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Sharp inequalities for hyperbolic functions and circular functions

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Abstract

In this paper, we obtain some new sharp bounds for the exponential functions whose powers involve hyperbolic functions and circular functions, respectively.

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1 Introduction

In [1], Stolarsky obtained the bounds for the exponential function whose power involves a hyperbolic function.

Theorem 1 *Let $x > 0$. Then*

$$1 < e^{x\coth x-1} < \cosh x. \tag{1.1}$$

On the other hand, Pittenger [2] and Stolarsky [3] got the lower bound for the function $e^{x\coth x-1}$ as follows.

Theorem 2 *Let $x > 0$. Then*

$$\left(\cosh\frac{2x}{3}\right)^{\frac{3}{2}} < e^{x\coth x-1}. \tag{1.2}$$

In fact, Zhu [4] and Kouba [5] showed a new sharp lower bound for the function $e^{x\coth x-1}$ as follows.

Theorem 3 *Let $x > 0$. Then*

$$\left(\frac{2(\cosh x)^{\frac{6}{5}}+1}{3}\right)^{\frac{5}{6}} < e^{x\coth x-1}. \tag{1.3}$$

Zhu [6] proved

$$\left(\cosh\frac{2x}{3}\right)^{\frac{3}{2}} < \left(\frac{2(\cosh x)^{\frac{6}{5}}+1}{3}\right)^{\frac{5}{6}}, \quad x > 0 \tag{1.4}$$

to illustrate that the inequality (1.3) is stronger than (1.2).

It should be pointed out that the paper of Yang et al. [7] has made a great deal of improvement on inequality (1.2). The subject of the present paper is to further study the inequality (1.3), and to obtain the following results.

Theorem 4 *Let* $p \neq 0$, $p_1 = (\ln(3/2))/(\ln(e/2)) = 1.3214\ldots$, *and* $x \in (0, +\infty)$. *Then we have*

(i) *when* $p \in [2, +\infty)$, *the double inequality*

$$e^{p(x\coth x-1)} < \frac{2\cosh^p x + 1}{3} < \frac{2}{3}\left(\frac{e}{2}\right)^p e^{p(x\coth x-1)} \tag{1.5}$$

holds, where the constants 1 *and* $(2/3)(e/2)^p$ *are the best possible*;

(ii) *when* $p \in (-\infty, 6/5]$, *we have*

(a) *if* $p \in (0, 6/5]$, *the double inequality* (1.5) *reverses, that is,*

$$\frac{2}{3}\left(\frac{e}{2}\right)^p e^{p(x\coth x-1)} < \frac{2\cosh^p x + 1}{3} < e^{p(x\coth x-1)} \tag{1.6}$$

holds, where the constants $(2/3)(e/2)^p$ *and* 1 *are the best possible*;

(b) *if* $p \in (-\infty, 0)$, *the left-hand side of the double inequality* (1.5) *holds too*;

(iii) *when* $p \in (6/5, 2)$, *we have*

(c) *if* $p \in [p_1, 2)$, *the left-hand side of inequality* (1.5) *holds*;

(d) *if* $p \in (6/5, p_1)$, *the left-hand side of inequality* (1.6) *holds.*

As straightforward consequences of Theorem 4, Theorem 5 which is due to Kouba [5] may be derived immediately.

Theorem 5 *Let* $p \neq 0$, *and* $p_1 = (\ln(3/2))/(\ln(e/2)) = 1.3214\ldots$. *Then*

(1) *the inequality*

$$\frac{2\cosh^p x + 1}{3} < e^{p(x\coth x-1)} \tag{1.7}$$

holds for all $x \in (0, +\infty)$ *if and only if* $p \in (0, 6/5]$;

(2) *the inequality* (1.7) *reverses for all* $x \in (0, +\infty)$ *if and only if* $p \in (-\infty, 0) \cup [p_1, +\infty)$.

The analog of Theorem 4 for the circular functions is the following result.

