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Equivalent theorem of approximation by linear combination of weighted Baskakov–Kantorovich operators in Orlicz spaces

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Abstract

In this paper, we introduce the Orlicz space corresponding to the Young function and, by virtue of the equivalent theorem between the modified K-functional and modulus of smoothness, establish the direct, inverse, and equivalent theorems for linear combination of the Jacobi weighted Baskakov–Kantorovich operators in the Orlicz spaces.

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1 Motivations and main results

In recent years, since the Orlicz spaces are more general than the classical L_p spaces, which are composed of measurable functions $f(x)$ such that $|f(x)|^p$ are integrable, there is growing interest in problems of approximation in Orlicz spaces.

For proceeding smoothly, we recall from [22] some definitions and related results.

A continuous convex function $\Phi(t)$ on $[0,\infty)$ is called a Young function if

$$\lim_{t\to 0^+}\frac{\Phi(t)}{t}=0 \quad\text{and}\quad \lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty.$$

For a Young function $\Phi(t)$, its complementary Young function is denoted by $\Psi(t)$.

It is clear that the convexity of $\Phi(t)$ leads to $\Phi(\alpha t)\le\alpha\Phi(t)$ for $\alpha\in[0,1]$. In particular, we have $\Phi(\alpha t)<\alpha\Phi(t)$ for $\alpha\in(0,1)$.

A Young function $\Phi(t)$ is said to satisfy the Δ_2-condition, denoted by $\Phi\in\Delta_2$, if there exist $t_0>0$ and $C\ge 1$ such that $\Phi(2t)\le C\Phi(t)$ for $t\ge t_0$.

Let $\Phi(t)$ be a Young function. We define the Orlicz class $L_\Phi[0,\infty)$ as the collection of all Lebesgue-measurable functions $u(x)$ on $[0,\infty)$. Since the integral

$$\rho(u,\Phi)=\int_0^\infty \Phi\bigl(|u(x)|\bigr)\,\mathrm{d}x$$

is finite, we define the Orlicz space $L^*_\Phi[0,\infty)$ as the linear hull of $L_\Phi[0,\infty)$ under the Luxembourg norm

$$\|u\|_{(\Phi)} = \inf_{\lambda>0}\left\{\lambda : \rho\left(\frac{u}{\lambda},\Phi\right) \le 1\right\}.$$

The Orlicz norm, which is equivalent to the Luxembourg norm on $L^*_\Phi[0,\infty)$, is given by

$$\|u\|_\Phi = \sup_{\rho(v,\Psi)\le 1}\left|\int_0^\infty u(x)v(x)\,\mathrm{d}x\right|$$

and satisfies

$$\|u\|_{(\Phi)} \le \|u\|_\Phi \le 2\|u\|_{(\Phi)}. \tag{1.1}$$

Throughout this paper, we use C to denote a constant independent of n and x, which may be not necessarily the same in different cases.

Let $f \in L^*_\Phi[0,\infty)$ and $r \in \mathbb{N}$. Then the weighted K-functional $K_{r,\varphi}(f,t^r)_{w,\Phi}$, the weighted modified K-functional $\bar{K}_{r,\varphi}(f,t^r)_{w,\Phi}$, and the weighted modulus of smoothness $\omega_{r,\varphi}(f,t)_{w,\Phi}$ are given respectively by

$$K_{r,\varphi}(f,t^r)_{w,\Phi} = \inf_g\bigl\{\|w(f-g)\|_\Phi + t^r\|w\varphi^r g^{(r)}\|_\Phi : g^{(r-1)} \in AC_{\mathrm{loc}}\bigr\},$$
$$\bar{K}_{r,\varphi}(f,t^r)_{w,\Phi} = \inf_g\bigl\{\|w(f-g)\|_\Phi + t^r\|w\varphi^r g^{(r)}\|_\Phi + t^{2r}\|wg^{(r)}\|_\Phi : g^{(r-1)} \in AC_{\mathrm{loc}}\bigr\},$$

and

$$\omega_{r,\varphi}(f,t)_{w,\Phi} = \begin{cases} \sup_{0<h\le t}\|w\Delta^r_{h\varphi}(f)\|_\Phi, & a = 0, \\ \sup_{0<h\le t}\|w\Delta^r_{h\varphi}(f)\|_{\Phi[t^*,\infty)} + \sup_{0<h\le t^*}\|w\vec{\Delta}^r_h(f)\|_{\Phi[0,12t^*]}, & a > 0, \end{cases}$$

where

$$\Delta^r_{h\varphi}(f,x) = \sum_{k=0}^r (-1)^k\binom{r}{k}f\left(x + r\left[\frac{h\varphi(x)}{2}\right] - kh\varphi(x)\right),$$
$$t^* = r^2t^2, \qquad \varphi(x) = \sqrt{x}, \qquad \varphi(x) = \sqrt{x(1+x)},$$
$$\varphi(x) = x, \qquad w(x) = x^a(1+x)^b$$

for $a, b \in \mathbb{R}$ is the Jacobi weight function, and $g^{(r-1)} \in AC_{\mathrm{loc}}$ means that g is $r-1$ times differentiable and $g^{(r-1)}$ is absolutely continuous in every closed finite interval $[c,d] \subset [0,\infty)$.

Between the weighted modulus of smoothness and the weighted modified $\bar{K}$-functional, there are the following equivalent theorems.

Theorem A ([13]) *Let $f \in L^*_\Phi[0,\infty)$ and $r \in \mathbb{N}$. Then there exist constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_{w,\Phi}}{C} \le \bar{K}_{r,\varphi}(f,t^r)_{w,\Phi} \le C\omega_{r,\varphi}(f,t)_{w,\Phi} \tag{1.2}$$

for $0 < t \le t_0$.

Theorem B ([12]) *Let $f \in L_{\Phi}^{*}[0,\infty)$ and $r \in \mathbb{N}$. Then there exist constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_{w,\Phi}}{C} \le K_{r,\varphi}\bigl(f,t^r\bigr)_{w,\Phi} \le C\omega_{r,\varphi}(f,t)_{w,\Phi} \tag{1.3}$$

for $0 < t \le t_0$.

