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Iterative unique positive solutions for a new class of nonlinear singular higher order fractional differential equations with mixed-type boundary value conditions

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Abstract

In this paper, we consider a new class of singular nonlinear higher order fractional boundary value problems supplemented with sum of Riemann–Stieltjes integral type and nonlocal infinite-point discrete type boundary conditions. The fractional derivative of different orders is involved in the nonlinear terms and boundary conditions, and the nonlinear terms are allowed to be singular in regard to not only time variable but also space variables. A unique positive solution is established by using the fixed point theorem of mixed monotone operator. In addition, some significant properties of the unique solution depending on the parameter λ are stated. In the end, two examples are worked out to illustrate our main results.

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Keywords: Fractional differential equations; Riemann–Stieltjes integral; Nonlocal infinite-point discrete boundary conditions; Uniqueness of solutions

1 Introduction

In this paper, we are investigating the following singular nonlinear higher order fractional boundary value problem (BVP for short):

$$\begin{cases} D_{0+}^{\gamma}z(t)+\lambda f(t,z(t),D_{0+}^{\nu_1}z(t),\ldots,D_{0+}^{\nu_{n-3}}z(t),D_{0+}^{\nu_{n-2}}z(t))=0, \quad 0<t<1,\\ z(0)=D_{0+}^{q_1}z(0)=\cdots=D_{0+}^{q_{n-2}}z(0)=0,\\ D_{0+}^{\gamma_0}z(1)=\sum_{i=1}^{p}a_i\int_{I_i}w_i(s)D_{0+}^{\alpha_i}z(s)\,dA_i(s)+\sum_{j=1}^{\infty}b_jD_{0+}^{\beta_j}z(\xi_j),\end{cases} \tag{1.1}$$

where D_{0+}^{γ} is the Riemann–Liouville fractional derivative of γ order, $\lambda>0$ is a parameter, $n-1<\gamma\le n$ $(n\ge 3)$, $k-1<\nu_k, q_k\le k$ $(k=1,2,\ldots,n-2)$, $\nu_{n-2}-q_k\le n-2-k$ $(k=1,2,\ldots,n-2)$, $1<\gamma-\nu_{n-2}\le 2$, $\nu_{n-2}\le\gamma_0\le n-1$, $\gamma-\gamma_0\ge 1$, $a_i\ge 0$ $(i=1,2,\ldots,p)$, $\nu_{n-2}\le\alpha_i\le\gamma_0$ $(i=1,2,\ldots,p)$, $b_j\ge 0$ $(j=1,2,\ldots)$, $\nu_{n-2}\le\beta_j\le\gamma_0$ $(j=1,2,\ldots)$, $0<\xi_1<\xi_2<\cdots<\xi_j<\cdots<1$; $I_i\subseteq[0,1]$ $(i=1,2,\ldots,p)$ is measurable; $w_i:(0,1)\to\mathbb{R}_+=[0,+\infty)$ is continuous with $w_i\in L^1(0,1)$, and $\int_0^1 w_i(s)z(s)\,dA_i(s)$ denotes the Riemann–Stieltjes integral, in which $A_i:I_i\to\mathbb{R}$ $(i=1,2,\ldots,p)$ is a function of bounded variation. $f:(0,1)\times(0,+\infty)^{n-1}\to\mathbb{R}_+$

($\mathbb{R}_+ = [0,+\infty)$) is continuous. A function $z \in C[0,1]$ is called a positive solution of BVP (1.1) if it satisfies (1.1) and $z(t) > 0$ for $t \in (0,1)$.

In recent years, the fractional differential equations have drawn the attention of many famous researchers, readers can refer to [1–41] and the references therein. It is caused by the applications of fractional differential equations in a proposed framework for describing significant phenomena, for example, the deflection of an elastic beam, the non-Newtonian fluid theory, the degrading of polymer materials, etc. Some interesting results can be found in [1–5].

