdot,t)\|_{L^p(\Omega)}$ for some $p > \frac{n}{2}$ is sufficient to guarantee boundedness of the solution.

Lemma 3.2 *Suppose that the initial data u_0, υ_0 and w_0 satisfy (1.5). Let $T \in (0,T_{\max}]$, $T < \infty$, $p \geq 1$. If the first and second components of the solution satisfy*

$$\sup_{t\in(0,T)}\bigl(\|u(\cdot,t)\|_{L^p(\Omega)} + \|\upsilon(\cdot,t)\|_{L^p(\Omega)}\bigr) < \infty,$$

for some $p > \frac{n}{2}$, *then*

$$\sup_{t\in(0,T)} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\upsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}}\right) < \infty.$$

Proof For each fixed $p > \frac{n}{2}$

$$\frac{np}{(n-p)_+} = \begin{cases} \infty & \text{if } p \geq n, \\ \frac{np}{(n-p)} > n & \text{if } \frac{n}{2} < p < n, \end{cases}$$

it is possible to find $p_0 > 1$ fulfilling

$$n < p_0 < \frac{np}{(n-p)_+}, \tag{3.1}$$

which enables us to choose $k > 1$ such that

$$n < kp_0 < \frac{np}{(n-p)_+}. \tag{3.2}$$

Applying the variation-of constants formula for u and $c_1 := \sup_{u>0}(u-u^2)$

$$\begin{aligned} u(\cdot,t) &= e^{t\Delta}u_0 - \chi_1 \int_0^t e^{(t-s)\Delta}\nabla\cdot\left(u\frac{\nabla w}{w}\right)ds + \mu_1\int_0^t e^{(t-s)\Delta}\left(u-u^2\right)ds \\ &\leq e^{t\Delta}u_0 - \chi_1 \int_0^t e^{(t-s)\Delta}\nabla\cdot\left(u\frac{\nabla w}{w}\right)ds + \mu_1 c_1 T \end{aligned}$$

we get

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + \chi_1\int_0^t \left\|e^{(t-s)\Delta}\nabla\cdot\left(u\frac{\nabla w}{w}\right)\right\|_{L^\infty(\Omega)} ds + \mu_1 c_1 T$$

for all $t \in (0,T)$. In view of the smooth estimates for the Neumann heat semigroup ([26], Lemma 3.1), we obtain c_2 satisfying

$$\begin{aligned} &\|u(\cdot,t)\|_{L^\infty(\Omega)} \\ &\quad \leq \|u_0\|_{L^\infty(\Omega)} + c_2\chi_1\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p_0}}\right)\left\|u\frac{\nabla w}{w}\right\|_{L^{p_0}(\Omega)} ds + \mu_1 c_1 T \end{aligned} \tag{3.3}$$

for all $t \in (0,T)$. Here by Hölder's inequality, the interpolation inequality, (2.1), Lemma 3.1, and (3.2), we can find c_3 such that

$$\begin{aligned} \left\|u\frac{\nabla w}{w}\right\|_{L^{p_0}(\Omega)} &\leq \frac{1}{d}\|u\|_{L^{k'p_0}(\Omega)}\|\nabla w\|_{L^{kp_0}(\Omega)} \\ &\leq \frac{1}{d}\|u\|^a_{L^\infty(\Omega)}\|u\|^{1-a}_{L^1(\Omega)}\|\nabla w\|_{L^{kp_0}} \\ &\leq \frac{1}{d}m_1^{1-a}c_3\|u\|^a_{L^\infty(\Omega)}, \end{aligned}$$

where $k' = \frac{k}{k-1}$, $a = 1 - \frac{1}{k'p_0} \in (0,1)$. Inserting this into (3.3), it follows that

$$\sup_{t\in(0,T)} \left\|u(\cdot,t)\right\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + c_4 \sup_{t\in(0,T)} \left\|u(\cdot,t)\right\|_{L^\infty(\Omega)}^a + \mu_1 c_1 T$$

for all $T \in (0, T_{\max}]$ with

$$c_4 = \frac{1}{d} m_1^{1-a} \chi_1 c_2 c_3 \int_0^\infty \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p_0}}\right) ds$$

is finite thanks to the left-hand side of (3.1).

