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f-asymptotically lacunary ideal equivalence of double sequences

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Abstract

In this study, we present the notions of f-asymptotically $\mathcal{I}_2$-equivalence, strongly f-asymptotically $\mathcal{I}_2$-equivalence, f-asymptotically lacunary $\mathcal{I}_2$-equivalence, and strongly f-asymptotically lacunary $\mathcal{I}_2$-equivalence of double sequences and investigate some relationships between them. Also, we examine some relationships between strongly f-asymptotically $\mathcal{I}_2$-equivalence and asymptotically $\mathcal{I}_2$-statistical equivalence and between strongly f-asymptotically lacunary $\mathcal{I}_2$-equivalence and asymptotically lacunary $\mathcal{I}_2$-statistical equivalence of double sequences.

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1 Introduction and background

Throughout the paper, $\mathbb{N}$ is the set of all positive integers, and $\mathbb{R}$ is the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [14] and Schoenberg [39]. Fridy and Orhan [15] studied lacunary statistical convergence. This concept was extended to the double sequences by Mursaleen and Edely [28]. A lot of development have been made in this area after the works of [1, 2, 13, 25–27].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko, Šalát, and Wilczyński [20] as a generalization of statistical convergence. Das et al. [4] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of development have been made in this area after the works of [5–11, 29–32, 40, 44].

Marouf [24] presented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [35] presented asymptotically statistically equivalent sequences for nonnegative summability matrices. Dündar et al. [12] defined asymptotically $\mathcal{I}_2^{\sigma}$-equivalent, asymptotically invariant equivalent, strongly asymptotically invariant equivalent, and p-strongly asymptotically invariant equivalent for double sequences. Ulusu and Dündar [42] introduced the concepts of asymptotically lacunary $\mathcal{I}_2$-invariant equivalence, asymptotically lacunary σ_2-equivalence, and asymptotically lacunary invariant S_2-equivalence of double sequences. Hazarika and Kumar [16] studied asymptotically double lacunary statistically equivalent sequences in ideal context.

The modulus function was introduced by Nakano [33]. Maddox [23], Pehlivan [37], and many authors used the modulus function f to define some new concepts and inclusion theorems. Kumar and Sharma [21] studied lacunary equivalent sequences by ideals and the modulus function. Also, several authors define some new concepts and give inclusion theorems using a modulus function f (see [17, 18]).

Now we recall the basic concepts and some definitions (see [3, 15, 19, 20, 22–24, 34–38, 41, 43]).

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout the paper, we let $\theta = \{k_r\}$ be a lacunary sequence.

A double sequence $\theta_2 = \{(k_r, j_u)\}$ is called a double lacunary sequence if there exist two increasing sequences of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ and $j_0 = 0$, $\bar{h}_u = j_u - j_{u-1} \to \infty$ as $r, u \to \infty$. We further use the following notations:

$$k_{ru} = k_r j_u, \qquad h_{ru} = h_r \bar{h}_u, \qquad I_{ru} = \bigl\{(k,j) : k_{r-1} < k \le k_r \text{ and } j_{u-1} < j \le j_u\bigr\},$$
$$q_r = \frac{k_r}{k_{r-1}} \quad \text{and} \quad q_u = \frac{j_u}{j_{u-1}}.$$

Throughout the paper, we let $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,

(ii) for all $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$,

(iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$, we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$, and a nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Throughout the paper, we let $\mathcal{I}$ be an admissible ideal.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called a strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

Throughout the paper, we let $\mathcal{I}_2$ be a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible.

We denote $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})\, (i,j \ge m(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal, and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_{k\to\infty} \frac{x_k}{y_k} = 1$ (denoted by $x \sim y$).

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistically equivalent of multiple L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S_L}{\sim} y$) and simply asymptotically statistically equivalent, if $L = 1$.

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega)}{\sim} y_k$) and simply strongly asymptotically equivalent with respect to the ideal $\mathcal{I}$ if $L = 1$.

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(N_\theta)}{\sim} y_k$) and simply strongly asymptotically lacunary $\mathcal{I}$-equivalent with respect to the ideal $\mathcal{I}$ if $L = 1$.

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistically equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for all $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{S}_\theta)}{\sim} y_k$) and simply asymptotically lacunary $\mathcal{I}$-statistically equivalent if $L = 1$.