For $f \in C([0,\infty))$, the classical Baskakov operators are defined in [3] as

$$V_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) v_{n,k}(x),$$

where $v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$, $k,n \in \mathbb{N}$. To approximate functions in the L_p-norm, Ditzian and Totik [5] defined the Kantorovich modifications

$$\tilde{V}_n(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_{k/(n-1)}^{(k+1)/(n-1)} f(u)\,\mathrm{d}u \tag{1.4}$$

and obtained the direct inequality

$$\|\tilde{V}_n f - f\|_p \le C\left[\omega_{2,\varphi}\left(f,\frac{1}{n^{1/2}}\right)_p + \frac{1}{n}\|f\|_p\right]$$

and the week converse inequality

$$\|\tilde{V}_n f - f\|_p = O\left(\frac{1}{n^{\alpha/2}}\right) \quad\Longleftrightarrow\quad \omega_{2,\varphi}(f,h)_p = O\bigl(h^{\alpha}\bigr)$$

for $\alpha < 2$, where $f \in L_p[0,\infty)$, $1 \le p < \infty$, and $\varphi(x) = \sqrt{x(1+x)}$.

There are many approximation results on operators of the Baskakov type in the space $C[0,\infty)$ or $L_p[0,\infty)$. See [1–12, 14–21, 23, 24, 28, 29] and closely related references therein. Gupta and Acu [10] discussed a uniform estimate and established a quantitative result for the modified Baskakov–Szász–Mirakyan operators. Kumar and Acar [14] introduced a modification of generalized Baskakov–Durrmeyer operators of the Stancu type and studied their approximation properties. Goyal and Agrawal [8] introduced the Bézier variant of the generalized Baskakov–Kantorovich operators, established a direct approximation theorem with the aid of the Ditzian–Totik modulus of smoothness, and studied the rate of convergence for the functions having the derivatives of bounded variation for these operators. Zhang and Zhu [29] studied preservation properties, such as monotonicity, convexity, and smoothness, as well as those under the average, of the Baskakov–Kantorovich operators. Gadjev [7] studied the approximation of functions by the Baskakov–Kantorovich operator in the space $L_p[0,\infty)$ and obtained the double inequality

$$\frac{1}{C}\|\tilde{V}_n f - f\|_p \le \tilde{K}\left(f,\frac{1}{n}\right)_p \le C\frac{\ell}{n}\bigl(\|\tilde{V}_n f - f\|_p + \|\tilde{V}_\ell f - f\|_p\bigr)$$

for $\ell \in \mathbb{N}$ with $\ell \geq C_1 n$, where C_1 is a positive constant, and

$$\begin{aligned}\tilde{K}(f,t)_p = \inf\Big\{ & \|f-g\|_p + t\|\tilde{D}g\|_p : f-g \in L_p[0,\infty), \\ & g \in \tilde{W}_p[0,\infty), \tilde{D} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\varphi^2(x)\frac{\mathrm{d}}{\mathrm{d}x}\right]\Big\}\end{aligned}$$

is a K-functional.

For $n, r \in \mathbb{N}$ such that $n \geq 2r$, the linear combinations of the Baskakov–Kantorovich operator are defined as

$$L_{n,r}(f;x) = \sum_{i=0}^{2r-1} c_i(n,r)\tilde{V}_{n_i}(f;x), \tag{1.5}$$

where the coefficients $c_i(n,r)$ only dependent of n, r and satisfy the following conditions:

$$n \leq n_0 \leq n_1 < \cdots < n_{2r-1} \leq C_n, \quad \sum_{i=0}^{2r-2} c_i(n,r) = 1,$$

$$\sum_{i=0}^{2r-2} \big|c_i(n,r)\big| \leq C, \tag{1.6}$$

$$\sum_{i=0}^{2r-1} c_i(n,r)\tilde{V}_{n_i}\big((t-x)^k;x\big) = 0, \quad k = 1,2,\ldots,2r-1. \tag{1.7}$$

There are few results on the linear combinations of the Baskakov–Kantorovich operators. In [11], we obtained approximation properties for linear combinations of modified summation operators of integral type in the Orlicz space. Basing on these conclusions, we discover in this paper approximation properties for linear combinations of the Baskakov–Kantorovich operators.

Our main results in this paper can be stated as the following three theorems.

Theorem 1.1 (Direct theorem) *Let* $f \in L_\Phi^*[0,\infty)$, $\Psi \in \Delta_2$, $\varphi(x) = \sqrt{x(1+x)}$, $n, a, b \in \mathbb{N}$, *and* $0 \leq a, b < n-1$. *Then*

$$\big\|w\big(L_{n,r}(f) - f\big)\big\|_\Phi \leq C\omega_{2r,\varphi}\left(f, \frac{1}{n^{1/2}}\right)_{w,\Phi}.$$

Theorem 1.2 (Inverse theorem) *Let* $f \in L_\Phi^*[0,\infty)$, $n \geq 2r$, $\varphi(x) = \sqrt{x(1+x)}$, $a, b \in \mathbb{N}$, *and* $0 \leq a, b < n-1$. *Then*

$$\omega_{2r,\varphi}\left(f, \frac{1}{n^{r/2}}\right)_{w,\Phi} \leq \frac{C}{n^r}\sum_{k=1}^{n} k^{r-1}\big\|w\big(L_{k,r}(f) - f\big)\big\|_\Phi.$$

Theorem 1.3 (Equivalence theorem) *Let* $f \in L_{\Phi}^{*}[0,\infty)$, $n \geq 2r$, $\varphi(x) = \sqrt{x(1+x)}$, $\Psi \in \Delta_2$, $a, b \in \mathbb{N}$, *and* $0 \leq a, b < n-1$. *Then*

$$\begin{aligned} &\left\| w\left(L_{n,r}(f) - f\right)\right\|_{\Phi} = O\left(\psi\left(\frac{1}{n^{1/2}}\right)\right), \quad n \to \infty \\ \iff \quad &\omega_{2r,\varphi}(f,t)_{w,\Phi} = O\left(\psi(t)\right), \quad t \to 0^{+}. \end{aligned}$$

These main results are stronger than the results mentioned before.

2 Proof of the direct theorem

To prove the direct theorem, we need several lemmas.

