Shi *Journal of Inequalities and Applications* (2019) 2019:225
https://doi.org/10.1186/s13660-019-2178-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the spectral norms of some circulant matrices with the trigonometric functions

Baijuan Shi[1*]

*Correspondence:
593800425@qq.com
[1]School of Mathematics, Northwest University, Xi'an, P.R. China

Abstract

In this paper, we use the properties of an r-circulant matrix and a geometric circulant matrix to study the spectral norms of the r-circulant matrix and the geometric circulant matrix involving trigonometric functions by some algebra methods.

Keywords: Geometric circulant matrix; r-circulant matrix; Trigonometric functions; Spectral norms; Euclidean norms

1 Introduction

In 1885, circulant matrix was first proposed by Muir, and he did some basic research. Until 1950–1955, Good et al. began to study the inverse, determinants and characteristic values of circulant matrices; these efforts have opened the door to study circulant matrices. A circulant matrix is a kind of matrix with a special structure, which has been widely used in algebra, geometry, signal processing and coding theory. In recent years, the circulant matrix is still a topic of focus in the research of matrix theory. Especially, some scholars studied the norms of r-circulant matrices and geometric circulant matrices with some famous numbers and polynomials, for example, on the spectral norms of circulant matrices, r-circulant matrices, geometric circulant matrices with Fibonacci number, Lucas number, generalized Fibonacci and Lucas numbers, generalized k-Horadam numbers, the biperiodic Fibonacci and Lucas numbers have been studied [1–13]. To the best of our knowledge, no one has studied the upper and lower estimate problems for the spectral norms involving trigonometric functions $\cos(\frac{k\pi}{n})$, $\sin(\frac{k\pi}{n})$ yet by using exponential sum.

A $n \times n$ r-circulant matrix C_r is defined by [8]

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

Kızılateş and Tuglu [9] defined geometric circulant matrices by the form

$$C_{r^*}=\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1}\\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2}\\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

Obviously, when the parameter satisfies $r=1$, we can get the classical circulant matrix. Inspired by [7], in this paper, we shall use identities of the trigonometric functions and power sums of $\cos(\frac{k\pi}{n})$, $\sin(\frac{k\pi}{n})$ to study the norms of the r-circulant matrices

$$A=\operatorname{Circ}_r\left(\cos\frac{0\cdot\pi}{n},\cos\frac{1\cdot\pi}{n},\cos\frac{2\cdot\pi}{n},\ldots,\cos\frac{(n-1)\cdot\pi}{n}\right),$$
$$B=\operatorname{Circ}_r\left(\sin\frac{0\cdot\pi}{n},\sin\frac{1\cdot\pi}{n},\sin\frac{2\cdot\pi}{n},\ldots,\sin\frac{(n-1)\cdot\pi}{n}\right),$$

and then we obtain the norms of geometric circulant matrices

$$P_{r^*}=\operatorname{Circ}_{r^*}\left(\cos\frac{0\cdot\pi}{n},\cos\frac{1\cdot\pi}{n},\cos\frac{2\cdot\pi}{n},\ldots,\cos\frac{(n-1)\cdot\pi}{n}\right),$$
$$R_{r^*}=\operatorname{Circ}_{r^*}\left(\sin\frac{0\cdot\pi}{n},\sin\frac{1\cdot\pi}{n},\sin\frac{2\cdot\pi}{n},\ldots,\sin\frac{(n-1)\cdot\pi}{n}\right).$$

Then we get some interesting and concise results which are stated by the following theorems.

