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Exclusion sets in Dashnic–Zusmanovich localization sets

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Abstract

A new eigenvalue localization set is given by excluding some proper subsets that do not contain any eigenvalues of matrices from Dashnic–Zusmanovich localization sets. As an application, a sufficient condition for non-singularity of matrices is obtained. In order to locate all eigenvalues of matrices precisely, another set including two positive integers s and k is presented. By adjusting the parameters s and k , one can locate all eigenvalues and judge the non-singularity of matrices accurately.

MSC: 15A18; 15A42

Keywords: Dashnic–Zusmanovich matrices; Eigenvalues; Localization sets; Exclusion sets; Non-singularity

1 Introduction

Let n be a positive integer, $n \geq 2$ and $N = \{1, 2, \dots, n\}$. $\mathbb{C}$ (or, respectively, $\mathbb{R}$) denotes the set of all complex (or, respectively, real) numbers, $\mathbb{C}^{n \times n}$ (or, respectively, $\mathbb{R}^{n \times n}$) denotes the set of all $n \times n$ complex (or, respectively, real) matrices and I stands for the identity matrix. Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ and $\sigma(A)$ be the set of all eigenvalues of A . Eigenvalue problems of matrices has a wide range of practical applications, such as image restoration [1], linear and multilinear algebra [2], higher order Markov chains [3], etc. In order to locate all eigenvalues of matrices, the authors in [4–13] found some regions including all eigenvalues of matrices in the complex plane. The first work is due to Geršgorin, who presented such a region called the Geršgorin disk theorem [5], which consists of n disks centered at the diagonal elements of the matrix.

Theorem 1 ([5, Geršgorin set]) *Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$. Then*

$$\sigma(A) \subseteq \Gamma(A) = \bigcup_{i \in N} \Gamma_i(A),$$

where

$$\Gamma_i(A) = \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i(A)\}$$

and

$$r_i(A) = \sum_{j \in N, j \neq i} |a_{ij}|.$$

Although Geršgorin set is concise, its result is not accurate enough. Hence, tighter sets than $\Gamma(A)$ are conjectured till now. The Dashnic–Zusmanovich localization set, which is tighter than Geršgorin set, provided by Dashnic–Zusmanovich (DZ) [14], is described as follows.

Theorem 2 ([14, DZ set]) *Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$. Then*

$$\sigma(A) \subseteq D(A) = \bigcap_{i \in N} \bigcup_{j \in N, j \neq i} D_{ij}(A),$$

where

$$D_{ij}(A) = \{z \in \mathbb{C} : |z - a_{ii}|(|z - a_{jj}| - r_j^i(A)) \leq r_i(A)|a_{ji}|\}$$

and

$$r_j^i(A) = r_j(A) - |a_{ji}|.$$

It is generally accepted that an eigenvalue localization set is connected with one kind of non-singular matrices [7, 8]. The non-singularity criterions for matrices derived from the Geršgorin set in Theorem 1 and DZ set in Theorem 2 are as follows.

Theorem 3 ([7]) *If $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is an SDD matrix, i.e., for each $i \in N$, we have*

$$|a_{ii}| > r_i(A),$$

then it is non-singular.

Theorem 4 ([7]) *If $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is a DZ matrix, i.e., there exists an index $i \in N$, for all $j \in N, j \neq i$,*

$$|a_{ii}|(|a_{jj}| - r_j^i(A)) > r_i(A)|a_{ji}|,$$

then it is non-singular.

By excluding some proper subsets that contain no eigenvalues of matrices from some existing eigenvalue localization sets, the authors in [15–18] obtained some tighter sets and some sufficient conditions for non-singularity of matrices. Inspired by these effective results, we in this paper first present a new eigenvalue localization set by excluding some proper subsets from DZ set, and we obtain a new sufficient condition for non-singularity of matrices. In order to precisely locate all eigenvalues of matrices, we in Sect. 3 present another set which includes two positive integers s and k , and show by an example that, by adjusting the parameters s and k , one can locate all eigenvalues and judge the non-singularity of matrices accurately.

2 Exclusion sets in Dashnic–Zusmanovich localization sets

In this section, we present a new eigenvalue inclusion set by excluding some proper subsets that contain no eigenvalues of matrices from DZ sets, and, as an application, we provide a sufficient condition to judge the non-singularity of matrices.