Arguing similarly, we see that

$$\sup_{t\in(0,T)} \left\|\upsilon(\cdot,t)\right\|_{L^\infty(\Omega)} \le \|\upsilon_0\|_{L^\infty(\Omega)} + c_6 \sup_{t\in(0,T)} \left\|\upsilon(\cdot,t)\right\|_{L^\infty(\Omega)}^a + \mu_2 c_5 T$$

for all $T \in (0, T_{\max}]$. The boundedness assertion concerning $\|w\|_{W^{1,\infty}(\Omega)}$ results from Lemma 2.4 and Lemma 3.1. Hence we complete the proof. □

We are in need of estimates of $\|u\|_{L^p(\Omega)}$ and $\|\upsilon\|_{L^p(\Omega)}$ for some $p > \frac{n}{2}$. The estimates will be based on the following observation.

Lemma 3.3 *For all $p, q \in \mathbb{R}$, on $(0, T_{\max})$ we have*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^p w^q \\ &\quad \le -(p-1)p\int_\Omega u^{p-2}w^q|\nabla u|^2 + \bigl(p(p-1)\chi_1 - 2pq\bigr)\int_\Omega u^{p-1}w^{q-1}\nabla u\cdot\nabla w \\ &\qquad + \bigl(pq\chi_1 - q(q-1)\bigr)\int_\Omega u^p w^{q-2}|\nabla w|^2 + p\mu_1\int_\Omega u^p w^q - (\mu_1 p + q)\int_\Omega u^{p+1}w^q, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \upsilon^p w^q \\ &\quad \le -(p-1)p\int_\Omega \upsilon^{p-2}w^q|\nabla \upsilon|^2 + \bigl(p(p-1)\chi_2 - 2pq\bigr)\int_\Omega \upsilon^{p-1}w^{q-1}\nabla \upsilon\cdot\nabla w \\ &\qquad + \bigl(pq\chi_2 - q(q-1)\bigr)\int_\Omega \upsilon^p w^{q-2}|\nabla w|^2 + p\mu_2\int_\Omega \upsilon^p w^q - (\mu_1 p + q)\int_\Omega \upsilon^{p+1}w^q. \end{aligned} \tag{3.5}$$

Proof A direct calculation shows that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^p w^q \\ &\quad = p\int_\Omega u^{p-1}u_t w^q + q\int_\Omega u^p w^{q-1} w_t \\ &\quad = p\int_\Omega u^{p-1}w^q \Delta u - p\chi_1\int_\Omega u^{p-1}w^q\nabla\cdot\left(u\frac{\nabla w}{w}\right) + p\int_\Omega u^{p-1}\bigl(\mu_1\bigl(u-u^2\bigr)\bigr)w^q \\ &\qquad + q\int_\Omega u^p w^{q-1}\Delta w - q\int_\Omega u^{p+1}w^q - q\int_\Omega u^p w^q \upsilon \\ &\quad \le -p\int_\Omega \nabla\bigl(u^{p-1}w^q\bigr)\cdot\nabla u + p\chi_1\int_\Omega \frac{u}{w}\nabla\bigl(u^{p-1}w^q\bigr) \end{aligned}$$