Theorem 1 *Let $A=C_r(\cos\frac{0\cdot\pi}{n},\cos\frac{1\cdot\pi}{n},\cos\frac{2\cdot\pi}{n},\ldots,\cos\frac{(n-1)\cdot\pi}{n})$ be an r-circulant matrix, then we have*

$$|r|\ge 1,\quad \frac{\sqrt{2}}{2}\le\|A\|_2\le\sqrt{\frac{n}{2}}\sqrt{(n-1)|r|^2+1};$$
$$|r|<1,\quad \frac{\sqrt{2}}{2}|r|\le\|A\|_2\le\frac{\sqrt{2}}{2}n.$$

Theorem 2 *Let $B=C_r(\sin\frac{0\cdot\pi}{n},\sin\frac{1\cdot\pi}{n},\sin\frac{2\cdot\pi}{n},\ldots,\sin\frac{(n-1)\cdot\pi}{n})$ be an r-circulant matrix, then we have*

$$|r|\ge 1,\quad \frac{\sqrt{2}}{2}\le\|B\|_2\le|r|\sqrt{\frac{n(n-1)}{2}};$$
$$|r|<1,\quad \frac{\sqrt{2}}{2}|r|\le\|B\|_2\le\sqrt{\frac{n(n-1)}{2}}.$$

Theorem 3 *Let $P_{r^*}=C_{r^*}(\cos\frac{0\cdot\pi}{n},\cos\frac{1\cdot\pi}{n},\cos\frac{2\cdot\pi}{n},\ldots,\cos\frac{(n-1)\cdot\pi}{n})$ be a geometric circulant matrix, we have*

$$|r|>1,\quad \frac{\sqrt{2}}{2}\le\|P_{r^*}\|_2\le\sqrt{\frac{n}{2}}\sqrt{\frac{1-|r|^{2n}}{1-|r|^2}};$$

$$|r|<1,\quad |r|^n\sqrt{N_1}\le \|P_{r^*}\|_2\le \frac{\sqrt{2}}{2}n,$$

where $N_1=\frac{1-r^{-2}-r^{-2n+2}}{4}+\frac{1-r^{-2n}}{2(1-r^{-2})}$.

Theorem 4 *Let* $R_{r^*}=C_{r^*}(\sin\frac{0\cdot\pi}{n},\sin\frac{1\cdot\pi}{n},\sin\frac{2\cdot\pi}{n},\dots,\sin\frac{(n-1)\cdot\pi}{n})$ *be a geometric circulant matrix, we have*

$$|r|>1,\quad \frac{\sqrt{2}}{2}\le \|R_{r^*}\|_2\le \sqrt{\frac{n}{2}}\sqrt{\frac{|r|^2-|r|^{2n}}{1-|r|^2}};$$

$$|r|<1,\quad |r|^n\sqrt{N_2}\le \|R_{r^*}\|_2\le \sqrt{\frac{n(n-1)}{2}},$$

where $N_2=\frac{1-r^{-2n}}{2(1-r^{-2})}-\frac{1-r^{-2}-r^{-2n+2}}{4}$.

2 Preliminaries

Definition 1 ([9]) Let any matrix $A=(a_{ij})\in M_{m\times n}(C)$, the spectral norm and the Euclidean norm of matrix A are defined by

$$\|A\|_2=\sqrt{\max_{1\le i\le n}\lambda_i\bigl(A^HA\bigr)},\qquad \|A\|_E=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}},\quad \text{respectively,}$$

where the $\lambda_i(A^HA)$ are the eigenvalues of matrices A^HA and A^H is the conjugate transpose of A.

The following important inequalities hold between the Euclidean norm and spectral norm:

$$\frac{1}{\sqrt{n}}\|A\|_E\le \|A\|_2\le \|A\|_E. \tag{1}$$

Definition 2 ([9]) Let both $A=(a_{ij})$ and $B=(b_{ij})$ be $m\times n$ matrices, then the Hadamard product of A and B is the $m\times n$ matrix of elementwise products, namely $A\circ B=(a_{ij}b_{ij})$.