A function $f : [0,\infty) \to [0,\infty)$ is called a modulus if

1. $f(x) = 0$ if and if only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$,
3. f is increasing, and
4. f is continuous from the right at 0.

A modulus may be unbounded (for example, $f(x) = x^p$, $0 < p < 1$) or bounded (for example, $f(x) = \frac{x}{x+1}$).

Let f be a modulus function. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon > 0$,

$$\left\{k \in \mathbb{N} : f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(f)}{\sim} y_k$) and simply f-asymptotically $\mathcal{I}$-equivalent if $L = 1$.

Let f be a modulus function. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega_f)}{\sim} y_k$)) and simply strongly f-asymptotically $\mathcal{I}$-equivalent if $L = 1$.

Let f be a modulus function. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically lacunary equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(N_\theta^f)}{\sim} y_k$)) and simply strongly f-asymptotically lacunary $\mathcal{I}$-equivalent if $L = 1$.

Two nonnegative double sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically strongly $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}$) and simply asymptotically $\mathcal{I}_2$-statistical equivalent if $L = 1$.

Two nonnegative double sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically $\mathcal{I}_2$-statistically equivalent of multiple L if for all $\varepsilon > 0$ and each $\gamma > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ k,j \leq m,n : \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_2(S)}{\sim} y_{kj}$) and simply asymptotically $\mathcal{I}_2$-statistically equivalent if $L = 1$.

Two nonnegative double sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}$) and simply strongly asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

Two nonnegative double sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically lacunary $\mathcal{I}_2$-statistically equivalent of multiple L if for all $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \left| \left\{ (k,j) \in I_{ru} : \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_2(S_\theta)}{\sim} y_{kj}$) and simply asymptotically $\mathcal{I}_2$-statistically equivalent if $L = 1$.

Lemma 1 ([37]) *Let f be a modulus, and let $0 < \delta < 1$. Then, for each $x \geq \delta$, we have $f(x) \leq 2f(1)\delta^{-1}x$.*

2 Main results

Definition 2.1 Let f be a modulus function. Two nonnegative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be f-asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(k,j) \in \mathbb{N} \times \mathbb{N} : f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_2^L(f)}{\sim} y_{kj}$) and simply f-asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

Definition 2.2 Let f be a modulus function. The two nonnegative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strongly f-asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every

$\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$) and simply strongly f-asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

Theorem 2.1 *Let f be a modulus function. Then $x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$.*

Proof Suppose that $x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}$, and let $\varepsilon > 0$ be given. Select $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. We can write

$$\begin{aligned} \frac{1}{mn} \sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &= \frac{1}{mn} \sum_{\substack{k,j=1 \\ |\frac{x_{kj}}{y_{kj}} - L| \leq \delta}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &\quad + \frac{1}{mn} \sum_{\substack{k,j=1 \\ |\frac{x_{kj}}{y_{kj}} - L| > \delta}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right), \end{aligned}$$

and so by Lemma 1

$$\frac{1}{mn} \sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right) \frac{1}{mn} \sum_{k,j=1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right|.$$

Thus, for any $\gamma > 0$,

$$\begin{aligned} &\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \gamma\right\} \\ &\quad \subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \frac{(\gamma - \varepsilon)\delta}{2f(1)}\right\}. \end{aligned}$$

Since $x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}$, it follows that the second set and thus the first set in the above expression belong to $\mathcal{I}_2$. This proves that $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$. □

Theorem 2.2 *If $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}$.*

Proof We showed that $x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$ in Theorem 2.1. Now we must show that $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}$.

Let $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$. Then we have $f(t) \geq \alpha t$ for all $t \geq 0$. Assume that $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$. Since

$$\frac{1}{mn} \sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \frac{1}{mn} \sum_{k,j=1}^{m,n} \alpha\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) = \alpha\left(\frac{1}{mn} \sum_{k,j=1}^{m,n} \left|\frac{x_{kj}}{y_{kj}} - L\right|\right),$$

it follows that for each $\varepsilon > 0$, we have

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) \geq \alpha\varepsilon \right\}.$$

Since $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$, it follows that the latter set and hence the former set in the above expression belong to $\mathcal{I}_2$. This proves that $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}$. □

Definition 2.3 Let f be a modulus function. Two nonnegative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be f-asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(k,j) \in I_{ru} : f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{\mathcal{I}_{\theta_2}^L(f)}{\sim} y_{kj}$)) and simply f-asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

Definition 2.4 Let f be a modulus function. Two nonnegative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strongly f-asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}_2$$

(denoted by $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$)) and simply strongly f-asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

Theorem 2.3 *Let f be a modulus function. Then,* $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$.