$$
\begin{aligned}
&\times \nabla w + p\mu_1 \int_\Omega u^p w^q - \mu_1 p \int_\Omega u^{p+1} w^q \\
&- q \int_\Omega \nabla\left(u^p w^{q-1}\right) \cdot \nabla w - q \int_\Omega u^{p+1} w^q \\
= &-p(p-1) \int_\Omega u^{p-2} w^q |\nabla u|^2 - pq \int_\Omega u^{p-1} w^{q-1} \nabla u \cdot \nabla w \\
&+ p(p-1)\chi_1 \int_\Omega u^{p-1} w^{q-1} \nabla u \cdot \nabla w + pq\chi_1 \int_\Omega u^q w^{q-2} |\nabla w|^2 + p\mu_1 \int_\Omega u^p w^q \\
&- \mu_1 p \int_\Omega u^{p+1} w^q - pq \int_\Omega u^{p-1} w^{q-1} \nabla u \cdot \nabla w \\
&- q(q-1) \int_\Omega u^p w^{q-2} |\nabla w|^2 - q \int_\Omega u^{p+1} w^q \\
= &-p(p-1) \int_\Omega u^{p-2} w^q |\nabla u|^2 + \left(p(p-1)\chi_1 - 2pq\right) \int_\Omega u^{p-1} w^{q-1} \nabla u \cdot \nabla w \\
&+ \left(pq\chi_1 - q(q-1)\right) \\
&\times \int_\Omega u^p w^{q-2} |\nabla w|^2 + p\mu_1 \int_\Omega u^p w^q - (\mu_1 p + q) \int_\Omega u^{p+1} w^q
\end{aligned}
$$

on $(0, T_{\max})$. In the same way, we obtain (3.5). □

Next, we transform (3.4), (3.5) into bounds on $\int_\Omega u^p w^q$, $\int_\Omega \upsilon^p w^q$, where we will, in fact, use a negative exponent q.

Lemma 3.4 *If $p > 1$ and $r > 0$ satisfy $p \in (1, \frac{1}{\max\{\chi_1^2, \chi_2^2\}})$ and $r \in (r_-, \min\{r_+, \min\{\mu_1, \mu_2\}p\})$, where*

$$
r_\pm = \frac{p-1}{2}\left(1 \pm \sqrt{1 - p\chi_i^2}\right) \quad (i = 1, 2),
$$

and $T \in (0, T_{\max}]$, $T < \infty$, then there is $C = C(T) > 0$ such that

$$
\int_\Omega u^p w^{-r} \le C, \qquad \int_\Omega \upsilon^p w^{-r} \le C \quad \text{on } (0, T).
$$

Proof Inserting $q = -r$ in Lemma 3.3, on $(0, T_{\max})$ we obtain

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega u^p w^{-r} \\
&\quad \le -p(p-1) \int_\Omega u^{p-2} w^{-r} |\nabla u|^2 + \left(p(p-1)\chi_1 + 2pr\right) \int_\Omega u^{p-1} w^{-r-1} \nabla u \cdot \nabla w \\
&\qquad - \left(pr\chi_1 + r(r+1)\right) \int_\Omega u^p w^{-r-2} |\nabla w|^2 + p\mu_1 \int_\Omega u^p w^{-r} + (r - \mu_1 p) \int_\Omega u^{p+1} w^{-r}.
\end{aligned}
$$

By Young's inequality, the second term can be estimated by

$$
\begin{aligned}
&\left| \left(p(p-1)\chi_1 + 2pr\right) \int_\Omega u^{p-1} w^{-r-1} \nabla u \cdot \nabla w \right| \\
&\quad \le p(p-1) \int_\Omega u^{p-2} w^{-r} |\nabla u|^2 + \frac{(p(p-1)\chi_1 + 2pr)^2}{4p(p-1)} \int_\Omega u^p w^{-r-2} |\nabla w|^2
\end{aligned}
$$

on $(0, T_{\max})$. Thus, on $(0, T_{\max})$ we have

$$\frac{d}{dt}\int_\Omega u^p w^{-r} \le \left(\frac{(p(p-1)\chi_1 + 2pr)^2}{4p(p-1)} - \big(pr\chi_1 + r(r+1)\big)\right)\int_\Omega u^p w^{-r-2}|\nabla w|^2 + p\mu_1\int_\Omega u^p w^{-r} + (r - \mu_1 p)\int_\Omega u^{p+1}w^{-r}.$$