Then we have the following inequalities:

$$\|A\circ B\|_2\le r_1(A)C_1(B), \tag{2}$$

$$r_1(A)=\max_{1\le i\le m}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2},\qquad C_1(B)=\max_{1\le j\le n}\sqrt{\sum_{i=1}^{m}|b_{ij}|^2}.$$

Lemma 1 ([7]) *For any positive integer* $n\ge 2$, *we have*

$$\sum_{k=0}^{n-1}\cos^2\left(\frac{k\pi}{n}\right)=\sum_{k=0}^{n-1}\sin^2\left(\frac{k\pi}{n}\right)=\frac{n}{2}.$$

Lemma 2 *For any positive integer* $n\ge 2$, *we can get*

$$\sum_{k=0}^{n-1}r^{-2k}\cos^2\left(\frac{k\pi}{n}\right)=\frac{1-r^{-2n}}{2(1-r^{-2})}+\frac{1-r^{-2}-r^{-2n+2}}{4}=N_1,$$

$$\sum_{k=0}^{n-1} r^{-2k}\sin^2\left(\frac{k\pi}{n}\right) = \frac{1-r^{-2n}}{2(1-r^{-2})} - \frac{1-r^{-2}-r^{-2n+2}}{4} = N_2.$$

Proof By the properties of $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, $e^{i\theta} = \cos\theta + i\sin\theta$, we can easily get $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$; let $e(x) = e^{2\pi ix}$, note that $e(1) = e(-1) = 1$, using the properties of the trigonometric sums $\sum_{k=0}^{n-1} e(\frac{k}{n}) = 0$. Hence,

$$\begin{aligned}\sum_{k=0}^{n-1} r^{-2k}\cos^2\left(\frac{k\pi}{n}\right) &= \sum_{k=0}^{n-1} r^{-2k}\frac{1+\cos(\frac{2k\pi}{n})}{2}\\ &= \frac{1-r^{-2n}}{2(1-r^{-2})} + \frac{1}{4}\sum_{k=0}^{n-1} r^{-2k}\left(e\left(\frac{k}{n}\right) + e\left(\frac{-k}{n}\right)\right).\end{aligned}$$

Taking

$$\begin{aligned}S_1 &= \sum_{k=0}^{n-1} r^{-2k} e\left(\frac{k}{n}\right)\\ &= r^{-2\cdot 0}1 + r^{-2\cdot 1}e\left(\frac{1}{n}\right) + r^{-2\cdot 2}e\left(\frac{2}{n}\right) + \cdots + r^{-2\cdot(n-2)}e\left(\frac{n-2}{n}\right) + r^{-2\cdot(n-1)}e\left(\frac{n-1}{n}\right),\\ e\left(\frac{1}{n}\right)S_1 &= r^{-2\cdot 0}e\left(\frac{1}{n}\right) + r^{-2\cdot 1}e\left(\frac{2}{n}\right) + r^{-2\cdot 2}e\left(\frac{3}{n}\right) + \cdots\\ &\quad + r^{-2\cdot(n-2)}e\left(\frac{n-1}{n}\right) + r^{-2\cdot(n-1)}e(1).\end{aligned}$$

Therefore,

$$\left(1 - e\left(\frac{1}{n}\right)\right)S_1 = 1 + r^{-2}\sum_{k=1}^{n-1} e\left(\frac{k}{n}\right) - r^{-2n+2} = 1 - r^{-2} - r^{-2n+2},$$

that is $S_1 = \sum_{k=0}^{n-1} r^{-2k}e(\frac{k}{n}) = \frac{1-r^{-2}-r^{-2n+2}}{1-e(\frac{1}{n})}$, as the same time, $\sum_{k=0}^{n-1} r^{-2k}e(\frac{-k}{n}) = \frac{1-r^{-2}-r^{-2n+2}}{1-e(\frac{-1}{n})}$.