Theorem 6 *Let* $p \neq 0$, $p_2 = \ln 3 = 1.0986\ldots$, *and* $x \in (0, \pi/2)$. *Then we have*

(i) *when* $p \in [6/5, +\infty)$, *the double inequality*

$$e^{p(x\cot x-1)} < \frac{2\cos^p x + 1}{3} < \left(\frac{e^p}{3}\right) e^{p(x\cot x-1)} \tag{1.8}$$

holds, where the constants 1 *and* $e^p/3$ *are the best possible*;

(ii) *when* $p \in (-\infty, 1]$, *we have*

(a) *if* $p \in (0, 1]$, *the double inequality* (1.8) *reverses, that is,*

$$\left(\frac{e^p}{3}\right) e^{p(x\cot x-1)} < \frac{2\cos^p x + 1}{3} < e^{p(x\cot x-1)} \tag{1.9}$$

holds, where the constants $e^p/3$ *and* 1 *are the best possible*;

(b) *if* $p \in (-\infty, 0)$, *the double inequality* (1.8) *holds too*;

(iii) *when* $p \in (1, 6/5)$, *we have*

(c) *if* $p \in (p_2, 6/5)$, *the right-hand side of inequality* (1.8) *holds*;

(d) *if* $p \in (1, p_2]$, *the right-hand side of inequality* (1.9) *holds*.

The following result which is due to Yang et al. [7] is a straightforward consequence of Theorem 6.

Theorem 7 *Let* $p \neq 0$, *and* $p_2 = \ln 3 = 1.0986\ldots$. *Then*

(A) *the inequality*

$$e^{p(x\cot x-1)} < \frac{2\cos^p x + 1}{3} \tag{1.10}$$

holds for all $x \in (0, \pi/2)$ *if and only if* $p \in (-\infty, 0) \cup [6/5, +\infty)$;

(B) *the inequality* (1.10) *reverses for all* $x \in (0, \pi/2)$ *if and only if* $p \in (0, p_2]$.

2 Lemmas

Lemma 1 ([8]) *For* $-\infty < a < b < \infty$, *let* $f, g : [a, b] \to \mathbb{R}$ *be continuous functions that are differentiable on* (a, b), *with* $f(a) = g(a) = 0$ *or* $f(b) = g(b) = 0$. *Assume that* $g'(x) \neq 0$ *for each* x *in* (a, b). *If* f'/g' *is increasing* (*decreasing*) *on* (a, b), *then so is* f/g.

Lemma 2 ([9]) *Let* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *be two real power series converging on* $(0, r)$ $(r > 0)$ *with* $b_n > 0$ *for all* n. *If the sequence* $\{a_n/b_n\}$ *is increasing* (*decreasing*) *for all* n, *then the function* $A(x)/B(x)$ *is also increasing* (*decreasing*) *on* $(0, r)$.

Lemma 3 *Let* $l(x)$ *be defined by*

$$l(x) = \frac{\ln \frac{\sinh 2x - 2x}{4(x\cosh x - \sinh x)}}{\ln \cosh x}. \tag{2.1}$$

Then $l(x)$ *is strictly increasing from* $(0, +\infty)$ *onto* $(1/5, 1)$.

Proof Let

$$l(x) = \frac{\ln \frac{\sinh 2x - 2x}{4(x\cosh x - \sinh x)}}{\ln \cosh x} =: \frac{f(x)}{g(x)} = \frac{f(x) - f(0^+)}{g(x) - g(0^+)},$$

where

$$f(x) = \ln \frac{\sinh 2x - 2x}{4(x\cosh x - \sinh x)}, \qquad g(x) = \ln \cosh x.$$

Then

$$\begin{aligned} f'(x) &= \frac{2\cosh 2x - 2}{\sinh 2x - 2x} + \frac{x \sinh x}{\sinh x - x\cosh x} \\ &= \sinh x \left(\frac{4 \sinh x}{\sinh 2x - 2x} + \frac{x}{\sinh x - x\cosh x} \right), \\ g'(x) &= \frac{\sinh x}{\cosh x}. \end{aligned}$$

