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Multivariate Bernstein inequalities for entire functions of exponential type in $L^p(\mathbb{R}^n)$ $(0 < p < 1)$

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Abstract

In (Rahman and Schmeisser in Trans. Amer. Math. Soc. 320: 91–103, 1990), the authors prove that the classical Bernstein inequality also holds for $0 < p \leq 1$. We extend their result for a differential operator induced by polynomials and find the several equivalent conditions to the Paley–Wiener theorem. As applications of the results, we also derive the Paley–Wiener type theorems for some special compact sets generated by number sequences, generated by polynomial, convex compact sets, in which we show that the Bernstein type inequalities have concrete upper bounds.

MSC: 46F12

Keywords: Bernstein's inequality; Paley–Wiener theorem; Fourier transform

1 Introduction and main theorems

Bernstein's inequality began with the problem of an estimate of an upper bound for derivatives of functions on the real line in 1912 ([7]). A generalization for the classical Bernstein inequality can be found in [8]: For any polynomial g of degree k,

$$\left\|g^{(m)}\right\|_p \leq k^m \|g\|_p, \tag{1}$$

where $1 \leq p \leq \infty$. The inequality is very useful in the field of approximation theory and differential equations. Even though there are innumerably many splendid studies related to Bernstein's inequality after [7] appeared, we only introduce directly related research results with this paper.

For example, the author in [1, p. 144, Theorem 3], derives that

$$\left\|(\sin\alpha)f' - \sigma(\cos\alpha)f\right\|_p \leq \sigma\|f\|_p \quad (p \geq 1),$$

for all real α, where f is an entire function of exponential type σ belonging to $L^p(\mathbb{R})$. As another result of the same kind, for real valued functions the authors show that

$$\left\|\left(\left(f'\right)^2 + \sigma^2(f)^2\right)^{1/2}\right\|_p \leq 2\sigma C_p\|f\|_p,$$

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where $C_p^{-p} = \frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\theta}|^p \, d\theta < 1$ ([12]). As a consequence of (1), we see that $\limsup_{m\to\infty} \|f^{(m)}\|_p^{1/m} \le \sup\{|\xi| : \xi \in \operatorname{supp}\hat{f}\}$, where $\hat{f}$ is the Fourier transform of f. In [2], the author proves that this inequality becomes the equality and also he derives a radial spectral formula in the following: If $1 \le p \le \infty$ and $f^{(m)} \in L^p(\mathbb{R})$ $(m = 0, 1, 2, \ldots)$, then there always exists the limit of $\|f^{(m)}\|_p^{1/m}$ and

$$\lim_{m\to\infty} \left\|f^{(m)}\right\|_p^{1/m} = \sup\left\{|\xi| : \xi \in \operatorname{supp}\hat{f}\right\}.$$

In particular, $\operatorname{supp}\hat{f} \subset [-\sigma, \sigma]$ if and only if $\limsup_{m\to\infty} \|f^{(m)}\|_p^{1/m} \le \sigma$.

The classical Paley–Wiener theorem gives a characterization of L^2-functions with their Fourier transforms compactly supported (the L^2 band-limited functions): Let $\sigma > 0$ and let f be an entire function of exponential type σ. Then $f \in L^2(\mathbb{R}^n)$ if and only if there exists $g \in L^2(\mathbb{R}^n)$ vanishing a.e. outside $[-\sigma, \sigma]$ such that $f = \hat{g}$ ([11]). As the generalized results for the Paley–Wiener theorem, we mention [13] and [9], in which the authors make an extension to the distribution supported in the closed ball and in convex compact, respectively.

In this paper, we focus on an extension of the inequality (1) to a differential operator for $0 < p < 1$ as a generalization of [12]. First, we establish necessary and sufficient conditions on the sequences of norm of derivatives of functions in $L_p(\mathbb{R}^n)$ such that their spectrum are contained in a fixed compact set in $\mathbb{R}^n$, refer to the main results of Theorems A, B, and C. In Theorem B and C, we provide the behavior of sequence of higher order derivatives, direction derivatives for the class of entire function of exponential type belong to $L^p(\mathbb{R}^n)$ spaces $(0 < p < 1)$ and about three applications of Theorem A. This paper is organized as follows: Sects. 2, 3, and 4 have the proof of each of the main theorems. In the last section, we provide the Paley–Wiener theorem for some special compact sets.

For simplicity, we introduce some notations: We denote the support of f by $\operatorname{supp} f$, the set of nonnegative integers by $\mathbb{Z}_*$ (also, $\mathbb{R}_*$ means the collection of all nonnegative real numbers) and a differential operator by $P(D)$ induced from a polynomial $P(x)$ in $\mathbb{R}^n$, where $D = -i\partial/\partial x$. For a multi-index $\alpha \in \mathbb{Z}_*^n$, put $|\alpha| = \sum_{j=1}^n |\alpha_j|$ $(\alpha = (\alpha_1, \ldots, \alpha_n))$.

Let $K \subset \mathbb{R}^n$ be compact and let $\delta > 0$. We write K_δ, $K_{(\delta)}$ as the real δ-neighborhood of K, the complex δ-neighborhood of K, respectively, i.e., $K_\delta = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \le \delta\}$ and $K_{(\delta)} = \{z \in \mathbb{C}^n : \operatorname{dist}(z, K) \le \delta\}$. In addition, throughout this paper, we assume that the function $f \in L^p(\mathbb{R}^n)$ has the bounded spectral if there is no any comment, e.g., since this condition implies differentiability properties of a function f.

Theorem A *Let $0 < p < 1$ and let $K \subset \mathbb{R}^n$ be compact. Then $\operatorname{supp}\hat{f} \subset K$ if and only if for any $\delta > 0$ there exists a constant $C_{p,K,\delta}$ independent of f, P such that*

$$\left\|P(D)f\right\|_p \le C_{p,K,\delta}\Big(\sup_{z\in K_{(\delta)}} \left|P(z)\right|\Big)\|f\|_p \tag{2}$$

for any polynomial P.

Theorem B *Let $0 < p < 1$, $0 < \lambda < 1$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}_*^n$. Then the following statements are equivalent:*

(i) $\operatorname{supp}\hat{f} \subset [-\sigma_1, \sigma_1] \times [-\sigma_2, \sigma_2] \times \cdots \times [-\sigma_n, \sigma_n]$.