So,

$$\begin{aligned}\sum_{k=0}^{n-1} r^{-2k}\cos^2\left(\frac{k\pi}{n}\right) &= \frac{1-r^{-2n}}{2(1-r^{-2})} + \frac{1-r^{-2}-r^{-2n+2}}{4}\left(\frac{1}{1-e(\frac{1}{n})} + \frac{1}{1-e(\frac{-1}{n})}\right)\\ &= \frac{1-r^{-2n}}{2(1-r^{-2})} + \frac{1-r^{-2}-r^{-2n+2}}{4} = N_1.\end{aligned}$$

Using the same methods, note that

$$\begin{aligned}S_2 &= \sum_{k=0}^{n-1} r^{-2k}\sin^2\left(\frac{k\pi}{n}\right) = \sum_{k=0}^{n-1} r^{-2k}\frac{1-\cos(\frac{2k\pi}{n})}{2}\\ &= \frac{1}{2}\sum_{k=0}^{n-1} r^{-2k} - \sum_{k=0}^{n-1} r^{-2k}\cos\left(\frac{2k\pi}{n}\right)\\ &= \frac{1-r^{-2n}}{2(1-r^{-2})} - \frac{1-r^{-2}-r^{-2n+2}}{4} = N_2.\end{aligned}$$

□

3 Proofs of theorems

Proof of Theorem 1 The matrix $A = C_r(\cos\frac{0\cdot\pi}{n}, \cos\frac{1\cdot\pi}{n}, \cos\frac{2\cdot\pi}{n}, \ldots, \cos\frac{(n-1)\cdot\pi}{n})$ is of the following form:

$$A = \begin{pmatrix} \cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cos\frac{2\cdot\pi}{n} & \cdots & \cos\frac{(n-1)\cdot\pi}{n} \\ r\cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cdots & \cos\frac{(n-2)\cdot\pi}{n} \\ r\cos\frac{(n-2)\cdot\pi}{n} & r\cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cdots & \cos\frac{(n-3)\cdot\pi}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r\cos\frac{1\cdot\pi}{n} & r\cos\frac{2\cdot\pi}{n} & r\cos\frac{3\cdot\pi}{n} & \cdots & \cos\frac{0\cdot\pi}{n} \end{pmatrix}_{n\times n}.$$

(i) From $|r| \geq 1$, using the definition of Euclidean norm and Lemma 1, we have

$$\begin{aligned} \|A\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\cos^2\left(\frac{k\cdot\pi}{n}\right) + \sum_{k=1}^{n-1} k|r|^2\cos^2\left(\frac{k\cdot\pi}{n}\right) \\ &\geq \sum_{k=0}^{n-1}(n-k)\cos^2\left(\frac{k\cdot\pi}{n}\right) + \sum_{k=1}^{n-1} k\cos^2\left(\frac{k\cdot\pi}{n}\right) \\ &= n\sum_{k=0}^{n-1}\cos^2\left(\frac{k\cdot\pi}{n}\right) = \frac{n^2}{2}, \end{aligned}$$

by (1), that is to say,

$$\|A\|_2 \geq \frac{1}{\sqrt{n}}\|A\|_E \geq \frac{\sqrt{2}}{2}.$$

Moreover, let the matrices E and F be defined by

$$E = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ r & r & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ r & r & r & \cdots & 1 \end{pmatrix}_{n\times n}$$

and

$$F = \begin{pmatrix} \cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cos\frac{2\cdot\pi}{n} & \cdots & \cos\frac{(n-1)\cdot\pi}{n} \\ \cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cdots & \cos\frac{(n-2)\cdot\pi}{n} \\ \cos\frac{(n-2)\cdot\pi}{n} & \cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cdots & \cos\frac{(n-3)\cdot\pi}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos\frac{1\cdot\pi}{n} & \cos\frac{2\cdot\pi}{n} & \cos\frac{3\cdot\pi}{n} & \cdots & \cos\frac{0\cdot\pi}{n} \end{pmatrix}_{n\times n},$$

then $A = E \circ F$. So $\|A\|_2 = \|E \circ F\|_2 \leq r_1(E)C_1(F)$,

$$r_1(E) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|e_{ij}|^2} = \sqrt{(n-1)r^2+1};$$

$$c_1(F) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |f_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} \cos^2 \frac{k\cdot\pi}{n}} = \sqrt{\frac{n}{2}}.$$