We compute

$$\begin{aligned}
\frac{f'(x)}{g'(x)} &= \frac{\sinh x(\frac{4\sinh x}{\sinh 2x-2x}+\frac{x}{\sinh x-x\cosh x})}{\frac{\sinh x}{\cosh x}} \\
&= \left(\frac{x}{\sinh x-x\cosh x}+4\frac{\sinh x}{-2x+\sinh 2x}\right)\cosh x \\
&= (\cosh x)\frac{-4\sinh^2 x-x\sinh 2x+2x^2+4x\cosh x\sinh x}{(-2x+\sinh 2x)(-\sinh x+x\cosh x)} \\
&= \frac{(2x^2\cosh x+2x\cosh^2 x\sinh x-4\cosh x\sinh^2 x)}{-(2x^2\cosh x-2x\cosh^2 x\sinh x-2x\sinh x+2\cosh x\sinh^2 x)} \\
&= \frac{(\cosh x-\cosh 3x+\frac{1}{2}x\sinh x+\frac{1}{2}x\sinh 3x+2x^2\cosh x)}{-(\frac{1}{2}\cosh 3x-\frac{1}{2}\cosh x-\frac{5}{2}x\sinh x-\frac{1}{2}x\sinh 3x+2x^2\cosh x)} \\
&= \frac{2\cosh x-2\cosh 3x+x\sinh x+x\sinh 3x+4x^2\cosh x}{-\cosh 3x+\cosh x+5x\sinh x+x\sinh 3x-4x^2\cosh x} \\
&:= \frac{A(x)}{B(x)},
\end{aligned}$$

where

$$\begin{aligned}
A(x) &= 2\cosh x-2\cosh 3x+x\sinh x+x\sinh 3x+4x^2\cosh x, \\
B(x) &= -\cosh 3x+\cosh x+5x\sinh x+x\sinh 3x-4x^2\cosh x.
\end{aligned}$$

Then

$$\begin{aligned}
A(x) &= \sum_{n=2}^{+\infty}\frac{6(n-2)3^{2n}+16n^2+26n+12}{(2n+2)!}x^{2n+2} =: \sum_{n=2}^{+\infty}a_n x^{2n+2}, \\
B(x) &= \sum_{n=2}^{+\infty}\frac{3(2n-1)3^{2n}-16n^2-14n+3}{(2n+2)!}x^{2n+2} =: \sum_{n=2}^{+\infty}b_n x^{2n+2},
\end{aligned}$$

where

$$\begin{aligned}
a_n &= \frac{6(n-2)3^{2n}+16n^2+26n+12}{(2n+2)!}, \\
b_n &= \frac{3(2n-1)3^{2n}-16n^2-14n+3}{(2n+2)!}.
\end{aligned}$$

We can obtain

$$\frac{a_n}{b_n} = 2\frac{3(n-2)3^{2n}+8n^2+13n+6}{3(2n-1)3^{2n}-16n^2-14n+3} =: 2s_n, \quad n\geq 2.$$

Now we will prove that $\{s_n\}_{n\geq 2}$ is strictly increasing, which means

$$s_n < s_{n+1} \quad\Longleftrightarrow\quad h(n)3^{2n}+32n^2+112n+81>0,$$

where

$$h(n) =: 81 \cdot 3^{2n} - \left(256n^3 + 224n^2 + 112n + 162\right) > 0$$

for $n \geq 2$ due to

$$\begin{aligned} h(n+1) - 9h(n) &= 2048n^3 + 1024n^2 - 320n + 704 \\ &= 20{,}544 + 28{,}352(n-2) + 13{,}312(n-2)^2 + 2048(n-2)^3 > 0 \end{aligned}$$

for $n \geq 2$ and $h(2) = 3231 > 0$. This leads to $s_n < s_{n+1}$ for $n \geq 2$. So $\{a_n/b_n\}_{n\geq 2}$ is strictly increasing. By Lemma 2, we know that $A(x)/B(x) = f'(x)/g'(x)$ is strictly increasing on $(0, +\infty)$. Then $l(x) = f(x)/g(x)$ is strictly increasing on $(0, +\infty)$ by Lemma 1.

Since

$$\lim_{x\to 0^+} \frac{\ln \frac{\sinh 2x - 2x}{4(x\cosh x - \sinh x)}}{\ln \cosh x} = \frac{1}{5}, \qquad \lim_{x\to +\infty} \frac{\ln \frac{\sinh 2x - 2x}{4(x\cosh x - \sinh x)}}{\ln \cosh x} = 1,$$

this completes the proof of Lemma 3. □

Lemma 4 *Let $x > 0$, B_{2n} be the even-indexed Bernoulli numbers (see [10]). Then the following power series expansions:*

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} 2^{2n} |B_{2n}| x^{2n-1}, \tag{2.2}$$

$$\sec^2 x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!} |B_{2n}| x^{2n-2}, \tag{2.3}$$

$$\tan x \sec^2 x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(n-1)}{(2n)!} |B_{2n}| x^{2n-3} \tag{2.4}$$

hold for all $x \in (-\pi/2, \pi/2)$.