Proof Let $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}$, and let $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. We can write

$$\frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) = \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ |\frac{x_{kj}}{y_{kj}} - L| \leq \delta}} f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) + \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ |\frac{x_{kj}}{y_{kj}} - L| > \delta}} f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right),$$

and so by Lemma 1

$$\frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} f\left(\left| \frac{x_{kj}}{y_{kj}} - L \right| \right) < \varepsilon + \left(\frac{2f(1)}{\delta} \right) \frac{1}{h_{ru}} \sum_{(k,j) \in I_r} \left| \frac{x_{kj}}{y_{kj}} - L \right|.$$

Thus, for each any $\gamma > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \gamma\right\}$$
$$\subseteq \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \frac{(\gamma - \varepsilon)\delta}{2f(1)}\right\}.$$

Since $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}$, it follows that the latter set and hence the former set in the above expression belong to $\mathcal{I}_2$. This proves that $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$. □

Theorem 2.4 *If* $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, *then* $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}$.

Proof We showed that $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$ in Theorem 2.3. Now we must show that $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}$.

Let $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$. Then we have $f(t) \geq \alpha t$ for all $t \geq 0$. Assume that $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$. From

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \alpha\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right)$$
$$= \alpha\left(\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right|\right)$$

it follows that for each $\varepsilon > 0$, we have

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}$$
$$\subseteq \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \alpha\varepsilon\right\}.$$

Since $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$, it follows that the latter set and hence the former set in the above expression belong to $\mathcal{I}_2$. This proves that $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$. □

Theorem 2.5 *Let f be a modulus function. If* $\liminf_{r,u} q_{r,u} > 1$, *then*

$$x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj} \quad \Rightarrow \quad x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}.$$

Proof Suppose that $\liminf_{r,u} q_{r,u} > 1$. Then there exists $\eta > 0$ such that $q_{r,u} \geq 1 + \eta$ for sufficiently large r, u. Then we have

$$\frac{h_{ru}}{k_r j_u} \geq \frac{\eta}{1 + \eta}.$$

Let $x_{kj} \stackrel{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$. For sufficiently large r, u, we have

$$\begin{aligned}\frac{1}{k_r j_u}\sum_{k,j=1,1}^{k_r j_u} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right) &\geq \frac{1}{k_r j_u}\sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\\ &= \left(\frac{h_{ru}}{k_r j_u}\right)\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\\ &\geq \frac{\eta}{1+\eta}\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right),\end{aligned}$$

which gives, for any $\varepsilon > 0$,

$$\begin{aligned}&\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\geq\varepsilon\right\}\\ &\quad\subseteq\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{k_r j_u}\sum_{k,j=1,1}^{k_r j_u} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\geq\frac{\varepsilon\eta}{1+\eta}\right\}.\end{aligned}$$

Since $x_{kj} \stackrel{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$, it follows that the latter set and hence the former set belong to $\mathcal{I}_2$. This shows that $x_{kj} \stackrel{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$. □

Theorem 2.6 *Let f be a modulus function. Then, $x_{kj} \stackrel{[\mathcal{I}_2^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \stackrel{\mathcal{I}_2(S)}{\sim} y_{kj}$.*

Proof Assume that $x_{kj} \stackrel{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$, and let $\varepsilon > 0$ be given. From

$$\begin{aligned}\frac{1}{mn}\sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right) &\geq \frac{1}{mn}\sum_{\substack{k,j=1\\ |\frac{x_{kj}}{y_{kj}}-L|\geq\varepsilon}}^{n} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\\ &\geq f(\varepsilon)\cdot\frac{1}{mn}\left|\left\{k\leq m, j\leq n:\left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon\right\}\right|\end{aligned}$$

it follows that for any $\gamma > 0$, we have

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\left\{k\leq m, j\leq n:\left|\frac{x_{kj}}{y_{kj}}-L\right|\geq\varepsilon\right\}\right|\geq\frac{\gamma}{f(\varepsilon)}\right\}\\ &\quad\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}}-L\right|\right)\geq\gamma\right\}.\end{aligned}$$