(ii) *For any $\alpha \in \mathbb{Z}_*^n$, $\|D^\alpha f\|_p \le \sigma^\alpha \|f\|_p$.*

Let $\eta \in \mathbb{R}^n$ be on the unit sphere. Let us recall that the directional derivative of f at x along η is defined by

$$D_\eta f(x) = \sum_{j=1}^{n} \eta_j \frac{\partial f}{\partial x_j}(x)$$

and the higher order directional derivative is defined by

$$D_\eta^m f(x) = D_\eta D_\eta^{m-1} f,$$

where $m \in \mathbb{Z}_*$.

Theorem C *Let* $0 < p < 1$, $0 < \lambda < 1$, *and* $r > 0$. *Put* $K_{\eta,r} = \{\xi \in \mathbb{R}^n : |\eta\xi| \leq r\}$. *Then two statements are equivalent*:

(i) $\operatorname{supp} \hat{f} \subset K_{\eta,r}$.

(ii) *For any* $m \in \mathbb{Z}_*$,

$$\left\| D_\eta^m f \right\|_p \leq r^m \|f\|_p. \tag{3}$$

If we consider the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ consists of Schwartz functions on $\mathbb{R}^n$, then, for the space

$$\mathcal{E}_p(K) = \left\{f \in L^p\left(\mathbb{R}^n\right) \cap \mathcal{S}'\left(\mathbb{R}^n\right) : \operatorname{supp} \hat{f} \subset K\right\} \quad (0 < p \leq 1),$$

the operator norms of $P(D)$ on $\mathcal{E}_p(K)$

$$\left\| P(D) \right\|_{\mathcal{E}_p(K) \to \mathcal{E}_p(K)} = \sup_{f \in \mathcal{E}_p(K), \|f\|_p \leq 1} \left\| P(D)f \right\|_p$$

are also deduced from Theorems A, B, and C.

2 Proof of Theorem A

We start with the Nikolskii inequality. This is useful to prove the necessity of Theorem A.

Proposition 2.1 (Nikolskii inequality [10]) *Let* $0 < p < q \leq \infty$ *and let* K *be compact. Then*

$$\|f\|_q \leq C_{p,q,K} \|f\|_p,$$

for all $f \in \mathcal{E}_p(K)$.

Proof of necessity of Theorem A Fix $0 < \delta < 1$ and consider the bump function ϕ in the class of the test-functions, with compact support, defined by $\phi(\xi) = 1$ if $\xi \in K_{\delta/4}$; and $\phi(\xi) = 0$ if $\xi \notin K_{\delta/2}$. Let $f \in \mathcal{E}_p(K)$. Since $\widehat{P(D)f} = P(\xi)\hat{f}(\xi)$, we have $\widehat{P(D)f} = P(\xi)\hat{f}(\xi) = \phi(\xi)P(\xi)\hat{f}(\xi)$. By the inversion formula for a convolution,

$$P(D)f = \mathcal{F}^{-1}(\phi P) * f, \tag{4}$$

where $\mathcal{F}^{-1}(\phi P)$ means the inverse Fourier transform of ϕP. (In this proof, we write the Fourier transform of f by $\mathcal{F}(f)$ instead of $\hat{f}$.) Define $\check{f}(x) = f(-x)$ and $f_y(x) = f(x+y)$ is the translation of f by y, e.g., $\check{f}_y(x) = f(y-x)$.

It follows that

$$\begin{aligned} \operatorname{supp}\mathcal{F}\bigl(\mathcal{F}^{-1}(\phi P)\check{f}_x\bigr) &\subset \operatorname{supp}(\phi P) + \operatorname{supp}\mathcal{F}(\check{f}_x) \\ &= \operatorname{supp}(\phi P) - \operatorname{supp}\mathcal{F}(f) \\ &\subset \operatorname{supp}\phi - \operatorname{supp}\mathcal{F}(f) \\ &\subset K_{\delta/2} - K \subset K_1 - K \end{aligned}$$

for any x. By Proposition 2.1 with $0 < p < 1$,

$$\bigl\|\mathcal{F}^{-1}(\phi P)\check{f}_x\bigr\|_1 \le C_{p,K}\bigl\|\mathcal{F}^{-1}(\phi P)\check{f}_x\bigr\|_p \tag{5}$$

for any x. By (4) and by (5),

$$\bigl|P(D)f(x)\bigr|^p \le C_{p,K}^p \int_{\mathbb{R}^n} \bigl|\mathcal{F}^{-1}(\phi P)(y)\bigr|^p \bigl|f(x-y)\bigr|^p \,dy$$

for any $x \in \mathbb{R}^n$. Consequently,

$$\int_{\mathbb{R}^n} \bigl|P(D)f(x)\bigr|^p \,dx \le C_{p,K}^p \int_{\mathbb{R}^n} \biggl(\int_{\mathbb{R}^n} \bigl|\mathcal{F}^{-1}\bigl(\phi(\xi)P(\xi)\bigr)(y)\bigr|^p \bigl|f(x-y)\bigr|^p \,dy\biggr) \,dx.$$

By Fubini's theorem, hence

$$\begin{aligned} \bigl\|P(D)f\bigr\|_p &\le C_{p,K}\bigl\|\mathcal{F}^{-1}(\phi P)\bigr\|_p \|f\|_p \\ &= (2\pi)^{-n} C_{p,K}\bigl\|\mathcal{F}(\phi P)\bigr\|_p \|f\|_p \\ &= (2\pi)^{-n} C_{p,K}\|\Phi\|_p \|f\|_p, \end{aligned} \tag{6}$$

where $\Phi(x) = \mathcal{F}(\phi P)(x)$.