Therefore, we have

$$\|A\|_2 \leq \sqrt{(n-1)r^2+1}\sqrt{\frac{n}{2}}.$$

Thus, we can obtain the inequality

$$\frac{\sqrt{2}}{2} \leq \|A\|_2 \leq \sqrt{\frac{n}{2}}\sqrt{(n-1)|r|^2+1}.$$

(ii) From $|r| < 1$,

$$\begin{aligned}
\|A\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\cos^2\left(\frac{k\cdot\pi}{n}\right) + \sum_{k=1}^{n-1} k|r|^2\cos^2\left(\frac{k\cdot\pi}{n}\right) \\
&\geq \sum_{k=0}^{n-1}(n-k)|r|^2\cos^2\left(\frac{k\cdot\pi}{n}\right) + \sum_{k=1}^{n-1} k|r|^2\cos^2\left(\frac{k\cdot\pi}{n}\right) \\
&= n|r|^2\sum_{k=0}^{n-1}\cos^2\left(\frac{k\cdot\pi}{n}\right) = \frac{|r|^2n^2}{2},
\end{aligned}$$

we can get

$$\|A\|_2 \geq \frac{1}{\sqrt{n}}\|A\|_E \geq \frac{\sqrt{2}}{2}|r|.$$

Moreover, for the matrices E and F as mentioned above, $A = E\circ F$. So $\|A\|_2 = \|E\circ F\|_2 \leq r_1(E)C_1(F) = \frac{\sqrt{2}}{2}n$.

Therefore, we have $\frac{\sqrt{2}}{2}|r| \leq \|A\|_2 \leq \frac{\sqrt{2}}{2}n$.

This proves Theorem 1. □

Now we prove Theorem 2.

Proof

$$B = \begin{pmatrix}
\sin\frac{0\cdot\pi}{n} & \sin\frac{1\cdot\pi}{n} & \sin\frac{2\cdot\pi}{n} & \cdots & \sin\frac{(n-1)\cdot\pi}{n} \\
r\sin\frac{(n-1)\cdot\pi}{n} & \sin\frac{0\cdot\pi}{n} & \sin\frac{1\cdot\pi}{n} & \cdots & \sin\frac{(n-2)\cdot\pi}{n} \\
r\sin\frac{(n-2)\cdot\pi}{n} & r\sin\frac{(n-1)\cdot\pi}{n} & \sin\frac{0\cdot\pi}{n} & \cdots & \sin\frac{(n-3)\cdot\pi}{n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r\sin\frac{1\cdot\pi}{n} & r\sin\frac{2\cdot\pi}{n} & r\sin\frac{3\cdot\pi}{n} & \cdots & \sin\frac{0\cdot\pi}{n}
\end{pmatrix}_{n\times n}.$$

(i) From $|r| \geq 1$, using the definition of Euclidean norm and Lemma 1, we have

$$\|B\|_E^2 = \sum_{k=0}^{n-1}(n-k)\sin^2\left(\frac{k\cdot\pi}{n}\right) + \sum_{k=1}^{n-1} k|r|^2\sin^2\left(\frac{k\cdot\pi}{n}\right)$$

$$\begin{aligned}
&\geq \sum_{k=0}^{n-1}(n-k)\sin^2\left(\frac{k\cdot\pi}{n}\right)+\sum_{k=1}^{n-1}k\sin^2\left(\frac{k\cdot\pi}{n}\right)\\
&= n\sum_{k=0}^{n-1}\sin^2\left(\frac{k\cdot\pi}{n}\right)=\frac{n^2}{2},
\end{aligned}$$

that is,

$$\|B\|_2 \geq \frac{1}{\sqrt{n}}\|B\|_E \geq \frac{\sqrt{2}}{2}.$$

Moreover, let the matrices C and D be defined by

$$C=\begin{pmatrix} \sin\frac{0\cdot\pi}{n} & 1 & 1 & \cdots & 1\\ r & \sin\frac{0\cdot\pi}{n} & 1 & \cdots & 1\\ r & r & \sin\frac{0\cdot\pi}{n} & \cdots & 1\\ \vdots & \vdots & \vdots & & \vdots\\ r & r & r & \cdots & \sin\frac{0\cdot\pi}{n}\end{pmatrix}_{n\times n}$$