Since $x_{kj} \stackrel{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$, it follows the latter set and hence the former set in the above expression belong to $\mathcal{I}_2$. Therefore $x_{kj} \stackrel{\mathcal{I}_2(S)}{\sim} y_{kj}$. □

Theorem 2.7 *Let f be a modulus function. If f is bounded, then $x_{kj} \stackrel{\mathcal{I}_2(S)}{\sim} y_{kj} \Leftrightarrow x_{kj} \stackrel{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$.*

Proof We showed that $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{\mathcal{I}_2(S)}{\sim} y_{kj}$ in Theorem 2.6. Now we must show that $x_{kj} \overset{\mathcal{I}_2(S)}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$.

Assume that f is bounded and let $x_{kj} \overset{\mathcal{I}_2(S)}{\sim} y_{kj}$. Since f is bounded, there exists a positive real number M such that $|f(x)| \leq M$ for all $x \geq 0$. We have

$$\begin{aligned}\frac{1}{mn}\sum_{k,j=1}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &= \frac{1}{mn}\sum_{\substack{k,j=1\\ |\frac{x_{kj}}{y_{kj}}-L|\geq\varepsilon}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right)\\ &\quad+\frac{1}{mn}\sum_{\substack{k,j=1\\ |\frac{x_{kj}}{y_{kj}}-L|<\varepsilon}}^{m,n} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right)\\ &\leq \frac{M}{mn}\left|\left\{k \leq m, j \leq n : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| + f(\varepsilon).\end{aligned}$$

This proves that $x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$. □

Theorem 2.8 *Let f be a modulus function. Then $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{\mathcal{I}(S_{\theta_2})}{\sim} y_{kj}$.*

Proof Assume that $x_k \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_k$, and let $\varepsilon > 0$ be given. From

$$\begin{aligned}\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &\geq \frac{1}{h_{ru}}\sum_{\substack{(k,j)\in I_{ru}\\ |\frac{x_{kj}}{y_{kj}}-L|\geq\varepsilon}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right)\\ &\geq f(\varepsilon)\cdot\frac{1}{h_{ru}}\left|\left\{(k,j)\in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right|\end{aligned}$$

it follows that for any $\gamma > 0$,

$$\begin{aligned}&\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}}\left|\left\{(k,j)\in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\}\\ &\quad\subseteq\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \geq \gamma f(\varepsilon)\right\}.\end{aligned}$$

Since $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$, the last set belongs to $\mathcal{I}_2$, and so by the definition of an ideal the first set belongs to $\mathcal{I}_2$. Therefore $x_{kj} \overset{\mathcal{I}(S_{\theta_2})}{\sim} y_{kj}$. □

Theorem 2.9 *Let f be a modulus function. Iff is bounded, then $x_{kj} \overset{\mathcal{I}(S_{\theta_2})}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$.*

Proof We showed that $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{\mathcal{I}(S_{\theta_2})}{\sim} y_{kj}$ in Theorem 2.8. Now we must show that $x_{kj} \overset{\mathcal{I}(S_{\theta_2})}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$.

Assume that f is bounded and let $x_{kj} \overset{\mathcal{I}(S_{\theta_2})}{\sim} y_{kj}$. Since f is bounded, there exists a positive real number M such that $|f(x)| \leq M$ for all $x \geq 0$. We have

$$\begin{aligned}\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) &= \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ |\frac{x_{kj}}{y_{kj}} - L| \geq \varepsilon}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &\quad + \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ |\frac{x_{kj}}{y_{kj}} - L| < \varepsilon}} f\left(\left|\frac{x_{kj}}{y_{kj}} - L\right|\right) \\ &\leq \frac{M}{h_{ru}} \left|\left\{(k,j) \in I_{ru} : \left|\frac{x_{kj}}{y_{kj}} - L\right| \geq \varepsilon\right\}\right| + f(\varepsilon).\end{aligned}$$

This proves that $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$. □

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