In [7], the authors considered the following nonlinear fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha} z(t) + f(t,z(t),Tz(t),Sz(t)) = 0, \quad 0 < t < 1, \\ z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} z(1) = \sum_{i=1}^{m} a_i D_{0^+}^{\gamma} z(\xi_i), \end{cases}$$

where D_{0+}^{α} is the Riemann–Liouville fractional derivative, $n-1 < \alpha \le n$ $(n \ge 3)$, $1 \le \beta \le n-2$, $0 \le \gamma \le \beta$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $Tz(t) = \int_0^t K(t,s)z(s)\,ds$, and $Sz(t) = \int_0^1 H(t,s)z(s)\,ds$. By using the Banach contraction mapping principle and the Krasnosel'skii fixed point theorem, they obtained the existence of nonnegative solutions for this problem.

By using the Banach contraction map principle and the theory of u_0-positive linear operator, Zhang and Zhong in [8] studied the following fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha} z(t) + f(t,z(t)) = 0, \quad 0 < t < 1, \\ z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} z(1) = \lambda \int_0^{\eta} h(s) D_{0^+}^{\gamma} z(s)\,ds, \end{cases}$$

where D_{0+}^{α} is the Riemann–Liouville derivative, $n-1 < \alpha \le n$ $(n \ge 3)$, $\beta \ge 1$, $\alpha - \beta > 1$, $0 < \eta \le 1$, $\lambda > 0$ is a parameter, $h \in L^1[0,1]$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. They got the existence and uniqueness of solutions for this problem.

Based on the reducing method of fractional orders, the Schauder fixed point theorem, and the upper and lower solutions method, Zhang, Liu, and Wu in [9] obtained an eigenvalue interval for the existence of positive solutions of the following fractional differential equation:

$$\begin{cases} -D_{0+}^{\alpha} u(t) = \lambda f(u(t), D_{0+}^{\mu_1} u(t), D_{0+}^{\mu_2} u(t), \ldots, D_{0+}^{\mu_{n-1}} u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^{\mu_i} u(0) = 0, \qquad D_{0+}^{\mu} u(1) = \sum_{j=1}^{p-2} a_j D_{0+}^{\mu} u(\xi_j), \quad 1 \le i \le n-2, \end{cases}$$

where D_{0+}^{α} is the Riemann–Liouville fractional derivative, $n-1 < \alpha \le n$ $(n \ge 3)$, $n-i-1 \le \alpha - \mu_i \le n-i$ $(i = 1,2,\ldots,n-2)$, $\mu - \mu_{n-1} > 0$, $\alpha - \mu_{n-1} \le 2$, $\alpha - \mu > 1$, $a_j \ge 0$ $(j = 1,2,\ldots,p-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$; $f : (0,+\infty)^n \to \mathbb{R}_+$ is continuous and is nonincreasing in $x_i > 0$ for $i = 1,2,\ldots,n$.

Inspired by the above-mentioned papers, we investigate BVP (1.1). As far as we know, BVP (1.1) has seldom been researched up to now, and the novelty of this paper lies in three aspects. Firstly, the boundary conditions are the combination of sum of Riemann–Stieltjes integral type boundary conditions and nonlocal infinite-point discrete type boundary conditions, which involves fractional derivative of different orders. This fact suggests BVP