and

$$D=\begin{pmatrix} \sin\frac{0\cdot\pi}{n} & \sin\frac{1\cdot\pi}{n} & \sin\frac{2\cdot\pi}{n} & \cdots & \sin\frac{(n-1)\cdot\pi}{n}\\ \sin\frac{(n-1)\cdot\pi}{n} & \sin\frac{0\cdot\pi}{n} & \sin\frac{1\cdot\pi}{n} & \cdots & \sin\frac{(n-2)\cdot\pi}{n}\\ \sin\frac{(n-2)\cdot\pi}{n} & \sin\frac{(n-1)\cdot\pi}{n} & \sin\frac{0\cdot\pi}{n} & \cdots & \sin\frac{(n-3)\cdot\pi}{n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \sin\frac{1\cdot\pi}{n} & \sin\frac{2\cdot\pi}{n} & \sin\frac{3\cdot\pi}{n} & \cdots & \sin\frac{0\cdot\pi}{n}\end{pmatrix}_{n\times n},$$

then $B=C\circ D$. So $\|B\|_2=\|C\circ D\|_2\leq r_1(C)C_1(D)$,

$$\begin{aligned}
r_1(C)&=\max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|c_{ij}|^2}=\sqrt{(n-1)r^2};\\
c_1(D)&=\max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|d_{ij}|^2}=\sqrt{\sum_{k=0}^{n-1}\sin^2\frac{k\cdot\pi}{n}}=\sqrt{\frac{n}{2}}.
\end{aligned}$$

Therefore, we have

$$\|B\|_2\leq |r|\sqrt{\frac{n(n-1)}{2}}.$$

Thus, we can obtain

$$\frac{\sqrt{2}}{2}\leq\|B\|_2\leq|r|\sqrt{\frac{n(n-1)}{2}}.$$

(ii) From $|r|<1$,

$$\|B\|_E^2=\sum_{k=0}^{n-1}(n-k)\sin^2\left(\frac{k\cdot\pi}{n}\right)+\sum_{k=1}^{n-1}k|r|^2\sin^2\left(\frac{k\cdot\pi}{n}\right)$$

$$\begin{aligned}
&\geq \sum_{k=0}^{n-1}(n-k)|r|^2\sin^2\left(\frac{k\cdot\pi}{n}\right)+\sum_{k=1}^{n-1}k|r|^2\sin^2\left(\frac{k\cdot\pi}{n}\right)\\
&= n|r|^2\sum_{k=0}^{n-1}\sin^2\left(\frac{k\cdot\pi}{n}\right)=\frac{|r|^2n^2}{2},
\end{aligned}$$

we can get

$$\|B\|_2 \geq \frac{1}{\sqrt{n}}\|B\|_E \geq \frac{\sqrt{2}}{2}|r|.$$

On the other hand, for the matrices C and D as mentioned above, $B = C \circ D$. So $\|B\|_2 = \|C \circ D\|_2 \leq r_1(C)C_1(D) = \sqrt{\frac{n(n-1)}{2}}$.

Therefore, we have $\frac{\sqrt{2}}{2}|r| \leq \|B\|_2 \leq \sqrt{\frac{n(n-1)}{2}}$.

This proves Theorem 2. □

Now we prove Theorem 3 and Theorem 4.