Lemma 2.1 *The operators* $\tilde{V}_n(f;x)$ *defined in* (1.4) *satisfy*

$$\tilde{V}_n(1;x) = 1 \quad \text{and} \quad \tilde{V}_n\left((t-x)^{2r};x\right) \leq \frac{C\delta_n^{2r}(x)}{n^r},$$

where $\delta_n^{2r}(x) = \max\{\varphi^{2r}(x), \frac{1}{n^r}\}$, $\varphi(x) = \sqrt{x(1+x)}$, $r \in \mathbb{N}$, *and* C *is a positive constant.*

Proof This follows from simple calculation. □

Lemma 2.2 ([5]) *If* t *locates between* x *and* u, *then*

$$\frac{(x-t)^{2r-1}}{\varphi^{2r}(t)} \leq \frac{|x-u|^{2r-1}}{\varphi^{2r-2}(x)} \frac{1}{x}\left(\frac{1}{1+x} + \frac{1}{1+u}\right).$$

Lemma 2.3 ([5]) *For* $w(x) = x^a(1+x)^b$ *and* $a, b \in \mathbb{R}$, *we have*

$$\frac{w(x)}{w(u)} \leq 2^{|b|}\left[\left(\frac{x}{u}\right)^a + \left(\frac{x}{u}\right)^b\right].$$

Lemma 2.4 *Let* $f \in L_{\Phi}^{*}[0,\infty)$, $w(x) = x^a(1+x)^b$, $a, b \in \mathbb{N}$, *and* $0 \leq a, b < n-1$. *Then*

$$\left\| wL_{n,r}(f)\right\|_{\Phi} \leq C\|wf\|_{\Phi}.$$

Proof By Lemma 2.3 we can write

$$\begin{aligned} \left|w(x)\tilde{V}_n(f;x)\right| &= \left|\sum_{k=0}^{\infty} v_{n,k}(x)w(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} f(u)\,\mathrm{d}u\right| \\ &\leq \sum_{k=0}^{\infty} v_{n,k}(x)(n-1)2^{|b|+1}\left[\left(\frac{x(n-1)}{k+1}\right)^a \right. \\ &\quad \left. + \left(\frac{x(n-1)}{k+1}\right)^b\right]\int_{k/(n-1)}^{(k+1)/(n-1)} w(u)\left|f(u)\right|\,\mathrm{d}u \\ &\triangleq I_1 + I_2. \end{aligned}$$

Using (1.1) and Jensen's inequality gives

$$\begin{aligned}
\|I_1\|_\Phi &\le 2\|I_1\|_{(\Phi)} \\
&= 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \Phi\Biggl(\Biggl|\sum_{k=0}^\infty v_{n,k}(x)(n-1)2^{|b|+1}\biggl[\frac{x(n-1)}{k+1}\biggr]^a \\
&\quad \times \int_{k/(n-1)}^{(k+1)/(n-1)} \frac{w(u)f(u)}{\lambda}\,\mathrm{d}u\Biggr|\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&\le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \Phi\Biggl(\sum_{k=0}^\infty v_{n-a,k+a}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{Cw(u)|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&= 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \Phi\Biggl(\sum_{k=a}^\infty v_{n-a,k}(x)(n-1)\int_{(k-a)/(n-1)}^{(k+1-a)/(n-1)} \frac{Cw(u)|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&= 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \Phi\Biggl(\sum_{k=a}^\infty v_{n-a,k}(x)(n-1)\int_{(k-a)/(n-1)}^{(k+1-a)/(n-1)} \frac{Cw(u)|f(u)|}{\lambda}\,\mathrm{d}u \\
&\quad + \sum_{k=0}^{a-1} v_{n-a,k}(x)(n-1)\int_{\max\{0,\frac{k-a}{n-1}\}}^{\max\{0,\frac{k+1-a}{n-1}\}} \frac{Cw(u)|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&\le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \sum_{k=0}^\infty v_{n-a,k}(x)\Phi\Biggl((n-1)\int_{\max\{0,\frac{k-a}{n-1}\}}^{\max\{0,\frac{k+1-a}{n-1}\}} \frac{Cw(u)|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&= 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \sum_{k=a}^\infty v_{n-a,k}(x)\Phi\Biggl((n-1)\int_{(k-a)/(n-1)}^{(k+1-a)/(n-1)} \frac{Cw(u)|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&\le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \sum_{k=a}^\infty v_{n-a,k}(x)(n-1)\int_{(k-a)/(n-1)}^{(k+1-a)/(n-1)} \Phi\biggl(\frac{Cw(u)|f(u)|}{\lambda}\biggr)\,\mathrm{d}u\,\mathrm{d}x \le 1\Biggr\} \\
&\le 2\inf_{\lambda>0}\Biggl\{\lambda : \sum_{k=0}^\infty \frac{n-1}{n-a-1}\int_{k/(n-1)}^{(k+1)/(n-1)} \Phi\biggl(\frac{Cw(u)|f(u)|}{\lambda}\biggr)\,\mathrm{d}u \le 1\Biggr\} \\
&\le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \Phi\biggl(\frac{Cw(u)|f(u)|}{\lambda}\biggr)\,\mathrm{d}u \le 1\Biggr\} \\
&= C\|wf\|_{(\Phi)} \le C\|wf\|_\Phi.
\end{aligned}$$

Similarly, we have

$$\|I_2\|_\Phi \le C\|wf\|_\Phi.$$

Consequently, we arrive at

$$\bigl\|w(x)\tilde{V}_n(f;x)\bigr\|_\Phi \le C\|wf\|_\Phi. \tag{2.1}$$

Combining (1.5), (1.6), and (2.1), it follows that

$$\begin{aligned}\|wL_{n,r}(f)\|_{\Phi} &= \left\|\sum_{i=0}^{2r-1} c_i(n,r)w\tilde{V}_{n_i}(f)\right\|_{\Phi} \le \sum_{i=0}^{2r-1}\|c_i(n,r)w\tilde{V}_{n_i}(f)\|_{\Phi} \\ &\le \sum_{i=0}^{2r-1}|c_i(n,r)|\|w\tilde{V}_{n_i}(f)\|_{\Phi} \le C\sum_{i=0}^{2r-1}|c_i(n,r)|\|wf\|_{\Phi} \le C\|wf\|_{\Phi}.\end{aligned}$$

The proof of Lemma 2.4 is complete. □

Lemma 2.5 ([13]) *For $f \in L_{\Phi}^{*}[0,\infty)$ and $\Psi \in \Delta_2$, we have*

$$\|\theta(f)\|_{\Phi} \le C\|f\|_{\Phi},$$

where

$$\theta(f;x) = \sup_{\substack{0\le t<\infty \\ t\ne x}} \frac{1}{t-x}\int_x^t f(u)\,\mathrm{d}u$$

is the Hardy–Littlewood function of $f(x)$.