Theorem 5 *Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$. Then*

$$\sigma(A) \subseteq \Omega(A) = \bigcap_{i \in N} \bigcup_{j \in N, j \neq i} (D_{ij}(A) \setminus \Omega_{ij}(A)),$$

where

$$\Omega_{ij}(A) = \{z \in \mathbb{C} : |z - a_{ii}|(|z - a_{jj}| + r_j^i(A)) < (2|a_{ij}| - r_i(A))|a_{ji}|\}.$$

Proof Let λ be an eigenvalue of A and $x = (x_1, \dots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ be its eigenvector. Then

$$Ax = \lambda x. \quad (1)$$

Let $|x_p| = \max_{j \in N} |x_j|$. Obviously, $|x_p| > 0$. For any $i \in N$, $i \neq p$, the p th formula of (1) can be written as

$$(\lambda - a_{pp})x_p = \sum_{s \neq p, i} a_{ps}x_s + a_{pi}x_i. \quad (2)$$

Taking the absolute value of (2) and using the triangle inequality, we have

$$|\lambda - a_{pp}||x_p| \leq \sum_{s \neq p, i} |a_{ps}||x_s| + |a_{pi}||x_i| \leq r_p^i(A)|x_p| + |a_{pi}||x_i|,$$

i.e.,

$$(|\lambda - a_{pp}| - r_p^i(A))|x_p| \leq |a_{pi}||x_i|. \quad (3)$$

If $|x_i| > 0$, then from the i th formula of (1), i.e.,

$$(\lambda - a_{ii})x_i = \sum_{t \neq i} a_{it}x_t, \quad (4)$$

we have

$$|\lambda - a_{ii}||x_i| \leq r_i(A)|x_p|. \quad (5)$$

Multiplying (3) and (5) and noting that $|x_p||x_i| > 0$, we have

$$|\lambda - a_{ii}|(|\lambda - a_{pp}| - r_p^i(A)) \leq r_i(A)|a_{pi}|. \quad (6)$$

If $|x_i| = 0$ in (3), then $|\lambda - a_{pp}| - r_p^i(A) \leq 0$ as $|x_p| > 0$, which implies that (6) also holds. Therefore, $\lambda \in D_{ip}(A)$. On the other hand, from (2) and (4), we have

$$a_{pi}x_i = (\lambda - a_{pp})x_p - \sum_{s \neq p, i} a_{ps}x_s$$

and

$$a_{ip}x_p = (\lambda - a_{ii})x_i - \sum_{t \neq i,p} a_{it}x_t,$$

which leads to

$$\begin{aligned} |a_{pi}||x_i| &\leq |\lambda - a_{pp}||x_p| + \sum_{s \neq p,i} |a_{ps}||x_s| \leq |\lambda - a_{pp}||x_p| + \sum_{s \neq p,i} |a_{ps}||x_p| \\ &= (|\lambda - a_{pp}| + r_p^i(A))|x_p| \end{aligned} \quad (7)$$

and

$$|a_{ip}||x_p| \leq |\lambda - a_{ii}||x_i| + \sum_{t \neq i,p} |a_{it}||x_t| \leq |\lambda - a_{ii}||x_i| + r_i^p(A)|x_p|,$$

i.e.,

$$(|a_{ip}| - r_i^p(A))|x_p| \leq |\lambda - a_{ii}||x_i|. \quad (8)$$

By (7) and (8), we have

$$|a_{pi}|(2|a_{ip}| - r_i(A)) \leq |\lambda - a_{ii}|(|\lambda - a_{pp}| + r_p^i(A)),$$

which implies that $\lambda \notin \Omega_{ip}(A)$. Hence, $\lambda \in (D_{ip}(A) \setminus \Omega_{ip}(A))$.

For some certain $i \in N$, $i \neq p$, since we do not know which p is appropriate to λ , we can only conclude that

$$\lambda \in \bigcup_{p \in N, i \neq p} (D_{ip}(A) \setminus \Omega_{ip}(A)).$$

Furthermore, by the arbitrariness of i , we have

$$\lambda \in \bigcap_{i \in N} \bigcup_{j \in N, j \neq i} (D_{ij}(A) \setminus \Omega_{ij}(A)).$$

The conclusion follows. $\square$

Now, a comparison theorem for Theorems 1, 2 and 5 is obtained.