By the choice of r, $r - \mu_1 p < 0$ and $\frac{(p(p-1)\chi_1+2pr)^2}{4p(p-1)} - (pr\chi_1 + r(r+1)) < 0$, because

$$\begin{aligned} r \in (r_-, r_+) \quad &\Longrightarrow \quad r^2 - (p-1)r + \frac{p(p-1)^2\chi_1^2}{4} < 0 \\ &\Longrightarrow \quad p(p-1)\chi_1^2 + 4pr\chi_1 + \frac{4r^2 p}{p-1} < 4pr\chi_1 + 4r^2 + 4r \\ &\Longrightarrow \quad \frac{(\chi_1 + \frac{2r}{p-1})^2 p(p-1)}{4(pr\chi_1 + r(r+1))} < 1 \\ &\Longrightarrow \quad \frac{(p(p-1)\chi_1 + 2pr)^2}{4(pr\chi_1 + r(r+1))} < p(p-1) \\ &\Longrightarrow \quad \frac{(p(p-1)\chi_1 + 2pr)^2}{4p(p-1)} - \big(pr\chi_1 + r(r+1)\big) < 0, \end{aligned}$$

we can conclude that

$$\frac{d}{dt}\int_\Omega u^p w^{-r} \le p\mu_1\int_\Omega u^p w^{-r}$$

on $(0, T_{\max})$ and hence

$$\int_\Omega u^p(\cdot,t)w^{-r}(\cdot,t) \le e^{p\mu_1 t}\int_\Omega u_0^p w_0^{-r}$$

for every $t \in (0, T_{\max})$. Thus for every $T \in (0, T_{\max}]$ with $T < \infty$ there is $C_1 := e^{p\mu_1 T} \times \int_\Omega u_0^p w_0^{-r}$, such that

$$\int_\Omega u^p(\cdot,t)w^{-r}(\cdot,t) \le C_1$$

for all $t \in (0, T)$. In the same way, we see that

$$\int_\Omega \upsilon^p(\cdot,t)w^{-r}(\cdot,t) \le C_2$$

for all $t \in (0, T)$. □

Aided by Lemma 3.4, we now can find a bound for $\|u(\cdot,t)\|_{L^p(\Omega)}$ and $\|\upsilon(\cdot,t)\|_{L^p(\Omega)}$.

Lemma 3.5 *Let* $p \in (1, \frac{1}{\max\{\chi_1^2,\chi_2^2\}})$ *be such that* $\min\{\mu_1,\mu_2\}p > \frac{p-1}{2}$ *and let* $T \in (0, T_{\max})$, $T < \infty$. *Then there is* $C = C(T) > 0$ *satisfying* $\|u(\cdot,t)\|_{L^p(\Omega)} \le C$, $\|\upsilon(\cdot,t)\|_{L^p(\Omega)} \le C$ *for all* $t \in (0, T)$.

Proof Let $r_\pm$ be the constants as in Lemma 3.4, that is,

$$r_\pm = \frac{p-1}{2}\left(1 \pm \sqrt{1-p\chi_i^2}\right) \quad (i=1,2).$$

Due to $p < \frac{1}{\max\{\chi_1^2,\chi_2^2\}}$, apparently we have $1 - p\chi_i^2 > 0$ and thus $r_- < r_+$. Moreover, since $\min\{\mu_1,\mu_2\}p > \frac{p-1}{2}$, $r_- < \frac{p-1}{2}$, it is ensured that $r_- < \min\{\mu_1,\mu_2\}p$. Accordingly, there is some $r \in (r_-, \min\{r_+, \min\{\mu_1,\mu_2\}p\})$. For such a number r by Lemma 3.4 there is $c_1 > 0$ satisfying

$$\int_\Omega u^p(\cdot,t)w^{-r}(\cdot,t) \le c_1 \quad \text{for all } t \in (0,T).$$