Now by estimating Φ properly, we will complete the proof. Put $p' = \lfloor \frac{1}{p} \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the floor function. Let $\beta \in \mathbb{Z}_*^n$ such that $\beta \le (p', \ldots, p') = \mathbf{p}'$, say, here the inequality means that every component of β is less than or equal to $\mathbf{p}'$ and $\beta!$ means a multi-index factorial. Since

$$\begin{aligned} \sup_x \bigl|x^\beta \Phi(x)\bigr| &= \sup_x \biggl|\int_{\mathbb{R}^n} D^\beta\bigl[\phi(\xi)P(\xi)\bigr] e^{ix\xi} \,d\xi\biggr| \\ &\le \int_{K_{\delta/2}} \bigl|D^\beta\bigl[\phi(\xi)P(\xi)\bigr]\bigr| \,d\xi = I, \quad \text{say}, \end{aligned} \tag{7}$$

the Leibniz rule yields

$$
\begin{aligned}
I &\le \int_{K_{\delta/2}} \left| \sum_{\gamma \le \beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} D^{\gamma}\phi(\xi) D^{\beta-\gamma} P(\xi) \right| d\xi \\
&\le 2^{2n} \sum_{\gamma \le \beta} \sup_{x \in K_{\delta/2}} \left| D^{\beta-\gamma} P(x) \right| \int_{K_{\delta/2}} \left| D^{\gamma}\phi(\xi) \right| d\xi \\
&\le 2^{2n} C_{p,K,\delta} \max_{\alpha \le \mathbf{p}'} \sup_{x \in K_{\delta/2}} \left| D^{\alpha} P(x) \right|, \qquad (8)
\end{aligned}
$$

where $C_{p,K,\delta} = \sum_{\gamma \le \mathbf{p}'} \int_{K_{\delta/2}} |D^{\gamma}\phi(\xi)|\, d\xi$.

By regarding $D^{\alpha}P(x)$ as a complex holomorphic polynomial and using Cauchy's integral formula, we can estimate its maximum modulus in $K_{\delta/2}$ as the maximum modulus of $P(z)$ in $K_{(\delta)}$. Thus there exists a constant $C_{K,\delta}$ which depends only on K, δ such that

$$
\sup_{K_{\delta/2}} \left| D^{\alpha} P(x) \right| \le C_{K,\delta} \sup_{K_{(\delta)}} \left| P(z) \right| \qquad (9)
$$

for any $\alpha \in \mathbb{Z}^n_*$ ($\alpha \le \mathbf{p}'$).

Combining (7), (8) with (9), we have

$$
\sup_{x} \left| x^{\beta} \Phi(x) \right| \le C'_{p,K,\delta} \sup_{K_{(\delta)}} \left| P(z) \right|, \qquad (10)
$$

where $C'_{p,K,\delta} = 2^{2n} C_{p,K,\delta} C_{K,\delta}$. From (10),

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \left| \Phi(x) \right|^p dx \\
&\quad \le \sup_{x} \left(\left(1+|x_1|\right)^{p'} \cdots \left(1+|x_n|\right)^{p'} \left| \Phi(x) \right| \right)^p \int_{\mathbb{R}^n} \frac{dx}{(1+|x_1|)^{p'p} \cdots (1+|x_n|)^{p'p}} \\
&\quad = C_p^p \sup_{x} \left(\left(1+|x_1|\right)^{p'} \cdots \left(1+|x_n|\right)^{p'} \left| \Phi(x) \right| \right)^p, \qquad (11)
\end{aligned}
$$

where $C_p^p = \int_{\mathbb{R}^n} \frac{dx}{(1+|x_1|)^{p'p} \cdots (1+|x_n|)^{p'p}} < \infty$.

Hence, by (11) and according to (10),

$$
\|\Phi\|_p \le 2^{p'-1+n} C_p C'_{p,K,\delta} \sup_{z \in K_{(\delta)}} \left| P(z) \right|. \qquad (12)
$$

By (6) with (12), the proof is complete. □

The following lemma is useful for the sufficiency of Theorem A.

Lemma 2.2 ([3]) *If* $\operatorname{supp}\hat{f}$ *is compact, then*

$$
\lim_{m \to \infty} \left\| P^m(D) f \right\|_1^{1/m} = \sup_{x \in \operatorname{supp}\hat{f}} \left| P(x) \right|.
$$

We prove the sufficiency of Theorem A by contradiction.

Proof of sufficiency of Theorem A Assume that there exists $x_0 \in H$ with $x_0 \notin K$, where $H = \operatorname{supp}\hat{f}$. We consider a polynomial $P(x) = t_0 - |x - x_0|^2$, where $t_0 = \sup_{x\in K} |x - x_0|^2 > 0$ and apply (2) with P^m for a positive integer m. In addition, by Proposition 2.1,

$$\left\|P^m(D)f\right\|_1^{1/m} \le \left(C_{p,H}C_{p,K,\delta}\|f\|_p\right)^{1/m} \sup_{z\in K_{(\delta)}} \left|P(z)\right|.$$

By applying the limsup on both sides, we have

$$\limsup_{m\to\infty}\left\|P^m(D)f\right\|_1^{1/m} \le \sup_{z\in K_{(\delta)}} \left|P(z)\right|.$$

Now letting $\delta \searrow 0$, we obtain

$$\limsup_{m\to\infty}\left\|P^m(D)f\right\|_1^{1/m} \le \sup_{x\in K}\left|P(x)\right|. \tag{13}$$

By Lemma 2.2 and by (13), we have the following contradiction:

$$t_0 = \left|P(x_0)\right| \le \sup_{x\in K}\left|P(x)\right| = \sup_{x\in K}\left(t_0 - |x - x_0|^2\right) \lneq t_0,$$

where the last inequality comes from the fact that $x_0 \notin K$. Therefore, the proof is complete. □

3 Proof of Theorem B

We recall the following lemma.

Lemma 3.1 ([6, Theorem 6]) *Let* $0 < p \le 1$ *and let* $f \in L^p(\mathbb{R}^n)$. *Then* $\lim_{\lambda\nearrow 1} \|f - {}_\lambda f\|_p = 0$, *where* ${}_\lambda f(x) = f(\lambda x)$ *denotes the dilation of* f *by* λ.

Proof of Theorem B Suppose (ii). According to (ii) with Proposition 2.1,

$$\limsup_{|\alpha|\to\infty}\left(\left\|D^\alpha f\right\|_1/\sigma^\alpha\right)^{1/|\alpha|} \le 1. \tag{14}$$

By Lemma 2.2,

$$\sup_{\xi\in\operatorname{supp}\hat{f}} \left|\xi^\beta\right| \le \limsup_{m\to\infty}\left\|D^{m\beta}f\right\|_1^{1/m} \tag{15}$$

for all $\beta \in \mathbb{Z}_*^n$.