Theorem 6 Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$. Then

$$\Omega(A) \subseteq D(A) \subseteq \Gamma(A).$$

Proof It is showed in Theorem 7 of [19] that $D(A) \subseteq \Gamma(A)$. For any $i, j \in N$, $j \neq i$, by $(D_{ij}(A) \setminus \Omega_{ij}(A)) \subseteq D_{ij}(A)$, obviously,

$$\bigcap_{i \in N} \bigcup_{j \in N, i \neq j} (D_{ij}(A) \setminus \Omega_{ij}(A)) \subseteq \bigcap_{i \in N} \bigcup_{j \in N, i \neq j} D_{ij}(A).$$

Hence, $\Omega(A) \subseteq D(A)$ holds. $\square$

Next, based on the fact that $\det(A) = 0$ if and only if $0 \in \sigma(A)$ for a matrix A , we can obtain the following condition for judging the non-singularity of matrices.

Corollary 1 *Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$. If there exists $i \in N$, for any $j \in N, j \neq i$, either*

$$|a_{ii}|(|a_{jj}| - r_j^i(A)) > r_i(A)|a_{ji}|, \quad (9)$$

or

$$|a_{ii}|(|a_{jj}| + r_j^i(A)) < (2|a_{ij}| - r_i(A))|a_{ji}|, \quad (10)$$

then A is non-singular.

Proof Let $0 \in \sigma(A)$. By Theorem 5, we have $0 \in \Omega(A)$, i.e., for each $i \in N$, there exists $j \in N, j \neq i$, such that

$$|a_{ii}|(|a_{jj}| - r_j^i(A)) \leq r_i(A)|a_{ji}|$$

and

$$|a_{ii}|(|a_{jj}| + r_j^i(A)) \geq (2|a_{ij}| - r_i(A))|a_{ji}|.$$

This contradicts (9) and (10). Consequently, $0 \notin \sigma(A)$, that is, A is non-singular. $\square$

Remark 1 (i) Let i and j be any two elements of N , and $i \neq j$. If $(2|a_{ij}| - r_i(A))|a_{ji}| > 0$, then, by $|a_{ij}| \leq r_i(A)$, we have $0 < 2|a_{ij}| - r_i(A) \leq r_i(A)$ and

$$\begin{aligned} |z - a_{ii}|(|z - a_{jj}| - r_j^i(A)) &\leq |z - a_{ii}|(|z - a_{jj}| + r_j^i(A)) \\ &< (2|a_{ij}| - r_i(A))|a_{ji}| \\ &\leq r_i(A)|a_{ji}|, \end{aligned}$$

which implies that

$$\Omega_{ij}(A) \subseteq D_{ij}(A) \quad \text{and} \quad (D_{ij}(A) \setminus \Omega_{ij}(A)) \subseteq D_{ij}(A). \quad (11)$$

That is to say, $\Omega_{ij}(A)$ is well defined. If $(2|a_{ij}| - r_i(A))|a_{ji}| \leq 0$, then $\Omega_{ij}(A) = \emptyset$. Obviously, (11) also holds. Here, $\Omega_{ij}(A)$ is called the exclusion set of $D_{ij}(A)$.

(ii) As has been shown in [7, 8], the wider the class of non-singular matrices is, the tighter eigenvalue localization set it will lead to. Obviously, this conclusion, in turn, holds true. By Corollary 1, one can conclude that the conditions of Corollary 1 for judging the non-singularity of matrices are weaker than those in Theorems 3 and 4.

Next, an example is given to show that $\Omega(A)$ can catch all eigenvalues of a matrix A more precisely than $\Gamma(A)$ and $D(A)$, and that Theorems 3 and 4 cannot be used to judge the non-singularity of A in some cases, but Corollary 1 works better.

Example 1 Let

$$A = \begin{bmatrix} 11 & 4 + \mathbf{i} & 0 & 15 - \mathbf{i} \\ -2 - \mathbf{i} & 10 & 5 - \mathbf{i} & 0 \\ 0 & 6 & 12 + \mathbf{i} & 4 \\ 16 & 2 & 0 & 11 \end{bmatrix}.$$

By computations, all eigenvalues of A are $26.4293 - 0.7552\mathbf{i}$, $-4.4930 + 0.6450\mathbf{i}$, $16.0853 + 0.0217\mathbf{i}$, $5.9784 + 1.0885\mathbf{i}$. Next, the eigenvalue location and the determination of non-singularity for A are considered.

(I) Eigenvalue inclusion sets for A .