For $t \in (0,T)$ it now holds true that

$$\begin{aligned}\|u(\cdot,t)\|_{L^p(\Omega)} &= \left(\int_\Omega u^p(\cdot,t)w^{-r}(\cdot,t)w^r(\cdot,t)\right)^{\frac{1}{p}} \le \left(\int_\Omega u^p(\cdot,t)w^{-r}(\cdot,t)\|w^r\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}} \\ &\le \|w_0\|_{L^\infty(\Omega)}^{\frac{r}{p}}\left(\int_\Omega u^p(\cdot,t)w^{-r}(\cdot,t)\right)^{\frac{1}{p}} \le \|w_0\|_{L^\infty(\Omega)}^{\frac{r}{p}} c_1^{\frac{1}{p}} =: C_1,\end{aligned}$$

because by Lemma 2.4, for every $t \in (0,T_{\max})$ we have $\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)}$.

Arguing similarly, we can obtain $\|\upsilon(\cdot,t)\|_{L^p(\Omega)} \le C_2$ for all $t \in (0,T)$. □

We can now use this to show global existence.

Proof of Theorem 1.1 Because $\max\{\chi_1,\chi_2\} < \sqrt{\frac{2}{n}}$, the interval $(\frac{n}{2}, \frac{1}{\max\{\chi_1^2,\chi_2^2\}})$ is nonempty. Since moreover $\frac{\frac{n}{2}-1}{2\cdot\frac{n}{2}} = \frac{n-2}{2n} < \min\{\mu_1,\mu_2\}$, it is possible to find $p \in (\frac{n}{2}, \frac{1}{\max(\chi_1^2,\chi_2^2)})$ such that $\frac{p-1}{2p} < \min\{\mu_1,\mu_2\}$, i.e. $\min\{\mu_1,\mu_2\}p > \frac{p-1}{2}$. By Lemma 3.5 for every such p and every $T \in (0,T_{\max}]$, $T < \infty$ there is $C(T) > 0$ with

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C(T) \quad \text{for } t \in (0,T).$$

If we suppose that T were finite, we could, herein, choose $T = T_{\max}$ and from Lemma 3.2 infer that

$$\sup_{t\in(0,T_{\max})}\left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\upsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}}\right) < \infty$$

in blatant contradiction to Lemma 2.1. □

4 The one-dimensional case

Now we deal with the global solutions of (1.4) when $n = 1$, $\alpha, \beta > 1$.

Lemma 4.1 *For $\alpha, \beta > 1$, $n = 1$ and $T \in (0,T_{\max}]$, $T < \infty$, there exists $C > 0$ such that*

$$w \ge \underline{w}(t) := \|w_0\|_{L^\infty(\Omega)} e^{-C(1+t)}, \quad (x,t) \in \Omega \times (0,T). \tag{4.1}$$

Proof We know from (2.2) that

$$z = e^{t\Delta}z_0 - \int_0^t e^{(t-s)\Delta}|\nabla z|^2\,ds + \int_0^t e^{(t-s)\Delta}(u+\upsilon)\,ds$$
$$\le e^{t\Delta}z_0 + \int_0^t e^{(t-s)\Delta}(u+\upsilon)\,ds \tag{4.2}$$

by the order preserving of the Neumann heat semigroup $\{e^{t\Delta}\}_{t\ge 0}$. For $\alpha, \beta > 1$ and $n = 1$, we have from (4.2)