Combining (14) and (15), we have

$$\sup_{\xi\in\operatorname{supp}\hat{f}} \left|\xi^\beta\right| \le \limsup_{m\to\infty}\left\|D^{m\beta}f\right\|_1^{1/m} \le \sigma^\beta$$

and this gives $\operatorname{supp}\hat{f} \subset [-\sigma_1,\sigma_1] \times [-\sigma_2,\sigma_2] \times \cdots \times [-\sigma_n,\sigma_n]$.

Next, (i) implies (ii), which follows from the Bernstein inequality for $0 < p \le 1$ ([12]). Therefore, the proof is complete. □

Remark 1 (1) Theorem B shows an L^p-boundedness of derivatives. In fact, we prove that the L^p-boundedness is equivalent to the following vanishing property:

$$\lim_{|\alpha|\to\infty} \|D^\alpha f\|_p/\sigma^\alpha = 0. \tag{16}$$

First, we prove that (i) of Theorem B implies (16). Indeed, from the definition of the function ${}_\lambda f$, we have $\operatorname{supp}\widehat{{}_\lambda f} = \lambda \operatorname{supp}\hat{f}$. Then

$$\operatorname{supp}\widehat{{}_\lambda f} \subset [-\lambda\sigma_1, \lambda\sigma_1] \times [-\lambda\sigma_2, \lambda\sigma_2] \times \cdots \times [-\lambda\sigma_n, \lambda\sigma_n]. \tag{17}$$

Since $D^\alpha {}_\lambda f(x) = \lambda^{|\alpha|} D^\alpha f(\lambda x)$, by a change of variables,

$$\|D^\alpha {}_\lambda f\|_p = \lambda^{|\alpha|-n/p} \|D^\alpha f\|_p \tag{18}$$

for all $\alpha \in \mathbb{Z}_*^n$. By (17), $\operatorname{supp}\widehat{f - {}_\lambda f} \subset [-\sigma_1, \sigma_1] \times [-\sigma_2, \sigma_2] \times \cdots \times [-\sigma_n, \sigma_n]$. Thus by the Bernstein inequality for $0 < p < 1$ ([12]),

$$\|D^\alpha (f - {}_\lambda f)\|_p \le \sigma^\alpha \|f - {}_\lambda f\|_p. \tag{19}$$

Also, by the triangle inequality,

$$\begin{aligned} \|D^\alpha f\|_p/\sigma^\alpha &\le 2^{1/p}\left(\|D^\alpha (f - {}_\lambda f)\|_p + \|D^\alpha {}_\lambda f\|_p\right)/\sigma^\alpha \\ &\le 2^{1/p}\left(\sigma^\alpha \|f - {}_\lambda f\|_p + \lambda^{|\alpha|-n/p}\|D^\alpha f\|_p\right)/\sigma^\alpha \end{aligned} \tag{20}$$

for all $\alpha \in \mathbb{Z}_*^n$, where the second inequality comes from (18). Hence,

$$\|D^\alpha f\|_p/\sigma^\alpha \le 2^{1/p}\|f - {}_\lambda f\|_p/\left(1 - 2^{1/p}\lambda^{|\alpha|-n/p}\right)$$

for all $\alpha \in \mathbb{Z}_*^n$. In addition, from the inequalities

$$2^{1/p}\lambda^{|\alpha|-n/p} \le \left(2\left(1 - \frac{1}{|\alpha|}\right)^{|\alpha|-n/p}\right)^{1/p} \le \left(\frac{2}{e}\left(1 - \frac{1}{|\alpha|}\right)^{-n/p}\right)^{1/p} \le \left(\frac{5}{2e}\right)^{1/p}$$

for all $|\alpha| \ge \max\{n/(p(1-(4/5)^{p/n})), (1-\lambda^p)^{-1}\}$, we have

$$\|D^\alpha f\|_p/\sigma^\alpha \le 2^{1/p}\|f - {}_\lambda f\|_p \Big/ \left(1 - \left(\frac{5}{2e}\right)^{1/p}\right) \tag{21}$$

for all $\alpha \in \mathbb{Z}_*^n$ with $|\alpha| \ge \max\{n/(p(1-(4/5)^{p/n})), 1/(1-\lambda^p)\}$.

Next, take $\limsup_{|\alpha|\to\infty}$ on both sides of (21) and then the right hand side of (21) is independent of α. Thus taking $\lim_{\lambda\nearrow 1}$ in (21), by Lemma 3.1 we have (16).

Reversely, since (16) implies (14), we can derive that (16) implies (i) of Theorem B.

(2) With the hypothesis of $\operatorname{supp}\hat{f} \subset [-\sigma_1, \sigma_1] \times [-\sigma_2, \sigma_2] \times \cdots \times [-\sigma_n, \sigma_n]$, the Bernstein inequality for $0 < p \le 1$ ([12]) says that $(\|D^\alpha f\|_p/(\sigma^\alpha \|f\|_p))_{\alpha\in\mathbb{Z}_*^n}$ is bounded by 1. On the other hand, (1) of Remark 1 gives a stronger result: $\lim_{|\alpha|\to\infty} \|D^\alpha f\|_p/(\sigma^\alpha \|f\|_p) = 0$.

(3) Let $0 < p < 1$, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathbb{R}^n_*$. If we define

$$\mathcal{M}_{\sigma,p} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \hat{f} \in C^{\ell}(\mathbb{R}^n), \operatorname{supp} \hat{f} \subset \prod_{j=1}^{n} [-\sigma_j, \sigma_j] \right\},$$

where $\ell = n(\lfloor 1/p \rfloor + 2)$, then the convergent ratio of $\{\|D^\alpha f\|_p / \sigma^\alpha\}_{\alpha \in \mathbb{Z}^n_*}$ is as follows:

$$\lim_{|\alpha| \to \infty} |\alpha|^a \left\| D^\alpha f \right\|_p / \sigma^\alpha = 0 \tag{22}$$

for all $0 < a < 1$ and for all $f \in \mathcal{M}_{\sigma,p}$. Indeed, we justify (22): Consider a function $G_\lambda(x)$ as follows:

$$G_\lambda(x) = f(x) - f(\lambda x), \quad x \in \mathbb{R}^n, \frac{1}{2} < \lambda < 1.$$