Proof The power series expansion (2.2) can be found in [11, equations 1.3.1.4(3)]. By (2.2) we have

$$(\sec x)^2 = (\tan x)' = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!} |B_{2n}| x^{2n-2}, \quad |x| < \frac{\pi}{2},$$

and

$$\begin{aligned} \tan x \sec^2 x &= \frac{1}{2}\left(\sec^2 x\right)' \\ &= \frac{1}{2} \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{(2n)!} |B_{2n}| x^{2n-3} \\ &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(n-1)}{(2n)!} |B_{2n}| x^{2n-3}, \quad |x| < \frac{\pi}{2}. \end{aligned}$$

□

Lemma 5 ([12, 13]) *Let B_{2n} be the even-indexed Bernoulli numbers. Then*

$$\frac{(2n+2)(2n+1)(2^{2n-1}-1)}{\pi^2(2^{2n+1}-1)} < \frac{|B_{2n+2}|}{|B_{2n}|} < \frac{(2n+2)(2n+1)(2^{2n}-1)}{\pi^2(2^{2n+2}-1)} \tag{2.5}$$

holds for $n = 1, 2, \ldots$.

Lemma 6 *Let $z(x)$ be defined by*

$$z(x) = \frac{\ln \frac{2x-\sin 2x}{4(\sin x - x\cos x)}}{\ln \cos x}. \tag{2.6}$$

Then $z(x)$ is strictly decreasing from $(0, \pi/2)$ onto $(0, 1/5)$.

Proof Let

$$z(x) = \frac{\ln \frac{2x-\sin 2x}{4(\sin x - x\cos x)}}{\ln \cos x} = \frac{p(x)}{q(x)} = \frac{p(x)-p(0^+)}{q(x)-q(0^+)},$$

where

$$p(x) = \ln \frac{2x-\sin 2x}{4(\sin x - x\cos x)}, \qquad q(x) = \ln \cos x.$$

Since

$$p'(x) = (\sin x)\left[\frac{4\sin x}{2x-\sin 2x} - \frac{x}{\sin x - x\cos x}\right], \qquad q'(x) = -\frac{\sin x}{\cos x},$$

we have

$$\begin{aligned}
\frac{p'(x)}{q'(x)} &= \frac{x\cos x}{\sin x - x\cos x} - \frac{2\sin x\cos x}{x - \cos x \sin x} \\
&= \frac{x^2\cos x + x\cos^2 x\sin x - 2\cos x\sin^2 x}{-x^2\cos x + x\cos^2 x\sin x + x\sin x - \cos x\sin^2 x} \\
&= \frac{(x^2\cos x + x\cos^2 x\sin x - 2\cos x\sin^2 x)/\cos^3 x}{(-x^2\cos x + x\cos^2 x\sin x + x\sin x - \cos x\sin^2 x)/\cos^3 x} \\
&= \frac{x^2\sec^2 x + x\tan x - 2\tan^2 x}{-x^2\sec^2 x + x\tan x + x\tan x\sec^2 x - \tan^2 x} \\
&=: \frac{C(x)}{D(x)}.
\end{aligned}$$

From Lemma 4 we obtain

$$\begin{aligned}
C(x) &= x^2\sec^2 x + x\tan x - 2\tan^2 x \\
&= \sum_{n=3}^{\infty}\left[\frac{(2n)(2^{2n}-1)|B_{2n}|}{(2n)!} - \frac{8(2^{2n+2}-1)(2n+1)|B_{2n+2}|}{(2n+2)!}\right]2^{2n}x^{2n} \\
&=: \sum_{n=3}^{\infty} c_n x^{2n},
\end{aligned}$$

$$\begin{aligned}
D(x) &= -x^2\sec^2 x + x\tan x + x\tan x\sec^2 x - \tan^2 x \\
&= \sum_{n=3}^{\infty}\left[\frac{2(2n+1)(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!} - \frac{(2^{2n}-1)|B_{2n}|}{(2n)!}\right](n-1)2^{2n+1}x^{2n} \\
&=: \sum_{n=3}^{\infty} d_n x^{2n}.
\end{aligned}$$