From Theorem 1, we have

$$\Gamma(A) = \{z \in \mathbb{C} : |z - 11| \leq 19.1564\}.$$

From Theorem 2, we have

$$D(A) = D_{24}(A) = \{z \in \mathbb{C} : |z - 10|(|z - 11| - 16) \leq 15.0662\}.$$

From Theorem 5, we have

$$\Omega(A) = [D_{12}(A) \cup D_{13}(A) \cup (D_{24}(A) \setminus \Omega_{41}(A)) \cup (D_{43}(A) \cap \Omega_{14}(A))],$$

where

$$D_{12}(A) = \{z \in \mathbb{C} : |z - 11|(|z - 10| - 5.0990) \leq 42.8356\},$$

$$D_{13}(A) = \{z \in \mathbb{C} : |z - 11|(|z - 12 - \mathbf{i}| - 10) \leq 0\},$$

$$D_{43}(A) = \{z \in \mathbb{C} : |z - 11|(|z - 12 - \mathbf{i}| - 18) \leq 72\},$$

$$\Omega_{14}(A) = \{z \in \mathbb{C} : |z - 11|(|z - 11| + 2) \leq 175.568\},$$

$$\Omega_{41}(A) = \{z \in \mathbb{C} : |z - 11|(|z - 11| + 4.1231) \leq 210.4662\}.$$

The eigenvalue localization sets $\Gamma(A)$, $D(A)$ and $\Omega(A)$ are drawn in Fig. 1, respectively, as red boundary, black boundary and yellow zones, and all eigenvalues are plotted as red asterisks. It is obvious that

$$\sigma(A) \subseteq \Omega(A) \subseteq D(A) \subset \Gamma(A),$$

another way of stating it is, $\Omega(A)$ can capture all eigenvalues of A more precisely than $D(A)$ and $\Gamma(A)$.

(II) The determination for non-singularity of A .

From Fig. 1, one can see that $0 \in \Gamma(A)$ and $0 \in D(A)$, but $0 \notin \Omega(A)$, that is, the sets $\Gamma(A)$ and $D(A)$ cannot be used to judge the non-singularity of A . However, by $0 \notin \Omega(A)$, one can conclude that A is non-singular.

Figure 1 Comparisons of $\Omega(A)$, $D(A)$ and $\Gamma(A)$

Furthermore, as

$$|a_{11}| = 11 < 19.1564 = r_1(A)$$

and

$$|a_{11}|(|a_{44}| - r_4^1(A)) = 99.0000 < 306.5024 = r_1(A)|a_{41}|,$$

$$|a_{22}|(|a_{44}| - r_4^2(A)) = -50.0000 < 14.6702 = r_2(A)|a_{42}|,$$

$$|a_{33}|(|a_{44}| - r_4^3(A)) = -84.2912 < 0 = r_3(A)|a_{43}|,$$

$$|a_{44}|(|a_{33}| - r_3^4(A)) = 66.4575 < 72.0000 = r_4(A)|a_{34}|,$$

we know that the conditions in Theorems 3 and 4 do not hold, that is, Theorems 3 and 4 do not work. However, by

$$|a_{11}|(|a_{22}| - r_2^1(A)) = 53.9108 > 42.8350 = r_1(A)|a_{21}|,$$

$$|a_{11}|(|a_{33}| - r_3^1(A)) = 22.4575 > 0 = r_1(A)|a_{31}|,$$

$$|a_{11}|(|a_{44}| + r_4^1(A)) = 143.0000 < 174.5631 = (2|a_{14}| - r_1(A))|a_{41}|,$$

we know that there exists an index $i = 1$, for $j = 2, 3, 4$, either (9) or (10) holds. Then we can conclude the non-singularity of A by Corollary 1.

3 An eigenvalue localization set with parameters

In this section, an eigenvalue localization set with parameters and its applications is considered.