Then, due to $\operatorname{supp} \hat{f} \subset K := \prod_{j=1}^n [-\sigma_j, \sigma_j]$, we have

$$G_\lambda(x) = (2\pi)^{-n} \int_K e^{-ix\xi} \left(\hat{f}(\xi) - \frac{1}{\lambda} \hat{f}\left(\frac{\xi}{\lambda}\right) \right) d\xi.$$

So,

$$\left| x^\alpha G_\lambda(x) \right| = (2\pi)^{-n} \left| \int_K e^{-ix\xi} \left(D^\alpha \hat{f}(\xi) - \frac{1}{\lambda^{1+|\alpha|}} (D^\alpha \hat{f})\left(\frac{\xi}{\lambda}\right) \right) d\xi \right|,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Hence,

$$\begin{aligned}
&\sup_{x \in \mathbb{R}} \left| x^\alpha G_\lambda(x) \right| \\
&\quad \le (2\pi)^{-n} \int_K \left| D^\alpha \hat{f}(\xi) - \frac{1}{\lambda^{1+|\alpha|}} (D^\alpha \hat{f})\left(\frac{\xi}{\lambda}\right) \right| d\xi \\
&\quad \le (2\pi)^{-n} \int_K \left(\left(\frac{1}{\lambda^{1+|\alpha|}} - 1 \right) \left| D^\alpha \hat{f}(\xi) \right| + \frac{1}{\lambda^{1+|\alpha|}} \left| D^\alpha \hat{f}(\xi) - (D^\alpha \hat{f})\left(\frac{\xi}{\lambda}\right) \right| \right) d\xi \\
&\quad \le (2\pi)^{-n} \left(\left(\frac{1}{\lambda^{1+|\alpha|}} - 1 \right) \left\| D^\alpha \hat{f} \right\|_\infty + \frac{1}{\lambda^{1+|\alpha|}} \left\| D^\alpha \hat{f}(\cdot) - (D^\alpha \hat{f})\left(\frac{\cdot}{\lambda}\right) \right\|_\infty \right) \int_K 1 \, d\xi.
\end{aligned}$$

By the mean value theorem,

$$\sup_{x \in \mathbb{R}} \left| x^\alpha G_\lambda(x) \right| \le (2\pi)^{-n} 4^{1+|\alpha|} (1 - \lambda) \left(\left\| D^\alpha \hat{f} \right\|_\infty + \sum_{j=1}^{n} \left\| D^{\alpha + e_j} \hat{f} \right\|_\infty \right) \int_K 1 \, d\xi$$

for all $1/2 < \lambda < 1$, where $e_j \in \mathbb{R}^n$ is the unit vector whose jth coordinate is 1.

Now putting $M = \lfloor 1/p \rfloor + 1$, we have

$$\begin{aligned}
\|G_\lambda\|_p &\le \sup_{x \in \mathbb{R}} \left| (1 + |x_1|)^M \cdots (1 + |x_n|)^M G_\lambda(x) \right| \left(\int_{\mathbb{R}^n} \frac{1}{(1 + |y_1|)^{pM} \cdots (1 + |y_n|)^{pM}} \, dy \right)^{1/p} \\
&\le \|f - f_\lambda\|_p \le C_p (1 - \lambda) \sum_{|\alpha| \le (M+1, M+1, \dots, M+1)} \left\| D^\alpha \hat{f} \right\|_\infty
\end{aligned}$$

for all $1/2 < \lambda < 1$. By Theorem B for $\lambda = (1 - \frac{1}{|\alpha|})^{1/p}$, we obtain

$$\|D^\alpha f\| \le C_{p,f}|\alpha|^{-1}\sigma^\alpha$$

for all $\alpha \in \mathbb{Z}_*^n$, $|\alpha| \ge n/(p(1-(4/5)^{p/n}))$, consequently,

$$\lim_{|\alpha|\to\infty} |\alpha|^a \|D^\alpha f\|_p/\sigma^\alpha = 0$$

for all $0 < a < 1$. Thus (22) holds.

4 Proof of Theorem C

Proof of Theorem C Suppose (i). We consider a real orthogonal matrix $A = (\alpha_{k,s})$ that satisfies

$$\alpha_{k,1} = \eta_k, \quad k = 1, \dots, n$$

and put

$$g(\xi) = f(x) \quad (x = A\xi).$$

By differentiation,

$$\frac{\partial}{\partial \xi_1} g(\xi) = \sum_{k=1}^n \frac{\partial f(x)}{\partial x_k}\frac{\partial x_k}{\partial \xi_1}.$$

It follows from $\frac{\partial x_k}{\partial \xi_1} = \eta_k$ $(k = 1, 2, \dots, n)$ that

$$\frac{\partial}{\partial \xi_1} g(\xi) = \sum_{k=1}^n \eta_k \frac{\partial f(x)}{\partial x_k} = D_\eta f(x).$$

Similarly,

$$\frac{\partial^m}{\partial \xi_1^m} g(\xi) = D_\eta^m f(x) \quad (m = 0, 2, \dots).$$

Thus,

$$\|D_\eta^m f(x)\|_p = \left\| \frac{\partial^m}{\partial \xi_1^m} g \right\|_p, \qquad \|g\|_p = r^m \|f\|_p, \tag{23}$$

here r^m may be 1. Note that $\hat{g}(\xi) = \hat{f}(A^t\xi)$ and so $|\xi_1| \le r$ for each $\xi \in \operatorname{supp}\hat{g}$. By the Bernstein inequality for $0 < p < 1$ ([12]) and by (23), we have

$$\|D_\eta^m f\|_p = \left\| \frac{\partial^m}{\partial \xi_1^m} g(\xi) \right\|_p \le r^m \|g\|_p = \|f\|_p.$$

Next, suppose (ii). By (ii), and by Proposition 2.1, we have

$$\limsup_{m\to\infty} \left\| D_\eta^m f \right\|_1^{1/m} / r \le 1. \tag{24}$$

By Lemma 2.2, we see that $\sup_{\xi\in \operatorname{supp}\hat{f}} |\eta\xi| \le r$, consequently, $\operatorname{supp}\hat{f} \subset K_{\eta,r}$. Therefore, the proof is complete. □