We consider the monotonicity of $C(x)/D(x)$, and compute that

$$\frac{c_n}{d_n} = \frac{1}{2(n-1)}\frac{(n+1)(2n)(2^{2n}-1)|B_{2n}| - 4(2^{2n+2}-1)|B_{2n+2}|}{(2^{2n+2}-1)|B_{2n+2}| - (n+1)(2^{2n}-1)|B_{2n}|}.$$

Then for $n \geq 3$

$$\begin{aligned}
&\frac{c_n}{d_n} > \frac{c_{n+1}}{d_{n+1}} \quad \Longleftrightarrow \\
&\frac{L}{M} =: \frac{(n+1)(2n)(2^{2n}-1)|B_{2n}| - 4(2^{2n+2}-1)|B_{2n+2}|}{(n-1)[(2^{2n+2}-1)|B_{2n+2}| - (n+1)(2^{2n}-1)|B_{2n}|]} \\
&\quad > \frac{(n+2)(2n+2)(2^{2n+2}-1)|B_{2n+2}| - 4(2^{2n+4}-1)|B_{2n+4}|}{n[(2^{2n+4}-1)|B_{2n+4}| - (n+2)(2^{2n+2}-1)|B_{2n+2}|]} \\
&\quad := \frac{X}{Y}.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{LY - MX}{2|B_{2n+2}|^2} \\
&\quad = -(n+2)\bigl(n^2-2n-1\bigr)\bigl(2^{2n+2}-1\bigr)^2 \\
&\qquad - \bigl(2^{2n+2}-1\bigr)\bigl(2^{2n}-1\bigr)(n+2)(n+1)\frac{|B_{2n}|}{|B_{2n+2}|} \\
&\qquad + (n+1)\bigl(n^2-2n+2\bigr)\bigl(2^{2n+4}-1\bigr)\bigl(2^{2n}-1\bigr)\frac{|B_{2n}|}{|B_{2n+2}|}\frac{|B_{2n+4}|}{|B_{2n+2}|} \\
&\qquad - 2\bigl(2^{2n+4}-1\bigr)\bigl(2^{2n+2}-1\bigr)\frac{|B_{2n+4}|}{|B_{2n+2}|},
\end{aligned}$$

by Lemma 5 we have

$$\begin{aligned}
&\frac{LY - MX}{2|B_{2n+2}|^2} \\
&\quad > -(n+2)\bigl(n^2-2n-1\bigr)\bigl(2^{2n+2}-1\bigr)^2 \\
&\qquad - \frac{(2^{2n+2}-1)(2^{2n}-1)(n+2)(n+1)\pi^2(2^{2n+1}-1)}{(2n+2)(2n+1)(2^{2n-1}-1)} \\
&\qquad + (n+1)\bigl(n^2-2n+2\bigr)\bigl(2^{2n+4}-1\bigr)\bigl(2^{2n}-1\bigr) \\
&\qquad \times \frac{\pi^2(2^{2n+2}-1)}{(2n+2)(2n+1)(2^{2n}-1)}\frac{(2n+4)(2n+3)(2^{2n+1}-1)}{\pi^2(2^{2n+3}-1)}
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2\bigl(2^{2n+4}-1\bigr)\bigl(2^{2n+2}-1\bigr)\frac{(2n+4)(2n+3)(2^{2n+2}-1)}{\pi^2(2^{2n+4}-1)} \\
&= \frac{(2^{2n+2}-1)(n+2)}{\pi^2(2^{2n+3}-1)(2^{2n}-2)(2n+1)}r(n),
\end{aligned}
$$

where

$$
r(n) = \bigl[u_1(n)2^{2n} - v_1(n)\bigr]2^{4n} + u_2(n)2^{2n} - v_2(n)
$$

with

$$
\begin{aligned}
u_1(n) &= \bigl(64\pi^2 - 512\bigr)n^2 - \bigl(1024 - 64\pi^2\bigr)n + 224\pi^2 - 16\pi^4 - 384, \\
v_1(n) &= 12\pi^2 n^3 + \bigl(146\pi^2 - 1216\bigr)n^2 - \bigl(2432 - 140\pi^2\bigr)n \\
&\quad + \bigl(568\pi^2 - 26\pi^4 - 912\bigr), \\
u_2(n) &= 24\pi^2 n^3 - \bigl(400 - 38\pi^2\bigr)n^2 - \bigl(800 - 26\pi^2\bigr)n \\
&\quad + 247\pi^2 - 11\pi^4 - 300, \\
v_2(n) &= \bigl(4\pi^2 - 32\bigr)n^2 + \bigl(64 - 4\pi^2\bigr)n - \bigl(14\pi^2 - \pi^4 - 24\bigr).
\end{aligned}
$$