We now in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Let

$$U = W_{\Phi}^{2r}\bigl\{g : g^{(2r-1)} \in AC_{\mathrm{loc}}, \varphi^{2r}g^{(2r)} \in L_{\Phi}^{*}[0,\infty)\bigr\}.$$

Taylor's formula with integral remainder of $g \in U$ gives

$$g(u) = \sum_{i=0}^{2r-1}\frac{1}{i!}(u-x)^i g^{(i)}(x) + R_{2r}(g;u,x),$$

where

$$R_{2r}(g;u,x) = \frac{1}{(2r-1)!}\int_x^u (u-t)^{2r-1}g^{(2r)}(t)\,\mathrm{d}t, \quad x\in[0,\infty).$$

From (1.7) it follows that

$$w(x)\bigl[L_{n,r}(g;x)-g(x)\bigr] = w(x)L_{n,r}\bigl(R_{2r}(g;u,x);x\bigr)$$

and

$$\bigl\|w\bigl(L_{n,r}(g)-g\bigr)\bigr\|_{\Phi} = \bigl\|wL_{n,r}\bigl(R_{2r}(g;\cdot,x),x\bigr)\bigr\|_{\Phi}. \tag{2.2}$$

We now estimate $|w(x)\tilde{V}_n(R_{2r}(g;u,x);x)|$. As $x \in [\frac{1}{n},\infty)$, we have $\delta_n^{2r}(x) = \varphi^{2r}(x)$. Applying Lemma 2.2 leads to

$$
\begin{aligned}
&\left|w(x)\tilde{V}_n\left(R_{2r}(g;u,x);x\right)\right| \\
&\quad= \left|w(x)\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{1}{(2r-1)!}\right. \\
&\qquad \left.\times \int_x^u (u-t)^{2r-1} g^{(2r)}(t)\,\mathrm{d}t\,\mathrm{d}u\right| \\
&\quad\leq w(x)\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{1}{(2r-1)!}\frac{(u-x)^{2r}}{\varphi^{2r-2}(x)} \\
&\qquad\times\left[\frac{1}{x(1+x)}+\frac{1}{x(1+u)}\right]\left[\frac{1}{w(u)}+\frac{1}{w(x)}\right]\mathrm{d}u\left|\theta\left(w\delta_n^{2r}g^{(2r)};x\right)\right| \\
&\quad\triangleq (J_1+J_2+J_3+J_4)\left|\theta\left(w\delta_n^{2r}g^{(2r)};x\right)\right|.
\end{aligned}
$$

From Cauchy's integral inequality [25, 26] and Lemma 2.1 it follows that

$$
\begin{aligned}
J_1 &= \sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{1}{(2r-1)!}\frac{(u-x)^{2r}}{\varphi^{2r}(x)}\frac{w(x)}{w(u)}\,\mathrm{d}u \\
&\leq \frac{1}{(2r-1)!}\frac{1}{\varphi^{2r}(x)}\left[\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{w^2(x)}{w^2(u)}\,\mathrm{d}u\right]^{1/2} \\
&\quad\times\left[\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} (u-x)^{4r}\,\mathrm{d}u\right]^{1/2} \\
&\leq \frac{1}{(2r-1)!}\frac{1}{\varphi^{2r}(x)}\frac{C\delta_n^{2r}(x)}{n^r} = \frac{C}{n^r},
\end{aligned}
$$

where

$$
\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{w^2(x)}{w^2(u)}\,\mathrm{d}u \leq C.
$$

Similarly, we can also obtain

$$
\begin{aligned}
J_2 &= \frac{1}{(2r-1)!}\frac{w(x)}{\varphi^{2r}(x)}\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{(u-x)^{2r}}{w(x)}\,\mathrm{d}u \\
&= \frac{1}{(2r-1)!}\frac{1}{\varphi^{2r}(x)}\tilde{V}_n\left((u-x)^{2r};x\right) \leq \frac{C}{n^r}.
\end{aligned}
$$

Now we estimate J_3. By Cauchy's integral inequality [25, 26] and Lemma 2.1 we derive that

$$\begin{aligned}
J_3 &= \frac{w(x)}{(2r-1)!\varphi^{2r-2}(x)x}\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}\frac{(u-x)^{2r}}{(1+u)w(u)}\,\mathrm{d}u \\
&= \frac{w(x)(1+x)}{(2r-1)!\varphi^{2r}(x)}\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}\frac{(u-x)^{2r}}{(1+u)w(u)}\,\mathrm{d}u \\
&= \frac{w_1(x)}{(2r-1)!\varphi^{2r}(x)}\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}\frac{(u-x)^{2r}}{w_1(u)}\,\mathrm{d}u \\
&\le \frac{1}{\varphi^{2r}(x)}\left[\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}\frac{w_1^2(x)}{w_1^2(u)}\,\mathrm{d}u\right]^{1/2} \\
&\quad\times\left[\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}(u-x)^{4r}\,\mathrm{d}u\right]^{1/2} \\
&\le \frac{C}{n^r},
\end{aligned}$$

where $w_1(x) = x^a(1+x)^{b+1}$.

Finally, we estimate J_4. Applying Cauchy's integral inequality [25, 26] and Lemma 2.1 yields

$$\begin{aligned}
J_4 &= \frac{1}{(2r-1)!}\frac{w(x)}{\varphi^{2r-2}(x)}\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}\frac{(u-x)^{2r}}{w(x)}\frac{1}{x(1+u)}\,\mathrm{d}u \\
&\le \frac{1}{x\varphi^{2r-2}(x)}\left[\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}(u-x)^{4r}\,\mathrm{d}u\right]^{1/2} \\
&\quad\times\left[\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}\frac{\mathrm{d}u}{(1+u)^2}\right]^{1/2} \\
&\le \frac{1}{x\varphi^{2r-2}(x)}\frac{C\delta_n^{2r}(x)}{n^r}\frac{\sqrt{2}}{1+x} \le \frac{C}{n^r},
\end{aligned}$$

where

$$\sum_{k=0}^{\infty}v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}\frac{\mathrm{d}u}{(1+u)^2} \le \frac{2}{(1+x)^2}.$$

From the previous inequalities and Lemma 2.5 it follows that

$$\begin{aligned}
\left\|w\tilde{V}_n\big(R_{2r}(g;\cdot,x);x\big)\right\|_{\Phi[\frac{1}{n},\infty)} &\le \frac{C}{n^r}\left\|\theta\big(w\delta_n^{2r}g^{(2r)}\big)\right\|_{\Phi[\frac{1}{n},\infty)} \\
&\le \frac{C}{n^r}\left\|w\delta_n^{2r}g^{(2r)}\right\|_{\Phi[\frac{1}{n},\infty)}.
\end{aligned} \tag{2.3}$$