Remark 2 (1) Theorem C also shows an L^p-boundedness of derivatives. Similar to (1) of Remark 1, we prove that the L^p-boundedness is equivalent to the following vanishing property:

$$\lim_{m\to\infty} \left\| D_\eta^m f \right\|_p / r^m = 0. \tag{25}$$

First, we show (ii) of Theorem C implies (25). Since $\operatorname{supp} \widehat{{}_\lambda f} \subset \lambda K_{\eta,r}$ and so $\operatorname{supp}\widehat{f - {}_\lambda f} \subset K_{\eta,r} \cup (\lambda K_{\eta,r}) = \mathcal{K}_\lambda$, say. From (ii),

$$\left\| D_\eta^m (f - {}_\lambda f) \right\|_p \le \Bigl(\sup_{\xi\in\mathcal{K}_\lambda} |\eta\xi| \Bigr)^m \| f - {}_\lambda f \|_p.$$

Since $\sup_{\xi\in\mathcal{K}_\lambda} |\eta\xi| \le \max\{\sup_{\xi\in K_{\eta,r}} |\eta\xi|, \sup_{\xi\in\lambda K_{\eta,r}} |\eta\xi|\} \le r$, we have

$$\left\| D_\eta^m (f - {}_\lambda f) \right\|_p \le r^m \| f - {}_\lambda f \|_p. \tag{26}$$

Also, $D_\eta^m {}_\lambda f(x) = \lambda^m D_\eta^m f(\lambda x)$ gives

$$\left\| D_\eta^m {}_\lambda f \right\|_p = \lambda^{m-n/p} \left\| D_\eta^m f \right\|_p \tag{27}$$

for all $m \in \mathbb{Z}_*$. Thus, by the triangle inequality, by (26), and by (27),

$$\begin{aligned} \left\| D_\eta^m f \right\|_p / r^m &\le 2^{1/p} \bigl(\left\| D_\eta^m (f - {}_\lambda f) \right\|_p + \left\| D_\eta^m {}_\lambda f \right\|_p \bigr) / r^m \\ &\le 2^{1/p} \bigl(r^m \| f - {}_\lambda f \|_p + \lambda^{m-n/p} \left\| D_\eta^m f \right\|_p \bigr) / r^m \end{aligned}$$

for all $m \in \mathbb{Z}_*$. On the other hand, the constant has the upper bound

$$2^{1/p} \lambda^{m-n/p} \le \left(2 \left(1 - \frac{1}{m} \right)^{m-n/p} \right)^{1/p} \le \left(\frac{2}{e} \left(1 - \frac{1}{m} \right)^{-n/p} \right)^{1/p} \le \left(\frac{5}{2e} \right)^{1/p}$$

for all $m \ge \max\{n/(p(1-(4/5)^{p/n})), (1-\lambda^p)^{-1}\}$. Hence, we get the desired inequality,

$$\left\| D_\eta^m f \right\|_p / r^m \le 2^{1/p} \| f - {}_\lambda f \|_p \Big/ \left(1 - \left(\frac{5}{2e} \right)^{1/p} \right), \tag{28}$$

where $m \ge \max\{n/(p(1-(4/5)^{p/n})), (1-\lambda^p)^{-1}\}$.

Now, take $\limsup_{|\alpha|\to\infty}$ on both sides of (28), and then the right hand side of (28) is independent of α. Thus taking $\lim_{\lambda\nearrow 1}$ in (28), by Lemma 3.1 we have (25).

Reversely, since (25) implies (24), we conclude the equivalence between Theorem C and (25).

(2) Let $0 < p < 1$, $r > 0$. Put

$$\mathcal{N}_p = \left\{f \in \mathcal{S}'\left(\mathbb{R}^n\right) : \hat{f} \in C^\ell\left(\mathbb{R}^n\right), \operatorname{supp}\hat{f} \subset K_{\eta,r}\right\},$$

where $\ell = n(\lfloor 1/p \rfloor + 2)$. Then from (1) of Remark 2, we have the following convergent ratio of $\{\|D_\eta^m f\|_p / r^m\}_{m \in \mathbb{Z}_*}$:

$$\lim_{m\to\infty} m^a \left\|D_\eta^m f\right\|_p / r^m = 0$$

for all $0 < a < 1$ and for all $f \in \mathcal{N}_p$.

5 Applications

In this section, we derive three results as applications of Theorem A: If the compact set K is specified in detail, we can reduce $K_{(\delta)}$ into K_δ, as appears in Theorem A. Let $\alpha \in \mathbb{Z}_*^n$ and let $0 \le \lambda_\alpha \le \infty$. We define the set generated by the number sequence $\{\lambda_\alpha\}$ as $G\{\lambda_\alpha\}$ consisting of all points $\xi \in \mathbb{R}^n$ such that

$$\left|\xi^\alpha\right| \le \lambda_\alpha \quad \text{for all } \alpha \in \mathbb{Z}_*^n.$$

Now for any set $E \subset \mathbb{R}^n$, put the g-hull of E by

$$g(E) = G\left\{\sup_E \left|\xi^\alpha\right|\right\}.$$

Then $E \subset g(E)$ readily. We say that E satisfies the g-property if $E = g(E)$. We note two facts that every set generated by a number sequence $G\{\lambda_\alpha\}$ has the g-property and vice versa and every symmetric compact convex set also has the g-property. For more information, refer to [4, 5].

Theorem 5.1 *Let $0 < p < 1$ and let $K \subset \mathbb{R}^n$ be compact satisfying the g-property. Then* $\operatorname{supp}\hat{f} \subset K$ *if and only if for any $\delta > 0$ there exists a constant $C_{p,K,\delta}$ independent of f, α such that*

$$\left\|D^\alpha f\right\|_p \le C_{p,K,\delta}\left(\sup_{x\in K_\delta}\left|x^\alpha\right|\right)\|f\|_p \tag{29}$$

for any $\alpha \in \mathbb{Z}_*^n$.

For any polynomial P and for $r > 0$, define an r-neighborhood with respect to P by

$$N_P(r) = \left\{x \in \mathbb{R}^n : \left|P(x)\right| \le r\right\}.$$

Theorem 5.2 *Let $0 < p < 1$ and let $K = N_P(r)$. Then $\operatorname{supp}\hat{f} \subset K$ if and only if for any $\delta > 0$ there exists a constant $C_{p,K,\delta}$ independent of f, m such that*

$$\left\|P^m(D)f\right\|_p \le C_{p,K,\delta}(r+\delta)^m\|f\|_p \tag{30}$$

for any $m \in \mathbb{Z}_*$.