Proof

$$P_{r^*} = \begin{pmatrix}
\cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cos\frac{2\cdot\pi}{n} & \cdots & \cos\frac{(n-1)\cdot\pi}{n}\\
r\cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cdots & \cos\frac{(n-2)\cdot\pi}{n}\\
r^2\cos\frac{(n-2)\cdot\pi}{n} & r\cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cdots & \cos\frac{(n-3)\cdot\pi}{n}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
r^{n-1}\cos\frac{1\cdot\pi}{n} & r^{n-2}\cos\frac{2\cdot\pi}{n} & r^{n-3}\cos\frac{3\cdot\pi}{n} & \cdots & \cos\frac{0\cdot\pi}{n}
\end{pmatrix}_{n\times n}.$$

(i) On the one hand, $|r| > 1$ and by using the definition of Euclidean norm, we can obtain

$$\begin{aligned}
\|P_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\cos^2\left(\frac{k\cdot\pi}{n}\right)+\sum_{k=1}^{n-1}i\left|r^{n-k}\right|^2\cos^2\left(\frac{k\cdot\pi}{n}\right)\\
&\geq \sum_{k=0}^{n-1}(n-k)\cos^2\left(\frac{k\cdot\pi}{n}\right)+\sum_{k=1}^{n-1}k\cos^2\left(\frac{k\cdot\pi}{n}\right)\\
&= n\sum_{k=0}^{n-1}\cos^2\left(\frac{k\cdot\pi}{n}\right)=\frac{n^2}{2}.
\end{aligned}$$

That is,

$$\|P_{r^*}\|_2 \geq \frac{1}{\sqrt{n}}\|P_{r^*}\|_E \geq \frac{\sqrt{2}}{2}n.$$

On the other hand, let the matrices S and Q be represented by

$$S = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1\\
r & 1 & 1 & \cdots & 1 & 1\\
r^2 & r & 1 & \cdots & 1 & 1\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1
\end{pmatrix}_{n\times n}$$

and

$$Q = \begin{pmatrix} \cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cos\frac{2\cdot\pi}{n} & \cdots & \cos\frac{(n-1)\cdot\pi}{n} \\ \cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cos\frac{1\cdot\pi}{n} & \cdots & \cos\frac{(n-2)\cdot\pi}{n} \\ \cos\frac{(n-2)\cdot\pi}{n} & \cos\frac{(n-1)\cdot\pi}{n} & \cos\frac{0\cdot\pi}{n} & \cdots & \cos\frac{(n-3)\cdot\pi}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos\frac{1\cdot\pi}{n} & \cos\frac{2\cdot\pi}{n} & \cos\frac{3\cdot\pi}{n} & \cdots & \cos\frac{0\cdot\pi}{n} \end{pmatrix}_{n\times n},$$

then $P_{r^*} = S \circ Q$. So $\|P_{r^*}\|_2 = \|S \circ Q\|_2 \leq r_1(S)C_1(Q)$,

$$r_1(S) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|s_{ij}|^2} = \sqrt{1 + |r|^2 + \cdots + \left|r^{n-1}\right|^2} = \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}},$$

$$c_1(Q) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|q_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1}\cos^2\left(\frac{k\cdot\pi}{n}\right)} = \sqrt{\frac{n}{2}}.$$

Therefore,

$$\|P_{r^*}\|_2 \leq r_1(S)c_1(Q) = \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}}\sqrt{\frac{n}{2}}.$$

(ii) For $|r| < 1$,

$$\begin{aligned}\|P_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\cos^2\left(\frac{k\cdot\pi}{n}\right) + \sum_{k=1}^{n-1}k\left|r^{n-k}\right|^2\cos^2\left(\frac{k\cdot\pi}{n}\right) \\ &\geq \sum_{k=0}^{n-1}(n-k)\left|r^{n-k}\right|^2\cos^2\left(\frac{k\cdot\pi}{n}\right) + \sum_{k=1}^{n-1}i\left|r^{n-k}\right|^2\cos^2\left(\frac{k\cdot\pi}{n}\right) \\ &= n|r|^{2n}\sum_{k=0}^{n-1}|r|^{-2k}\cos^2\left(\frac{k\cdot\pi}{n}\right) = n|r|^{2n}N_1.\end{aligned}$$

So

$$\|P_{r^*}\|_2 \geq \frac{1}{\sqrt{n}}\|P_{r^*}\|_E \geq |r|^n\sqrt{N_1},$$

where $N_1 = \frac{1-r^{-2n}}{2(1-r^{-2})} + \frac{1-r^{-2}-r^{-2n+2}}{4}$.