$$\begin{aligned}
&\|z\|_{L^\infty(\Omega)} \\
&\le \|e^{t\Delta}z_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}(u-\overline{u})\|_{L^\infty(\Omega)}\,ds + \int_0^t \|e^{(t-s)\Delta}\overline{u}\|_{L^\infty(\Omega)}\,ds \\
&\quad + \int_0^t \|e^{(t-s)\Delta}(\upsilon-\overline{\upsilon})\|_{L^\infty(\Omega)}\,ds + \int_0^t \|e^{(t-s)\Delta}\overline{\upsilon}\|_{L^\infty(\Omega)}\,ds \\
&\le \|z_0\|_{L^\infty(\Omega)} + \frac{m_1}{|\Omega|}t + \frac{m_2}{|\Omega|}t + k_1\int_0^t \bigl(1+(t-s)^{-\frac{1}{2}}\bigr)e^{-\lambda(t-s)}\|u-\overline{u}\|_{L^1(\Omega)}\,ds \\
&\quad + k_1\int_0^t \bigl(1+(t-s)^{-\frac{1}{2}}\bigr)e^{-\lambda(t-s)}\|\upsilon-\overline{\upsilon}\|_{L^1(\Omega)}\,ds \\
&\le \|z_0\|_{L^\infty(\Omega)} + \frac{m_1}{|\Omega|}t + \frac{m_2}{|\Omega|}t + 2k_1m_1\int_0^t \bigl(1+(t-s)^{-\frac{1}{2}}\bigr)e^{-\lambda(t-s)}\,ds \\
&\quad + 2k_1m_2\int_0^t \bigl(1+(t-s)^{-\frac{1}{2}}\bigr)e^{-\lambda(t-s)}\,ds \\
&\le C(1+t),
\end{aligned} \tag{4.3}$$

where $\overline{u} := \frac{1}{|\Omega|}\int_\Omega u\,dx$, $\overline{\upsilon} := \frac{1}{|\Omega|}\int_\Omega \upsilon\,dx$, λ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under the homogeneous Neumann boundary conditions. The estimates (4.3) with $z := -\ln(\frac{w}{\|w_0\|_{L^\infty(\Omega)}})$ yield (4.1). □

There is just a more easy case of getting an L^p-estimate of u, υ with some $p > 1$ directly by the L^1-estimate.

Lemma 4.2 *For any $T \le T_{\max}$, $T < \infty$. Let $n = 1$, $\alpha, \beta > 1$ and $p > 1$. Then there exists $C > 0$ such that*

$$\|u\|_{L^p(\Omega)} \le C\left(1 + t + \int_0^t \left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(p+\alpha-1)}{\alpha-1}} ds\right)^{\frac{1}{p}}, \quad t \in (0,T), \tag{4.4}$$

$$\|\upsilon\|_{L^p(\Omega)} \le C\left(1 + t + \int_0^t \left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(p+\beta-1)}{\beta-1}} ds\right)^{\frac{1}{p}}, \quad t \in (0,T). \tag{4.5}$$

Proof Due to the variation-of-constants formula and the semigroup estimates of [26], Lemma 1.3, we have the existence of $k_1 > 0$ and $\lambda > 0$ such that

$$\|\nabla w(\cdot,t)\|_{L^p(\Omega)}$$

$$
\begin{aligned}
&\le \|\nabla w_0\|_{L^p(\Omega)} + k_1 \int_0^t \bigl(1+(t-s)^{-1+\frac{1}{2p}}\bigr)e^{-\lambda(t-s)}\bigl\|(u+\upsilon)w\bigr\|_{L^1(\Omega)}\,ds \\
&\le \|\nabla w_0\|_{L^p(\Omega)} + k_1\|w_0\|_{L^\infty(\Omega)}m_1 \int_0^\infty \bigl(1+(t-s)^{-1+\frac{1}{2p}}\bigr)e^{-\lambda(t-s)}\,ds \\
&\quad + k_1\|w_0\|_{L^\infty(\Omega)}m_2 \int_0^\infty \bigl(1+(t-s)^{-1+\frac{1}{2p}}\bigr)e^{-\lambda(t-s)}\,ds \\
&\le c_1
\end{aligned}
\tag{4.6}
$$

holding for all $t \in (0,T)$, $s \le t$.