Theorem 5.3 *Let* $0 < p < 1$ *and let* $K \subset \mathbb{R}^n$ *be convex and compact. Then* $\operatorname{supp}\hat{f} \subset K$ *if and only if for any* $\delta > 0$ *there exists a constant* $C_{p,K,\delta}$ *independent of* f, P, m *such that*

$$\left\|P^m(D)f\right\|_p \le C_{p,K,\delta}\Bigl(\sup_{x\in K_\delta}\left|P(x)\right|^m\Bigr)\|f\|_p \tag{31}$$

for any real polynomial P *of degree* 1 *and for any* $m \in \mathbb{Z}_*$.

To prove Theorem 5.1, we need a lemma: The inclusion of $K_\delta \subset K_{(\delta)}$, implies

$$\sup_{K_{(\delta)}}\left|z^\alpha\right| \ge \sup_{K_\delta}\left|x^\alpha\right| \quad \text{for all } \alpha \in \mathbb{Z}_*^n.$$

Moreover, for $z \in K_{(\delta)}$ there exist $\xi \in K$ and $\eta \in \mathbb{C}^n$ such that $z = \xi + \eta$, $|\eta| \le \delta$ and so $|z_j| \le |\xi_j| + |\eta_j|$ $(1 \le j \le n)$. Put $x = (\xi_1 + |\eta_1|\operatorname{sign}(\xi_1),\dots,\xi_n + |\eta_n|\operatorname{sign}(\xi_n))$. Clearly, $x \in K_\delta$ and $|x_j| = |\xi_j| + |\eta_j| \ge |z_j|$ for all $1 \le j \le n$. Thus for each $z \in K_{(\delta)}$, there exists $x \in K_\delta$ such that

$$\left|z^\alpha\right| \le \left|x^\alpha\right| \quad \text{for all } \alpha \in \mathbb{Z}_*^n.$$

Therefore, we conclude the following.

Lemma 5.4 *If* K *is compact on* $\mathbb{R}^n$, *then, for any* $\delta > 0$,

$$\sup_{K_{(\delta)}}\left|z^\alpha\right| = \sup_{K_\delta}\left|x^\alpha\right| \quad \textit{for all } \alpha \in \mathbb{Z}_*^n.$$

Proof of Theorem 5.1 Fix $\delta > 0$. By Theorem A, there exists a constant $C_{p,K,\delta} < \infty$ such that

$$\begin{aligned}\left\|D^\alpha f\right\|_p &\le C_{p,K,\delta}\sup_{z\in K_{(\delta)}}\left|z^\alpha\right|\|f\|_p\\ &\le C_{p,K,\delta}\sup_{x\in K_\delta}\left|x^\alpha\right|\|f\|_p\end{aligned} \tag{32}$$

for all $\alpha \in \mathbb{Z}_*^n$, where the second inequality follows from Lemma 5.4. This proves the necessity.

To see the sufficiency, on the contrary, assume that there exists $x_0 \in \operatorname{supp}\hat{f}$ with $x_0 \notin K$. Since K has the g-property, we find $\alpha \in \mathbb{Z}_*^n$ such that

$$\left|x_0^\alpha\right| \gneq \sup_{x\in K}\left|x^\alpha\right|. \tag{33}$$

By the hypothesis of (29) with $m\alpha$ $(m = 1, 2, \dots)$,

$$\left\|D^{m\alpha}f\right\|_p \le C_{p,K,\delta}\Bigl(\sup_{x\in K_\delta}\left|x^{m\alpha}\right|\Bigr)\|f\|_p.$$

According to Proposition 2.1, applying limsup, we have

$$\limsup_{m\to\infty}\left\|D^{m\alpha}f\right\|_1^{1/m} \le \sup_{x\in K_\delta}\left|x^\alpha\right|,$$

and taking $\delta \searrow 0$, we get

$$\limsup_{m\to\infty} \left\| D^{m\alpha} f \right\|_p^{1/m} \le \sup_{x\in K} \left| x^\alpha \right|. \tag{34}$$

Also, by Lemma 2.2 with $P(x) = x^\alpha$,

$$\liminf_{m\to\infty} \left\| D^{m\alpha} f \right\|_p^{1/m} \ge \left| x_0^\alpha \right|. \tag{35}$$

Thus, the equations of (34) and (35) yield

$$\left| x_0^\alpha \right| \le \sup_{x\in K} \left| x^\alpha \right|.$$

This contradicts (33). Therefore, the proof is complete. □

Since any symmetric convex compact set satisfies the g-property ([4, 5]), Theorem 5.1 produces the corollary:

Corollary 5.5 *Assume K is a symmetric convex compact set in $\mathbb{R}^n$, $0 < p < 1$. Then, for any $\delta > 0$, there exists a constant $C_{p,K,\delta} < \infty$ such that*

$$\left\| D^\alpha f \right\|_p \le C_{p,K,\delta} \Big(\sup_{x\in K_\delta} \left| x^\alpha \right| \Big) \|f\|_p$$

for all $\alpha \in \mathbb{Z}_*^n$.

Let us note that a symmetric convex compact set is a typical example for a compact set that has the g-property. Since D^α is simpler than $P(D)$, in Corollary 5.5 the supremum runs over K_δ instead of $K_{(\delta)}$.

Proof of Theorem 5.2 We first prove the necessity. For any $\delta > 0$, by continuity, there is $\delta' > 0$ so that

$$\sup_{K_{(\delta')}} \left| P(z) \right| \le r + \delta,$$

since $\sup_K |P(x)| = r$. By Theorem A, there exists a constant $C_{p,K,\delta'}$ such that

$$\begin{aligned} \left\| P^m(D) f \right\|_p &\le C_{p,K,\delta'} \Big(\sup_{K_{(\delta')}} \left| P^m(z) \right| \Big) \|f\|_p \\ &\le C_{p,K,\delta'} (r+\delta)^m \|f\|_p. \end{aligned}$$

To prove the sufficiency, suppose that there exists $x_0 \in \operatorname{supp}\hat{f}$ and $x_0 \notin K = N_P(r)$. Then $|P(x_0)| > r$. From hypothesis (30) and by Proposition 2.1,

$$\limsup_{m\to\infty} \left\| P^m(D) f \right\|_1^{1/m} \le r + \delta. \tag{36}$$

Also, by Lemma 2.2,

$$\liminf_{m\to\infty}\|P^m(D)f\|_p^{1/m} \geq |P(x_0)|. \tag{37}$$

Hence, by (36), by (37), and by assumption,

$$r < |P(x_0)| \leq r + \delta$$

for any $\delta > 0$. Letting $\delta \searrow 0$, we reach a contradiction of the inequality. This gives the proof. □

By Theorems 5.1 and 5.2, we have a corollary.