Then we have $c_n/d_n > c_{n+1}/d_{n+1}$ for $n \geq 3$ when proving

$$
u_1(n)2^{2n} - v_1(n) > 0 \quad \Longleftrightarrow \quad 2^{2n} > \frac{v_1(n)}{u_1(n)}, \tag{2.7}
$$

$$
u_2(n)2^{2n} - v_2(n) > 0 \quad \Longleftrightarrow \quad 2^{2n} > \frac{v_2(n)}{u_2(n)}. \tag{2.8}
$$

Now we use mathematical induction to prove (2.7). When $n = 3$, (2.7) clearly holds. Assuming that (2.7) holds for $n = m$, that is,

$$
2^{2m} > \frac{v_1(m)}{u_1(m)}. \tag{2.9}
$$

Next, we prove that (2.7) is valid for $n = m + 1$. By (2.9) we have

$$
2^{2(m+1)} = 4 \cdot 2^{2m} > 4\frac{v_1(m)}{u_1(m)},
$$

in order to complete the proof of (2.7) it suffices to show that

$$
4\frac{v_1(m)}{u_1(m)} > \frac{v_1(m+1)}{u_1(m+1)} \quad \Longleftrightarrow \quad 4v_1(m)u_1(m+1) - v_1(m+1)u_1(m) > 0.
$$

In fact,

$$
4v_1(m)u_1(m+1) - v_1(m+1)u_1(m) = k(m),
$$

where

$$
\begin{aligned}
k(m) &= \bigl(13{,}452{,}288\pi^4 - 184{,}343{,}040\pi^2 - 230{,}976\pi^6 + 1248\pi^8 + 728{,}082{,}432\bigr) \\
&\quad + \bigl(11{,}200{,}896\pi^4 - 166{,}279{,}680\pi^2 - 103{,}872\pi^6 + 663{,}994{,}368\bigr)(m-3)
\end{aligned}
$$

$$
\begin{aligned}
&+ \bigl(3{,}983{,}424\pi^4 - 59{,}525{,}376\pi^2 - 16{,}608\pi^6 + 225{,}067{,}008\bigr)(m-3)^2 \\
&+ \bigl(738{,}048\pi^4 - 10{,}274{,}304\pi^2 - 576\pi^6 + 33{,}619{,}968\bigr)(m-3)^3 \\
&+ \bigl(68{,}736\pi^4 - 801{,}792\pi^2 + 1{,}867{,}776\bigr)(m-3)^4 \\
&+ \bigl(2304\pi^4 - 18{,}432\pi^2\bigr)(m-3)^5 \\
> {}& 0
\end{aligned}
$$

for $m \geq 3$.

Similarly, we can prove (2.8). By (2.7) and (2.8) we find that $\{c_n/d_n\}_{n\geq 3}$ is a monotonic decreasing sequence. Then we arrive at the conclusion that $p'(x)/q'(x) = C(x)/D(x)$ is decreasing on $(0,\pi/2)$ by Lemma 2. By Lemma 1 we see that $z(x)$ is decreasing on $(0,\pi/2)$.

Since

$$
z(0^+) = \frac{1}{5}, \qquad z\left(\left(\frac{\pi}{2}\right)^-\right) = 0,
$$

this completes the proof of Lemma 6. □

3 The proofs of main results

3.1 The proof of Theorem 4

Proof Let

$$
G(x) = \frac{2\cosh^p x + 1}{3e^{p(x\coth x-1)}}, \quad x > 0.
$$

Then

$$
G(+\infty) = \begin{cases} \frac{2}{3}(\frac{e}{2})^p, & p > 0, \\ +\infty, & p < 0, \end{cases}
$$