For $x \in [0, \frac{1}{n})$, we have $\delta_n^{2r}(x) = \frac{1}{n^r}$. Accordingly,

$$\begin{aligned}
&\left|w(x)\tilde{V}_n\big(R_{2r}(g;u,x);x\big)\right| \\
&\quad\leq \sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{w(x)n^r}{(2r-1)!}(u-x)^{2r}\left[\frac{1}{w(u)}+\frac{1}{w(x)}\right]\mathrm{d}u\left|\theta\big(w\delta_n^{2r}g^{(2r)},x\big)\right| \\
&\quad\triangleq (E_1+E_2)\left|\theta\big(w\delta_n^{2r}g^{(2r)};x\big)\right|.
\end{aligned}$$

Using Cauchy's integral inequality [25, 26] and Lemma 2.1 yields

$$\begin{aligned}
E_1 &= \frac{w(x)n^r}{(2r-1)!}\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{(u-x)^{2r}}{w(u)}\,\mathrm{d}u \\
&\leq \frac{n^r}{(2r-1)!}\left[\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} (u-x)^{4r}\,\mathrm{d}u\right]^{1/2} \\
&\quad\times\left[\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{w^2(x)}{w^2(u)}\,\mathrm{d}u\right]^{1/2} \\
&\leq \frac{n^r}{(2r-1)!}\frac{C}{n^r}\delta_n^{2r}(x) = \frac{C}{n^r}
\end{aligned}$$

and

$$\begin{aligned}
E_2 &= \frac{w(x)n^r}{(2r-1)!}\sum_{k=0}^{\infty} v_{n,k}(x)(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)} \frac{(u-x)^{2r}}{w(x)}\,\mathrm{d}u \\
&= \frac{n^r}{(2r-1)!}\tilde{V}_n\big((u-x)^{2r};x\big) \leq \frac{C}{n^r}.
\end{aligned}$$

Therefore we have

$$\left|w(x)\tilde{V}_n\big(R_{2r}(g;u,x);x\big)\right| \leq \frac{C}{n^r}\left|\theta\big(w\delta_n^{2r}g^{(2r)};x\big)\right|$$

and

$$\begin{aligned}
\left\|w\tilde{V}_n\big(R_{2r}(g;\cdot,x);x\big)\right\|_{\Phi[0,\frac{1}{n})} &\leq \frac{C}{n^r}\left\|\theta\big(w\delta_n^{2r}g^{(2r)}\big)\right\|_{\Phi[0,\frac{1}{n})} \\
&\leq \frac{C}{n^r}\left\|w\delta_n^{2r}g^{(2r)}\right\|_{\Phi[0,\frac{1}{n})}.
\end{aligned} \tag{2.4}$$

Hence, by virtue of (2.3) and (2.4), we derive

$$\left\|w\tilde{V}_n\big(R_{2r}(g;\cdot,x);x\big)\right\|_{\Phi[0,\infty)} \leq \frac{C}{n^r}\left\|w\delta_n^{2r}g^{(2r)}\right\|_{\Phi[0,\infty)},$$

and, consequently,

$$\begin{aligned}
\left\|wL_{n,r}\big(R_{2r}(g;\cdot,x);x\big)\right\|_{\Phi} &\leq \sum_{i=0}^{2r-1}\left\|c_i(n,r)w\tilde{V}_{n_i}\big(R_{2r}(g;\cdot,x);x\big)\right\|_{\Phi} \\
&\leq \frac{C}{n^r}\sum_{i=0}^{2r-1}\left|c_i(n,r)\right|\left\|w\delta_n^{2r}g^{(2r)}\right\|_{\Phi} \leq \frac{C}{n^r}\left\|w\delta_n^{2r}g^{(2r)}\right\|_{\Phi}.
\end{aligned}$$

Combining this inequality with (1.2), (2.2), and Lemma 2.4 results in

$$\begin{aligned}\|w(L_{n,r}(f)-f)\|_{\Phi} &\le \|w(L_{n,r}(f-g)-(f-g))\|_{\Phi}+\|w(L_{n,r}(g)-g)\|_{\Phi}\\ &\le C\|w(f-g)\|_{\Phi}+\frac{C}{n^r}\|w\delta_n^{2r}g^{(2r)}\|_{\Phi}\le C\omega_{2r,\varphi}\left(f,\frac{1}{\sqrt{n}}\right)_{w,\Phi}.\end{aligned}$$

The proof of the direct theorem is complete. □

3 Proofs of the inverse and equivalence theorems

We first prepare several lemmas for proving Theorems 1.2 and 1.3.

Lemma 3.1 *If $f\in L_{\Phi}^{*}[0,\infty)$, $n\ge 2r$, $a,b\in\mathbb{N}$, and $0\le a,b<n-1$, then*

$$\|w\varphi^{2r}L_{n,r}^{(2r)}(f)\|_{\Phi}\le Cn^r\|wf\|_{\Phi}.$$

Proof From [7, Eq. (16)] it follows that

$$\tilde{V}_n^{(2r)}(f;x)=\frac{(n+2r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\Delta^{2r}a_k(n-1)v_{n+2r,k}(x),$$

where

$$a_k(n-1)=(n-1)\int_{k/(n-1)}^{(k+1)/(n-1)}f(u)\,\mathrm{d}u,\qquad \Delta a_k=a_{k+1}-a_k,\qquad \Delta^m a_k=\Delta(\Delta^{m-1}a_k).$$