Moreover, for the matrices S and Q as mentioned above, in this case, $P_{r^*} = S \circ Q$. So $\|P_{r^*}\|_2 = \|S \circ Q\|_2 \leq r_1(S)C_1(Q)$,

$$r_1(S) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|s_{ij}|^2} = \sqrt{n},$$

$$c_1(Q) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|q_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1}\cos^2\left(\frac{k\cdot\pi}{n}\right)} = \sqrt{\frac{n}{2}},$$

$\|P_{r^*}\|_2 \leq \frac{\sqrt{2}}{2}n$.

Therefore, we have

$$|r|^n\sqrt{N_1} \le \|P_{r^*}\|_2 \le \frac{\sqrt{2}}{2}n.$$

By the same methods, using Lemma 2 and Theorem 2, we can get Theorem 4.

This completes all of the theorems. □

Remark Lemma 2 of this paper gave a new method to compute the power sums of the trigonometric functions.

4 Conclusion

By the same methods as of this paper, we can also get determinants and norms of some other special circulant matrices involving trigonometric functions $\cos(\frac{k\pi}{n})$, $\sin(\frac{k\pi}{n})$.

Acknowledgements
The author is grateful to anonymous referees and the associate editor for their careful reading, helpful comments, and constructive suggestions, which improved the presentation of the results.

Funding
This work is supported by N.S.F. (11771351).

Competing interests
The author declares to have no competing interests.

Authors' contributions
The author contributed to each part of this work seriously and read and approved the final version of the manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 11 December 2018 Accepted: 12 August 2019 Published online: 23 August 2019

References

1. Solak, S.: On the norms of circulant matrices with the Fibonacci and Lucas numbers. Appl. Math. Comput. **160**, 125–132 (2005)
2. Shen, S.Q., Cen, J.M.: On the bounds for the norms of r-circulant matrices with Fibonacci and Lucas numbers. Appl. Math. Comput. **216**, 2891–2897 (2010)
3. Yazlik, Y., Taskara, N.: On the norms of an r-circulant matrix with the generalized k-Horadam numbers. J. Inequal. Appl. **2013**, 394 (2013)
4. Bahsi, M.: On the norms of circulant matrices with the generalized Fibonacci and Lucas numbers. TWMS J. Pure Appl. Math. **6**(1), 84–92 (2015)
5. Köme, C., Yazlik, Y.: On the spectral norms of r-circulant matrices with the biperiodic Fibonacci and Lucas numbers. J. Inequal. Appl. **2017**, 192 (2017)
6. Yazlik, Y., Taskara, N.: Spectral norm, eigenvalues and determinant of circulant matrix involving the generalized k-Horadam numbers. Ars Comb. **104**, 505–512 (2012)
7. Lv, X.X., Shen, S.M.: On Chebyshev polynomials and their applications. Adv. Differ. Equ. **2017**, 343 (2017)
8. Davis, P.J.: Circulant Matrices. Wiley, New York (1979)
9. Kizilates, C., Tuglu, N.: On the bounds for the spectral norms of geometric circulant matrices. J. Inequal. Appl. **2016**, 312 (2016)
10. Tuglu, N., Kizilates, C.: On the norms of circulant and r-circulant matrices with the hyperharmonic Fibonacci numbers. J. Inequal. Appl. **2015**, 253 (2015)
11. He, C., Ma, J., Zhang, K., Wang, Z.: The upper bound estimation on the spectral norm of r-circulant natrices with the Fibonacci and Lucas numbers. J. Inequal. Appl. **2015**, 72 (2015)
12. Köme, C., Yazlik, Y.: On the determinants and inverses of r-circulant matrices with biperiodic Fibonacci and Lucas numbers. Filomat **32**(10), 3637–3650 (2018)
13. Bahsi, M.: On the norms of circulant matrices with the generalized Fibonacci and Lucas numbers. TWMS J. Pure Appl. Math. **6**(1), 84–92 (2015)