and

$$
G'(x) = \frac{p}{3}\frac{Q(x)}{e^{p(x\coth x-1)}}, \tag{3.1}
$$

where

$$
\begin{aligned}
Q(x) &= \frac{2(x\cosh x - \sinh x)}{\sinh^2 x}\cosh^{p-1} x - \frac{\cosh x\sinh x - x}{\sinh^2 x} \\
&= \frac{2(x\cosh x - \sinh x)}{\sinh^2 x}\left[\cosh^{p-1} x - \frac{\cosh x\sinh x - x}{2(x\cosh x - \sinh x)}\right] \\
&= \frac{2(x\cosh x - \sinh x)(\ln\cosh x)}{\sinh^2 x}\,\frac{\cosh^{p-1} x - \frac{\cosh x\sinh x - x}{2(x\cosh x-\sinh x)}}{\ln(\cosh^{p-1} x) - \ln\frac{\cosh x\sinh x - x}{2(x\cosh x - \sinh x)}} \\
&\quad\times\left(p - 1 - \frac{\ln\frac{\sinh 2x - 2x}{4(x\cosh x - \sinh x)}}{\ln\cosh x}\right)
\end{aligned}
$$

$$=: 2\frac{(x\cosh x-\sinh x)(\ln\cosh x)}{\sinh^2 x}\frac{\cosh^{p-1}x-\frac{\cosh x\sinh x-x}{2(x\cosh x-\sinh x)}}{\ln(\cosh^{p-1}x)-\ln\frac{\cosh x\sinh x-x}{2(x\cosh x-\sinh x)}} \times \bigl(p-1-l(x)\bigr) \tag{3.2}$$

with

$$l(x)=\frac{\ln\frac{\sinh 2x-2x}{4(x\cosh x-\sinh x)}}{\ln\cosh x}.$$

We consider the following three cases.

Case 1: $p \geq 2$.

From Lemma 3, we get $\max_{x\in(0,+\infty)} l(x)=1$. So $p-1-l(x)>0$. This leads to $Q(x)>0$ by (3.2), and $G'(x)>0$ by (3.1). Then

$$G\bigl(0^+\bigr)<G(x)<G(+\infty),$$

this is the double inequality (1.5).

Case 2: $p \leq 6/5$.

From Lemma 3, we get $\min_{x\in(0,+\infty)} l(x)=1/5$. So $p-1-l(x)<0$. This leads to $Q(x)<0$.

Subcase 2.1: $0<p\leq 6/5$. In this case, $G'(x)<0$ by (3.1). Then

$$G(+\infty)<G(x)<G\bigl(0^+\bigr),$$

this is the double inequality (1.6).

Subcase 2.2: $p<0$. We have $G'(x)>0$ by (3.1). In view of $G(+\infty)=+\infty$, the left-hand side of inequality (1.5) holds too.

Case 3: $6/5<p<2$.

Let $r(x):=l(x)+1-p$. Then

$$r\bigl(0^+\bigr)=l\bigl(0^+\bigr)+1-p=\frac{6}{5}-p<0,$$
$$r(+\infty)=l(+\infty)+1-p=2-p>0,$$

and there is the unique point $\xi\in(0,+\infty)$ such that $r(x)<0$ holds for all $x\in(0,\xi)$ and $r(x)>0$ holds for all for $x\in(\xi,+\infty)$. That is, $p-1-l(x)>0$ holds for all $x\in(0,\xi)$ and $p-1-l(x)>0$ holds for all $x\in(\xi,+\infty)$. By (3.2) and (3.1), we have $G'(x)>0$ for all $x\in(0,\xi)$ and $G'(x)<0$ holds for all $x\in(\xi,+\infty)$. Then

$$\min(G\bigl(0^+\bigr),G(+\infty)<G(x)<G(\xi).$$

Subcase 3.1: $p_1=(\ln(3/2))/(\ln(e/2))<p<2$. In this case, $1<(2/3)(e/2)^p$, that is, $G(0^+)<G(+\infty)$ holds, so $\min(G(0^+),G(+\infty))=G(0^+)$. This leads to the left-hand side of inequality (1.5).

Subcase 3.2: $6/5<p<p_1=(\ln(3/2))/(\ln(e/2))$. In this case, $1>(2/3)(e/2)^p$, that is, $G(0^+)>G(+\infty)$ holds, so $\min(G(0^+),G(+\infty))=G(+\infty)$. This leads to the left-hand side of inequality (1.6).