By Lemma 2.3 it follows that

$$\begin{aligned}&w(x)\varphi^{2r}(x)\tilde{V}_n^{(2r)}(f;x)\\ &\quad=\frac{(n+2r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\Delta^{2r}a_k(n-1)v_{n+2r,k}(x)x^r(1+x)^r w(x)\\ &\quad\le Cn^r\sum_{k=0}^{\infty}\left|\sum_{i=0}^{2r}\binom{2r}{i}(-1)^i a_{k+(2r-i)}(n-1)\right|v_{n,k+r}(x)w(x)\\ &\quad\le Cn^r\sum_{i=0}^{2r}\binom{2r}{i}\sum_{k=0}^{\infty}v_{n,k+r}(x)w(x)(n-1)\int_{(k+2r-i)/(n-1)}^{(k+1+2r-i)/(n-1)}\frac{w(u)|f(u)|}{w(u)}\,\mathrm{d}u\\ &\quad\le Cn^r\sum_{i=0}^{2r}\binom{2r}{i}\sum_{k=0}^{\infty}v_{n,k+r}(x)2^{|b|+1}\left[\left(\frac{(n-1)x}{k+2r-i+1}\right)^a\right.\\ &\qquad\left.+\left(\frac{(n-1)x}{k+2r-i+1}\right)^b\right](n-1)\int_{(k+2r-i)/(n-1)}^{(k+1+2r-i)/(n-1)}w(u)|f(u)|\,\mathrm{d}u\\ &\quad\triangleq\sum_{i=0}^{2r}\binom{2r}{i}(F_1+F_2),\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}
F_1 &= \sum_{k=0}^{\infty} v_{n,k+r}(x)\left[\frac{(n-1)x}{k+2r-i+1}\right]^a (n-1)\int_{(k+2r-i)/(n-1)}^{(k+1+2r-i)/(n-1)} Cn^r w(u)\left|f(u)\right| \mathrm{d}u \\
&\leq \sum_{k=0}^{\infty} v_{n-a,k+a+r}(x)(n-1)\int_{(k+2r-i)/(n-1)}^{(k+1+2r-i)/(n-1)} Cn^r w(u)\left|f(u)\right| \mathrm{d}u,
\end{aligned}$$

$$\begin{aligned}
\|F_1\|_{\Phi} &\leq 2\|F_1\|_{(\Phi)} \\
&\leq 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^{\infty} \Phi\Biggl(\sum_{k=0}^{\infty} v_{n-a,k+a+r}(x)(n-1) \\
&\qquad \times \int_{(k+2r-i)/(n-1)}^{(k+1+2r-i)/(n-1)} Cn^r w(u)\frac{|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \leq 1\Biggr\} \\
&= 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^{\infty} \Phi\Biggl(\sum_{k=a+r}^{\infty} v_{n-a,k}(x)(n-1) \\
&\qquad \times \int_{(k+r-a-i)/(n-1)}^{(k+1+r-a-i)/(n-1)} Cn^r w(u)\frac{|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \leq 1\Biggr\} \\
&\leq 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^{\infty} \Phi\Biggl(\sum_{k=a+r}^{\infty} v_{n-a,k}(x)(n-1)\int_{(k+r-a-i)/(n-1)}^{(k+1+r-a-i)/(n-1)} Cn^r w(u)\frac{|f(u)|}{\lambda}\,\mathrm{d}u \\
&\qquad + \sum_{k=0}^{a+r-1} v_{n-a,k}(x)(n-1)\int_{\max\{0,\frac{k+r-a-i}{n-1}\}}^{\max\{0,\frac{k+1+r-a-i}{n-1}\}} Cn^r w(u)\frac{|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \leq 1\Biggr\} \\
&= 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^{\infty} \Phi\Biggl(\sum_{k=0}^{\infty} v_{n-a,k}(x)(n-1) \\
&\qquad \times \int_{\max\{0,\frac{k+r-a-i}{n-1}\}}^{\max\{0,\frac{k+1+r-a-i}{n-1}\}} Cn^r w(u)\frac{|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \leq 1\Biggr\} \\
&\leq 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^{\infty} \sum_{k=0}^{\infty} v_{n-a,k}(x) \\
&\qquad \times \Phi\Biggl((n-1)\int_{\max\{0,\frac{k+r-a-i}{n-1}\}}^{\max\{0,\frac{k+1+r-a-i}{n-1}\}} Cn^r w(u)\frac{|f(u)|}{\lambda}\,\mathrm{d}u\Biggr)\,\mathrm{d}x \leq 1\Biggr\} \\
&\leq 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^{\infty} \sum_{k=a+i-r}^{\infty} v_{n-a,k}(x)(n-1) \\
&\qquad \times \int_{(k+r-a-i)/(n-1)}^{(k+1+r-a-i)/(n-1)} \Phi\biggl(Cn^r w(u)\frac{|f(u)|}{\lambda}\biggr)\,\mathrm{d}u\,\mathrm{d}x \leq 1\Biggr\} \\
&\leq 2\inf_{\lambda>0}\Biggl\{\lambda : \sum_{k=a+i-r}^{\infty} \frac{n-1}{n-a-1}\int_{(k+r-a-i)/(n-1)}^{(k+1+r-a-i)/(n-1)} \Phi\biggl(Cn^r w(u)\frac{|f(u)|}{\lambda}\biggr)\,\mathrm{d}u \leq 1\Biggr\} \\
&\leq 2\inf_{\lambda>0}\Biggl\{\lambda : \sum_{k=0}^{\infty} \int_{k/(n-1)}^{(k+1)/(n-1)} \Phi\biggl(Cn^r w(u)\frac{|f(u)|}{\lambda}\biggr)\,\mathrm{d}u \leq 1\Biggr\} \\
&\leq Cn^r\|wf\|_{(\Phi)} \leq Cn^r\|wf\|_{\Phi},
\end{aligned}$$

and, similarly,

$$\|F_2\|_\Phi \le Cn^r \|wf\|_\Phi. \tag{3.2}$$

Employing inequalities between (3.1) and (3.2) yields

$$\begin{aligned}
\left\|w\varphi^{2r}(x)\tilde{V}_n^{(2r)}(f)\right\|_\Phi &\le \left\|\sum_{i=0}^{2r}\binom{2r}{i}(F_1+F_2)\right\|_\Phi \\
&\le \left\|\sum_{i=0}^{2r}\binom{2r}{i}F_1\right\|_\Phi + \left\|\sum_{i=0}^{2r}\binom{2r}{i}F_2\right\|_\Phi \\
&\le \sum_{i=0}^{2r}\binom{2r}{i}\left(\|F_1\|_\Phi + \|F_2\|_\Phi\right) \le Cn^r\|wf\|_\Phi.
\end{aligned}$$

Combining this inequality with (1.5) and (1.6) results in

$$\left\|w\varphi^{2r}L_{n,r}^{(2r)}(f)\right\|_\Phi = \left\|\sum_{i=0}^{2r-1}c_i(n,r)w\varphi^{2r}\tilde{V}_{n_i}^{(2r)}(f,x)\right\|_\Phi \le Cn^r\|wf\|_\Phi.$$

The proof of Lemma 3.1 is complete. □

Lemma 3.2 *Let $f \in L_\Phi^*[0,\infty)$, $n \ge 2r$, $a,b \in \mathbb{N}$, and $0 \le a,b < n-1$. Then*

$$\left\|w\varphi^{2r}L_{n,r}^{(2r)}(f)\right\|_\Phi \le C\left\|w\varphi^{2r}f^{(2r)}\right\|_\Phi.$$