By Young's inequality with Lemma 4.1 and (4.6), we have

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx \\
&= \int_\Omega u^{p-1}\biggl(\Delta u - \chi_1\nabla\cdot\biggl(u\frac{\nabla w}{w}\biggr) + \mu_1 u - \mu_1 u^\alpha\biggr)\,dx \\
&= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx + \chi_1(p-1)\int_\Omega u^{p-1}\frac{\nabla u\cdot\nabla w}{w}\,dx \\
&\quad + \mu_1\int_\Omega u^p\,dx - \mu_1\int_\Omega u^{p+\alpha-1}\,dx \\
&\le \frac{p-1}{4}\int_\Omega u^p\frac{|\nabla w|^2}{w^2}\,dx + \mu_1\int_\Omega u^p\,dx - \mu_1\int_\Omega u^{p+\alpha-1}\,dx \\
&\le \frac{p-1}{4\|w_0\|_{L^\infty(\Omega)}e^{-2C(1+t)}}\int_\Omega u^p|\nabla w|^2\,dx + \mu_1\int_\Omega u^p\,dx - \mu_1\int_\Omega u^{p+\alpha-1}\,dx \\
&\le c_2\biggl(\frac{1}{\|w_0\|_{L^\infty(\Omega)}e^{-C(1+t)}}\biggr)^{\frac{2(p+\alpha-1)}{\alpha-1}}\int_\Omega |\nabla w|^{\frac{2(p+\alpha-1)}{\alpha-1}}\,dx + c_3 \\
&\le c_4\biggl(\biggl(\frac{1}{\|w_0\|_{L^\infty(\Omega)}e^{-C(1+t)}}\biggr)^{\frac{2(p+\alpha-1)}{\alpha-1}}c_1^{\frac{2(p+\alpha-1)}{\alpha-1}} + 1\biggr), \quad t\in(0,T),
\end{aligned}
$$

then

$$
\begin{aligned}
\int_\Omega u^p\,dx &\le \int_\Omega u_0^p\,dx + c_4pt + c_4pc_1^{\frac{2(p+\alpha-1)}{\alpha-1}}\int_0^t\biggl(\frac{1}{\|w_0\|_{L^\infty(\Omega)}e^{-C(1+s)}}\biggr)^{\frac{2(p+\alpha-1)}{\alpha-1}}\,ds \\
&\le C\biggl(1+t+\int_0^t\biggl(\frac{1}{\|w_0\|_{L^\infty(\Omega)}e^{-C(1+s)}}\biggr)^{\frac{2(p+\alpha-1)}{\alpha-1}}\,ds\biggr), \quad t\in(0,T).
\end{aligned}
$$

In the same way, we can easily get (4.5). □

We can now use this to show global existence.

Proof of Theorem 1.2 When $n = 1$ with $\alpha, \beta > 1$. We have from Lemma 4.1, Lemma 4.2 with $p = 2$

$$
\begin{aligned}
&\|\nabla w\|_{L^\infty(\Omega)} \\
&\quad \le \|\nabla w_0\|_{L^\infty(\Omega)} + k_1\int_0^t\bigl(1+(t-s)^{-\frac{3}{4}}\bigr)e^{-\lambda(t-s)}\bigl\|\bigl((u+\upsilon)w\bigr)\bigr\|_{L^2(\Omega)}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\nabla w_0\|_{L^\infty(\Omega)} + k_1 C\|w_0\|_{L^\infty(\Omega)}\left(1+t+\int_0^t \left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(\alpha+1)}{\alpha-1}} ds\right)^{\frac{1}{2}} \\
&\quad \times \int_0^\infty \left(1+(t-s)^{-\frac{3}{4}}\right)e^{-\lambda(t-s)}\,ds \\
&\quad + k_1 C\|w_0\|_{L^\infty(\Omega)}\left(1+t+\int_0^t \left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(\beta+1)}{\beta-1}} ds\right)^{\frac{1}{2}} \\
&\quad \times \int_0^\infty \left(1+(t-s)^{-\frac{3}{4}}\right)e^{-\lambda(t-s)}\,ds \\
&\leq \|\nabla w_0\|_{L^\infty(\Omega)} + 2k_1 C\|w_0\|_{L^\infty(\Omega)}\left(1+t+\int_0^t \left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(\gamma+1)}{\gamma-1}} ds\right)^{\frac{1}{2}} \\
&\quad \times \int_0^\infty \left(1+(t-s)^{-\frac{3}{4}}\right)e^{-\lambda(t-s)}\,ds \\
&\leq c_5\left(1+t+\int_0^t \left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(\gamma+1)}{\gamma-1}} ds\right)^{\frac{1}{2}}, \quad t\in(0,T),
\end{aligned}
\tag{4.7}
$$