Theorem 7 Let $A = sI - B \in \mathbb{C}^{n \times n}$ and $s \in \mathbb{C}$. Given an arbitrary positive integer k , then

$$\sigma(A) \subseteq \Omega_k^s(A) = \bigcap_{i \in N} \bigcup_{j \in N, j \neq i} (D_{ij}^s(B^k) \setminus \Omega_{ij}^s(B^k)),$$

where

$$D_{ij}^s(B^k) = \{z \in \mathbb{C} : |(s-z)^k - (B^k)_{ii}| (|(s-z)^k - (B^k)_{jj}| - r_j^i(B^k)) \leq r_i(B^k) |(B^k)_{ji}|\}$$

and

$$\begin{aligned} \Omega_{ij}^s(B^k) = \{z \in \mathbb{C} : & |(s-z)^k - (B^k)_{ii}| (|(s-z)^k - (B^k)_{jj}| + r_j^i(B^k)) \\ & < (2|(B^k)_{ij}| - r_i(B^k)) |(B^k)_{ji}|\}. \end{aligned}$$

Proof Let $\lambda \in \sigma(A)$. Given an arbitrary positive integer k , suppose that $\lambda \notin \Omega_k^s(A)$. Then there exists $i \in N$, for all $j \in N, j \neq i$, $\lambda \notin (D_{ij}^s(B^k) \setminus \Omega_{ij}^s(B^k))$, that is, $\lambda \notin D_{ij}^s(B^k)$ or $\lambda \in \Omega_{ij}^s(B^k)$, i.e.,

$$|(s-\lambda)^k - (B^k)_{ii}| (|(s-\lambda)^k - (B^k)_{jj}| - r_j^i(B^k)) > r_i(B^k) |(B^k)_{ji}|,$$

or

$$|(s-\lambda)^k - (B^k)_{ii}| (|(s-\lambda)^k - (B^k)_{jj}| + r_j^i(B^k)) < (2|(B^k)_{ij}| - r_i(B^k)) |(B^k)_{ji}|.$$

By Corollary 1, $(s-\lambda)^k I - B^k$ is non-singular, which implies that $(s-\lambda)^k$ is not an eigenvalue of B^k .

On the other hand, let x be an eigenvector corresponding to λ , i.e., $Ax = \lambda x$, which leads to that $Bx = (s-\lambda)x$. Furthermore, we have $B^k x = (s-\lambda)^k x$, which implies that $(s-\lambda)^k$ is an eigenvalue of B^k . This is a contradiction. Hence, $\lambda \in \Omega_k^s(A)$. $\square$

By Theorem 7, the following general method for non-singularity of matrices is obtained.

Corollary 2 Let $A = sI - B \in \mathbb{C}^{n \times n}$ and $s \in \mathbb{C}$. Given an arbitrary positive integer k , if there exists an index i , for all $j \in N, j \neq i$, either

$$(|s^k - (B^k)_{ii}| - r_i^j(B^k)) |s^k - (B^k)_{jj}| > |(B^k)_{ij}| r_j(B^k), \quad (12)$$

or

$$(|s^k - (B^k)_{ii}| + r_i^j(B^k)) |s^k - (B^k)_{jj}| < |(B^k)_{ij}| (2|(B^k)_{ji}| - r_j(B^k)), \quad (13)$$

then A is non-singular.

Remark 2 Taking $k = 1$ and arbitrary complex number s in Theorem 7 and Corollary 2, then they degenerate, respectively, into Theorem 5 and Corollary 1. Hence, Theorem 7 and Corollary 2 can be viewed as generalizations of Theorem 5 and Corollary 1. Furthermore, by selecting appropriate parameters s and k , one may locate all eigenvalues and judge the non-singularity of matrices precisely.

Example 2 Consider again the matrix A in Example 1. Taking $s = 11$ and $k = 2$, the sets $\Omega(A)$ and $\Omega_2^{11}(A)$ are drawn in Fig. 2, respectively, as yellow zones and blue zones. All eigenvalues are plotted as red asterisks. It can be seen from Fig. 2 that the set $\Omega_2^{11}(A)$ can be used to more precisely locate all eigenvalues of A and judge the non-singularity of A .

4 Conclusions

In this paper, we present a new eigenvalue inclusion set $\Omega(A)$ by excluding some proper subsets that contain no eigenvalues of matrices from Dashnic–Zusmanovich localization sets $D(A)$, and we prove that $\Omega(A)$ is tighter than $D(A)$ for a matrix A . After that, by the set $\Omega(A)$, we obtain a sufficient condition for judging the non-singularity of matrices. To catch all eigenvalues of matrices precisely, we put forward another eigenvalue inclusion set $\Omega_k^s(A)$ including two parameters s and k . By selecting these two positive integer s and k appropriately, one can locate all eigenvalues of matrices and judge the non-singularity of matrices precisely. However, how to choose s and k to make $\Omega_k^s(A)$ works better? This question at present is far from being solved.

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Availability of data and materials

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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