Proof Since

$$\begin{aligned}
&\varphi^{2r}(x)\tilde{V}_n^{(2r)}(f;x) \\
&\quad = \frac{(n+2r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\Delta^{2r}a_k(n-1)v_{n+2r,k}(x)x^r(1+x)^r \\
&\quad = \frac{(n+2r-1)!}{(n-1)!}\sum_{k=0}^{\infty}\Delta^{2r}a_k(n-1)v_{n+2r,k}(x)\sum_{i=0}^{r}\binom{r}{i}x^{2r-i} \\
&\quad = \sum_{i=0}^{r}\binom{r}{i}\sum_{k=0}^{\infty}\Delta^{2r}a_k(n-1)v_{n+i,k+2r-i}(x)\frac{(k+2r-i)!(n+i-1)!}{k!(n-1)!} \\
&\quad = \sum_{i=0}^{r}\binom{r}{i}\sum_{k=0}^{\infty}\frac{(k+2r-i)!(n+i-1)!}{k!(n-1)!}(n-1)\int_0^{1/(n-1)}\int_0^{1/(n-1)}\cdots \\
&\qquad \int_0^{1/(n-1)}f^{(2r)}\left(\frac{k}{n-1}+t_1+t_2+\cdots+t_{2r}\right)\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_{2r}v_{n+i,k+2r-i}(x) \\
&\quad \le C\sum_{k=0}^{\infty}(n-1)^{2r}\int_0^{1/(n-1)}\int_0^{1/(n-1)}\cdots \\
&\qquad \int_0^{1/(n-1)}\sum_{i=0}^{r}\binom{r}{i}\left(\frac{k}{n-1}+t_1+t_2+\cdots+t_{2r}\right)^{2r-i}
\end{aligned}$$

$$
\begin{aligned}
&\times \left|f^{(2r)}\left(\frac{k}{n-1}+t_1+t_2+\cdots+t_{2r}\right)\right| \mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_{2r} v_{n+i,k+2r-i}(x) \\
&\leq C\sum_{k=0}^{\infty}(n-1)^{2r}\int_0^{1/(n-1)}\int_0^{1/(n-1)}\cdots \\
&\int_0^{1/(n-1)}\varphi^{2r}\left(\frac{k}{n-1}+t_1+t_2+\cdots+t_{2r}\right) \\
&\times \left|f^{(2r)}\left(\frac{k}{n-1}+t_1+t_2+\cdots+t_{2r}\right)\right| \mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_{2r}\sum_{i=0}^{r} v_{n+i,k+2r-i}(x),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&w(x)\varphi^{2r}(x)\tilde{V}_n^{(2r)}(f;x) \\
&\leq C\sum_{i=0}^{r}\sum_{k=0}^{\infty} v_{n+i,k+2r-i}(x)(n-1)^{2r}\int_0^{1/(n-1)}\cdots \\
&\int_0^{1/(n-1)} w\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\varphi^{2r}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right) \\
&\times \left|f^{(2r)}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\right| \frac{w(x)}{w(\frac{k}{n-1}+t_1+\cdots+t_{2r})}\,\mathrm{d}t_1\cdots\mathrm{d}t_{2r} \\
&\leq C\sum_{i=0}^{r}\sum_{k=0}^{\infty} v_{n+i,k+2r-i}(x)(n-1)^{2r}2^{b+1}\left[\left(\frac{n-1}{k+1}\right)^a x^a+\left(\frac{n-1}{k+1}\right)^b x^b\right] \\
&\times \int_0^{1/(n-1)}\cdots\int_0^{1/(n-1)} w\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\varphi^{2r}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right) \\
&\times \left|f^{(2r)}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\right| \mathrm{d}t_1\cdots\mathrm{d}t_{2r} \triangleq G_1+G_2,
\end{aligned}
$$

where

$$
\begin{aligned}
G_1 &= C\sum_{i=0}^{r}\sum_{k=0}^{\infty}(n-1)^{2r} v_{n+i,k+2r-i}(x)2^{b+1}\left(\frac{n-1}{k+1}\right)^a x^a\int_0^{1/(n-1)}\cdots\int_0^{1/(n-1)} \\
&\times w\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\varphi^{2r}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right) \\
&\times \left|f^{(2r)}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\right| \mathrm{d}t_1\cdots\mathrm{d}t_{2r} \\
&\leq C\sum_{i=0}^{r}\sum_{k=0}^{\infty} v_{n+i-a,k+2r+a-i}(x)(n-1)^{2r}\int_0^{1/(n-1)}\cdots \\
&\cdots\int_0^{1/(n-1)} w\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right) \\
&\times \varphi^{2r}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\left|f^{(2r)}\left(\frac{k}{n-1}+t_1+\cdots+t_{2r}\right)\right| \mathrm{d}t_1\cdots\mathrm{d}t_{2r},
\end{aligned}
$$