where $\gamma := \min\{\alpha,\beta\}$, λ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under the homogeneous Neumann boundary conditions. By the variation-of-constants formula for u and the order preserving of the Neumann heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ with the positivity of u, we know

$$
\begin{aligned}
&u(\cdot,t) \\
&\quad = e^{t\Delta}u_0 - \chi_1\int_0^t e^{(t-s)\Delta}\nabla\cdot\left(u\frac{\nabla w}{w}\right)ds + \mu_1\int_0^t e^{(t-s)\Delta}\left(u-u^\alpha\right)ds \\
&\quad \leq e^{t\Delta}u_0 - \chi_1\int_0^t e^{(t-s)\Delta}\nabla\cdot\left(u\frac{\nabla w}{w}\right)ds + \mu_1\int_0^t e^{(t-s)\Delta}u\,ds, \quad t\in(0,T).
\end{aligned}
\tag{4.8}
$$

Furthermore, according to [26], Lemma 3.1, with (4.8), (2.1), Lemma 2.4, Lemma 4.1, (4.4) and (4.7), we get for $p=2$

$$
\begin{aligned}
&\|u\|_{L^\infty(\Omega)} \\
&\quad \leq \|u_0\|_{L^\infty(\Omega)} + k_2\chi_1\int_0^t \left(1+(t-s)^{-\frac{3}{4}}\right)e^{-\lambda(t-s)}\left\|u\frac{\nabla w}{w}\right\|_{L^2(\Omega)} ds \\
&\qquad + \mu_1 k_2\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-\lambda(t-s)}\|u-\bar{u}\|_{L^1(\Omega)}\,ds + \frac{\mu_1 m_1}{|\Omega|}t \\
&\quad \leq \|u_0\|_{L^\infty(\Omega)} \\
&\qquad + k_2\chi_1\frac{1}{\|w_0\|_{L^\infty(\Omega)}e^{-C(1+t)}}\int_0^t \left(1+(t-s)^{-\frac{3}{4}}\right)e^{-\lambda(t-s)}\|u\|_{L^2(\Omega)}\|\nabla w\|_{L^\infty(\Omega)}\,ds \\
&\qquad + 2\mu_1 k_2 m_1\int_0^\infty \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-\lambda(t-s)}\,ds + \frac{\mu_1 m_1}{|\Omega|}t \\
&\quad \leq c_6\left(1+\frac{1}{\|w_0\|_{L^\infty(\Omega)}e^{-C(1+t)}}\right)\left(1+t+\int_0^t \left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(\gamma+1)}{\gamma-1}} ds\right), \\
&\qquad t\in(0,T),
\end{aligned}
\tag{4.9}
$$

where $\gamma := \min\{\alpha, \beta\}$, $\overline{u} := \frac{1}{|\Omega|}\int_\Omega u\,dx$, λ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under the homogeneous Neumann boundary conditions.

In the same way, we see that

$$\begin{aligned} &\|v\|_{L^\infty(\Omega)} \\ &\quad\le c_7\left(1+\frac{1}{\|w_0\|_{L^\infty(\Omega)}e^{-C(1+t)}}\right)\left(1+t+\int_0^t\left(\frac{e^{C(1+s)}}{\|w_0\|_{L^\infty(\Omega)}}\right)^{\frac{2(\gamma+1)}{\gamma-1}}ds\right), \\ &\quad t\in(0,T). \end{aligned} \tag{4.10}$$

Combining Lemma 2.4, (4.7), (4.9), (4.10) with Lemma 2.1, we complete the proof of Theorem 1.2. □

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Authors' contributions
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