The proof of Theorem 4 is complete. □

3.2 The proof of Theorem 6

Proof Let

$$F(x)=\frac{2\cos^p x+1}{3e^{p(x\cot x-1)}},\quad 0<x<\frac{\pi}{2}.$$

Then

$$F\left(\left(\frac{\pi}{2}\right)^-\right)=\frac{e^p}{3}$$

and

$$\begin{aligned}F'(x)&=\frac{p}{3}\frac{2(\sin x-x\cos x)[-\ln(\cos x)]}{(\sin^2 x)\exp(p(x\cot x-1))}\frac{\cos^{p-1}x-\frac{x-\cos x\sin x}{2(\sin x-x\cos x)}}{(p-1)\ln\cos x-\ln\frac{(x-\cos x\sin x)}{2(\sin x-x\cos x)}}\\&\quad\times\left(p-1-\frac{\ln\frac{2x-\sin 2x}{4(\sin x-x\cos x)}}{\ln\cos x}\right)\\&=\frac{p}{3}\frac{2(\sin x-x\cos x)[-\ln(\cos x)]}{(\sin^2 x)\exp(p(x\cot x-1))}\frac{\cos^{p-1}x-\frac{x-\cos x\sin x}{2(\sin x-x\cos x)}}{(p-1)\ln\cos x-\ln\frac{(x-\cos x\sin x)}{2(\sin x-x\cos x)}}\\&\quad\times\left[p-1-z(x)\right],\end{aligned}\tag{3.3}$$

where

$$z(x)=\frac{\ln\frac{2x-\sin 2x}{4(\sin x-x\cos x)}}{\ln\cos x}.$$

We consider the following three cases.

Case 1: $p\geq 6/5$.

From Lemma 6, we get $\max_{x\in(0,+\infty)} z(x)=1/5$. So $p-1-z(x)>0$ holds. This leads to $F'(x)<0$ by (3.3). Then

$$F\left(0^+\right)<F(x)<F\left(\left(\frac{\pi}{2}\right)^-\right),$$

which is the double inequality (1.8).

Case 2: $p\leq 1$.

From Lemma 6, we get $\min_{x\in(0,+\infty)} z(x)=0$. So $p-1-z(x)<0$ holds.

Subcase 2.1: $0<p\leq 1$. In this case, $F'(x)<0$ by (3.3). Then

$$F\left(\left(\frac{\pi}{2}\right)^-\right)<F(x)<F\left(0^+\right),$$

which is the double inequality (1.9).

Subcase 2.2: $p<0$. We have $F'(x)>0$ by (3.3), the double inequality (1.8) holds too.

Case 3: $1<p<6/5$.

Let $q(x):=z(x)-p+1$. Then $q(0^+)=z(0^+)-p+1=6/5-p>0$, $q((\pi/2)^-)=z((\pi/2)^-)-p+1=1-p<0$. There is a unique point $\eta\in(0,\pi/2)$ such that $q(x)>0$ holds for all $x\in(0,\eta)$

and $q(x) < 0$ holds for all for $x \in (\eta, \pi/2)$. That is, $p - 1 - z(x) < 0$ holds for all $x \in (0, \eta)$ and $p - 1 - z(x) > 0$ holds for all $x \in (\eta, \pi/2)$. By (3.3), we have $F'(x) < 0$ for all $x \in (0, \eta)$ and $F'(x) > 0$ for all $x \in (\eta, \pi/2)$. Then

$$F(\eta) < F(x) < \max\bigl(F(0^+), F\bigl((\pi/2)^-\bigr)\bigr).$$

Subcase 3.1: $p_2 = \ln 3 = 1.0986 < p < 6/5$. In this case, $1 < e^p/3$, that is, $F(0^+) < F((\pi/2)^-)$, so $\max(F(0^+), F((\pi/2)^-)) = F((\pi/2)^-)$. This leads to the right-hand side of inequality (1.8).

Subcase 3.2: $1 < p \leq p_2 = \ln 3 = 1.0986$. In this case, $1 \geq e^p/3$, that is, $F(0^+) \geq F((\pi/2)^-)$, so $\max(F(0^+), F((\pi/2)^-)) = F(0^+)$. This leads to the right-hand side of inequality (1.9).

The proof of Theorem 6 is complete. □

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The author provided the questions and gave the proof for all results. He read and approved this manuscript.

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