and, by Jensen's inequality,

$$
\begin{aligned}
&\|G_1\|_\Phi \le 2\|G_1\|_{(\Phi)} \\
&\quad \le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \Phi\Biggl(C\sum_{i=0}^{r}\sum_{k=0}^{\infty} v_{n+i-a,k+a+2r-i}(x)(n-1)^{2r}\int_0^{1/(n-1)} \mathrm{d}t_1 \cdots \\
&\qquad \int_0^{1/(n-1)} \mathrm{d}t_{2r-1}\int_{k/(n-1)}^{(k+1)/(n-1)} w(t_1+\cdots+t_{2r-1}+t_{2r}) \\
&\qquad \times \varphi^{2r}(t_1+\cdots+t_{2r-1}+t_{2r})\frac{|f^{(2r)}(t_1+\cdots+t_{2r-1}+t_{2r})|}{\lambda}\,\mathrm{d}t_{2r}\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&\quad \le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \Phi\Biggl(\sum_{i=0}^{r}\Biggl[\sum_{k=2r+a-i}^{\infty} v_{n+i-a,k}(x)(n-1)^{2r}\int_0^{1/(n-1)} \mathrm{d}t_1 \cdots \\
&\qquad \int_0^{1/(n-1)} \mathrm{d}t_{2r-1}\int_{(k-2r-a+i)/(n-1)}^{(k-2r-a+1+i)/(n-1)} w(t_1+\cdots+t_{2r-1}+t_{2r}) \\
&\qquad \times \varphi^{2r}(t_1+\cdots+t_{2r-1}+t_{2r})\frac{C|f^{(2r)}(t_1+\cdots+t_{2r-1}+t_{2r})|}{\lambda}\,\mathrm{d}t_{2r} \\
&\qquad + \sum_{k=0}^{2r+a-i-1} v_{n+i-a,k}(x)(n-1)^{2r}\int_0^{1/(n-1)} \mathrm{d}t_1 \cdots \int_0^{1/(n-1)} \mathrm{d}t_{2r-1} \\
&\qquad \times \int_{\max\{0,\frac{k-2r-a+i}{n-1}\}}^{\max\{0,\frac{k-2r-a+1+i}{n-1}\}} w(t_1+\cdots+t_{2r-1}+t_{2r})\varphi^{2r}(t_1+\cdots+t_{2r-1}+t_{2r}) \\
&\qquad \times \frac{C|f^{(2r)}(t_1+\cdots+t_{2r-1}+t_{2r})|}{\lambda}\,\mathrm{d}t_{2r}\Biggr]\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&\quad \le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}\sum_{k=0}^{\infty} v_{n+i-a,k}(x)(n-1)^{2r-1}\int_0^{1/(n-1)} \mathrm{d}t_1 \cdots \\
&\qquad \int_0^{1/(n-1)} \mathrm{d}t_{2r-1}\Phi\Biggl((n-1)\int_{\max\{0,\frac{k-2r-a+i}{n-1}\}}^{\max\{0,\frac{k-2r-a+1+i}{n-1}\}} C2^r w(t_1+\cdots+t_{2r-1}+t_{2r}) \\
&\qquad \times \varphi^{2r}(t_1+\cdots+t_{2r-1}+t_{2r})\frac{|f^{(2r)}(t_1+\cdots+t_{2r-1}+t_{2r})|}{\lambda}\,\mathrm{d}t_{2r}\Biggr)\,\mathrm{d}x \le 1\Biggr\} \\
&\quad \le 2\inf_{\lambda>0}\Biggl\{\lambda : \int_0^\infty \frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}\sum_{k=0}^{\infty} v_{n+i-a,k+2r+a-i}(x)(n-1)^{2r}\int_0^{1/(n-1)} \mathrm{d}t_1 \cdots \\
&\qquad \int_0^{1/(n-1)} \mathrm{d}t_{2r-1}\int_{k/(n-1)}^{(k+1)/(n-1)} \Phi\Biggl(Cw(t_1+\cdots+t_{2r-1}+t_{2r}) \\
&\qquad \times \varphi^{2r}(t_1+\cdots+t_{2r-1}+t_{2r})\frac{|f^{(2r)}(t_1+\cdots+t_{2r-1}+t_{2r})|}{\lambda}\Biggr)\,\mathrm{d}t_{2r}\,\mathrm{d}x \le 1\Biggr\} \\
&\quad \le 2\inf_{\lambda>0}\Biggl\{\lambda : \frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}C_1\sum_{k=0}^{\infty}(n-1)^{2r-1}\int_0^{1/(n-1)} \mathrm{d}t_1 \cdots \int_0^{1/(n-1)} \mathrm{d}t_{2r-1} \\
&\qquad \times \int_{k/(n-1)}^{(k+1)/(n-1)} \Phi\Biggl(Cw(t_1+\cdots+t_{2r-1}+t_{2r})\varphi^{2r}(t_1+\cdots+t_{2r-1}+t_{2r})
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{|f^{(2r)}(t_1+\cdots+t_{2r-1}+t_{2r})|}{\lambda}\Bigg)\,\mathrm{d}t_{2r} \le 1\Bigg\} \\
&\le 2\inf_{\lambda>0}\Bigg\{\lambda : \sum_{k=0}^{\infty}(n-1)^{2r-1}\int_0^{1/(n-1)}\mathrm{d}t_1\cdots\int_0^{1/(n-1)}\mathrm{d}t_{2r-1} \\
&\times \int_{k/(n-1)}^{(k+1)/(n-1)}\Phi\Bigg(w(t_1+\cdots+t_{2r-1}+t_{2r})\varphi^{2r}(t_1+\cdots+t_{2r-1}+t_{2r})C \\
&\times C_1\frac{|f^{(2r)}(t_1+\cdots+t_{2r-1}+t_{2r})|}{\lambda}\Bigg)\,\mathrm{d}t_{2r} \le 1\Bigg\} \\
&\le 2\inf_{\lambda>0}\Bigg\{\lambda : \int_0^{\infty}\Phi\Bigg(Cw(t)\varphi^{2r}(t)\frac{|f^{(2r)}(t)|}{\lambda}\Bigg)\,\mathrm{d}t \le 1\Bigg\} \\
&\le C\big\|w\varphi^{2r}f^{(2r)}\big\|_{\Phi},
\end{aligned}
$$

where $C_1 \ge 1$. Similarly, we have

$$
\|G_2\|_{\Phi} \le C\big\|w\varphi^{2r}f^{(2r)}\big\|_{\Phi}.
$$

Consequently, it follows that

$$
\big\|w\varphi^{2r}\tilde{V}_n^{(2r)}(f)\big\|_{\Phi} \le C\big\|w\varphi^{2r}f^{(2r)}\big\|_{\Phi}.
$$

Combining this inequality with (1.5) and (1.6) yields

$$
\big\|w\varphi^{2r}L_{n,r}^{(2r)}(f)\big\|_{\Phi} = \Bigg\|\sum_{i=0}^{2r-1}c_i(n,r)w\varphi^{2r}\tilde{V}_{n_i}^{(2r)}(f)\Bigg\|_{\Phi} \le C\big\|w\varphi^{2r}f^{(2r)}\big\|_{\Phi}.
$$

The proof of Lemma 3.2 is complete. □

We now in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2 From Lemmas 3.1 and 3.2 and [27, Theorem 2.2] we obtain

$$
K_{2r,\varphi}\Big(f,\frac{1}{n^{r/2}}\Big)_{w,\Phi} \le \frac{C}{n^r}\sum_{k=1}^{n}k^{r-1}\big\|w\big(L_{k,r}(f)-f\big)\big\|_{\Phi}.
$$

Application of inequality (1.3) concludes the inverse theorem. □

Proof of Theorem 1.3 Using the so-called order function $\psi(t) = t^{\alpha}|\ln t|^{\beta}e^{|\ln t|^{\gamma}}$ for $0 < \alpha < 1$, $\beta \in \mathbb{R}$, and $\gamma < 1$ and combining Theorems 1.1 and 1.2 conclude the equivalence theorem. □

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Authors' contributions
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