Corollary 5.6 *Let $r > 0$ and let P_j be polynomial $(1 \leq j \leq q)$. Put $K = H \bigcap_{j=1}^{q} N_{P_j}(r)$, where H, $N_{P_j}(r)$ are compact. Then $\operatorname{supp}\hat{f} \subset K$ if and only if for any $\delta > 0$ there exists a constant $C_{p,K,\delta}$ such that*

$$\left\| D^\alpha \prod_{j=1}^{q} P_j^{m_j}(D)f \right\|_p \leq C_{p,K,\delta}(r+\delta)^{\sum_{j=1}^{q} m_j}\Bigl(\sup_{z\in K_{(\delta)}}|z^\alpha|\Bigr)\|f\|_p$$

for all $m_j \in \mathbb{Z}_*$ $(1 \leq j \leq q)$, for all $\alpha \in \mathbb{Z}_*^n$.

We are ready to derive Theorem 5.3 with a lemma.

Lemma 5.7 *Let P be a real polynomial of degree 1. If $E \subset \mathbb{R}^n$ is any set, then*

$$\sup_{z\in E_{(\delta)}}|P(z)| = \sup_{x\in E_\delta}|P(x)|.$$

Proof From $E_\delta \subset E_{(\delta)}$, we have $\sup_{z\in E_{(\delta)}}|P(z)| \geq \sup_{x\in E_\delta}|P(x)|$. To complete the proof, we need prove that $\sup_{z\in E_{(\delta)}}|P(z)| \leq \sup_{x\in E_\delta}|P(x)|$. Indeed, let $z \in E_{(\delta)}$. Then there are $x \in E$ and $r\eta \in \mathbb{C}^n$ $(0 \leq r \leq \delta, |\eta| = 1)$ such that $z = x + r\eta$. Since we can replace P with $-P$, we may assume that $P(x) \geq 0$. Taking $y = x + r(\operatorname{sign}(\partial_{x_1}P)|\eta_1|, \dots, \operatorname{sign}(\partial_{x_n}P)|\eta_n|) \in E_r \subset E_\delta$ so that

$$|P(y)| = P(x) + r\sum_{j=1}^{n}|\partial_{x_j}P||\eta_j|.$$

By the triangle inequality,

$$|P(z)| \leq P(x) + r\sum_{j=1}^{n}|\partial_{x_j}P||\eta_j|.$$

The previous two inequalities show that for each $z \in E_{(\delta)}$, there is $y \in E_\delta$ such that $|P(z)| \leq |P(y)|$. Therefore, taking supremums successively, we justify the lemma. □

Proof of Theorem 5.3 The necessity follows readily. Indeed, let $\delta > 0$. By Theorem A, for some $C_{p,K,\delta}$, we have

$$\begin{aligned}\|P^m(D)f\|_p &\le C_{p,K,\delta}\Bigl(\sup_{z\in K_{(\delta)}}|P(z)|^m\Bigr)\|f\|_p\\ &\le C_{p,K,\delta}\Bigl(\sup_{z\in K_\delta}|P(z)|^m\Bigr)\|f\|_p\end{aligned}$$

for a real polynomial P of degree 1 and for all $m \in \mathbb{Z}_*$, where the second inequality comes from Lemma 5.7.

It remains to prove the sufficiency. On the contrary, assume that there exists $x_0 \in H$ with $x_0 \notin K$, where $H = \operatorname{supp}\hat{f}$. Since K is convex and compact, we easily find a linear P such that

$$|P(x_0)| > \sup_{x\in K}|P(x)|. \tag{38}$$

By (31) and Proposition 2.1,

$$\|P^m(D)f\|_1 \le C_{p,H}C_{p,K,\delta}\Bigl(\sup_{x\in K_\delta}|P^m(x)|\Bigr)\|f\|_p$$

for all $m \in \mathbb{Z}_*$. So,

$$\limsup_{m\to\infty}\|P^m(D)f\|_1^{1/m} \le \sup_{x\in K_\delta}|P(x)|. \tag{39}$$

By (39) and by Lemma 2.2,

$$|P(x_0)| \le \sup_{x\in K_\delta}|P(x)|,$$

and, putting $\delta \searrow 0$, we get

$$|P(x_0)| \le \sup_{x\in K}|P(x)|.$$

This contradicts (38), therefore, the proof is complete. □

From Theorems 5.1 and 5.3, we have the following.

Corollary 5.8 *Let K be convex compact in $\mathbb{R}^n$, $0 < p < 1$. Then, for every $\delta > 0$, there exists a constant $C_{p,K,\delta}$ independent of f, P, m such that*

$$\|P^m(D)f\|_p \le C_{p,K,\delta}\Bigl(\sup_{x\in K_\delta}|P(x)|^m\Bigr)\|f\|_p$$

for all $P(x)$ having degree 1 and for all $m \in \mathbb{Z}_*$.

Corollary 5.9 *Let $0 < p < 1$. Suppose that K_1 is convex and compact and that K_2 compact satisfying the g-property. Then $\operatorname{supp}\hat{f} \subset K_1 \cap K_2 = K$, say, if and only if for any $\delta > 0$ there* *exists a constant $C_{p,K,\delta}$ such that*

$$\left\| P^m(D)D^\alpha f \right\|_p \le C_{p,K,\delta} \Big(\sup_{x\in K_\delta} \left| P^m(x) \right| \Big) \|f\|_p$$

for any real polynomials P of degree 1 *and for any $m \in \mathbb{Z}_*$.*

Remark 3 According to the Nikolskii inequality all L^p–L^p inequalities for differential operators in this paper can be extended to the L^p–L^q inequalities for $0 < p < q < \infty$.

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