(1.1) is more general than the above-mentioned literature. For instance, let $I_1 = (0,1)$, $I_i = 0$ $(i = 2,3,\ldots,p)$, and $b_j = 0$ $(j = 1,2,\ldots)$, then the boundary condition of BVP (1.1) reduces to the boundary condition in [16]; if $b_j = 0$ $(j = 1,2,\ldots)$, $\alpha_i = 0$ $w_i = 1$ $(i = 2,3,\ldots,p)$, then the boundary condition is equal to [37]; and if $I_1 = (0,\eta)$ $(\eta \in (0,1))$, $I_i = 0$ $(i = 2,3,\ldots,p)$, the boundary condition of BVP (1.1) is the same as [18]. Meanwhile the work to check the properties of the corresponding Green's function is too hard. Secondly, the nonlinearity f contains different orders of fractional derivative of the unknown function. In general, many papers consider these kinds of boundary value problem in the space $E = \{u \in C[0,1] : D_{0+}^{\nu_i} u \in C[0,1], i = 1,2,\ldots,n-2\}$, which makes the study extremely difficult. In this paper, we use the reducing method to transform BVP (1.1) into a relatively low-order equivalent problem, which could be considered in the space $C[0,1]$, and is a good way to do this. Some interesting results of the reducing method can be found in [9, 17, 23, 25, 27, 29, 31] and the references therein. Thirdly, there is much to be learned about the theory and applications of mixed monotone operator, recently. Especially, many papers have taken it into the research for fractional boundary value problems. Some interesting results can be found in [10, 11, 15, 21–23, 25, 27] and the references therein. Thus, in this paper, by using the fixed point theorem of mixed monotone operator, we obtain the uniqueness of positive solution under the assumption that f may be singular with respect to both the time and space variables. It is worth mentioning that some important properties of the unique solution rely on the parameter λ.

The paper is organized as follows. In Sect. 2, we present some preliminary setting, derive the corresponding Green's function, and transform BVP (1.1) into a relatively low-order equivalent problem, in which the nonlinear term has no fractional derivatives. In Sect. 3, we pay particular attention to establishing the uniqueness of positive solutions and consider some relative properties of the unique positive solution. In Sect. 4, two examples are devoted to our main results.

2 Preliminaries and lemmas

Let E be a Banach space and P be a cone in E. P is said to be normal if there exists a constant $N > 0$ such that, for any $u, v \in E$, $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$, the smallest constant, which satisfies this inequality, is called the normality constant of P. Then E is partially ordered by P, i.e., $u \leq v$ if and only if $v - u \in P$. For any $u, v \in E$, the notation $u \sim v$ means that there exist constants $\lambda > 0$ and $\mu > 0$ such that $\lambda u \leq v \leq \mu u$. Obviously, $\sim$ is an equivalence relation. For fixed $e \in P_e$ and $e > \theta$, we denote $P_e = \{u \in E : u \sim e\} = \{u \in E : \omega e \leq u \leq \frac{1}{\omega} e, 0 < \omega < 1\}$. It is easy to see that $P_e \subset P$ is a component of P.

Definition 2.1 ([11]) Let E be a Banach space and $D \subset E$. The operator $A : D \times D \to E$ is called a mixed monotone operator if $A(u,v)$ is increasing in $u \in D$ and decreasing in $v \in D$, i.e., u_i, $v_i \in D$ $(i = 1,2)$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1,v_1) \leq A(u_2,v_2)$. An element $u \in D$ is called a fixed point of A if $A(u,u) = u$.

Lemma 2.2 ([10, 12]) *Let P be a normal cone in the Banach space E, and $A, B : P_e \times P_e \to P_e$ be two mixed monotone operators which satisfy the following conditions*:

(i) *For any $\mu \in (0,1)$, there exists $\varphi(\mu) \in (\mu, 1]$ such that*

$$A(\mu u, \mu^{-1} v) \geq \varphi(\mu) A(u,v), \quad \forall u, v \in P_e.$$

(ii) *For any* $\mu \in (0,1)$, $u, v \in P_e$,

$$B(\mu u, \mu^{-1} v) \geq \mu B(u, v).$$

(iii) *There exists a constant* $\kappa > 0$ *such that* $A(u,v) \geq \kappa B(u,v)$, $\forall u, v \in P_e$.

Then there exists a unique fixed point $u^* \in P_e$ *such that* $A(u^*, u^*) + B(u^*, u^*) = u^*$. *And for any initial values* $u_0, v_0 \in P_e$, *by constructing successively the sequences as follows*:

$$u_n = A(u_{n-1}, v_{n-1}) + B(u_{n-1}, v_{n-1}), \qquad v_n = A(v_{n-1}, u_{n-1}) + B(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots,$$

we have $u_n \to u^*$ *and* $v_n \to u^*$ *in E, as* $n \to \infty$.

Lemma 2.3 ([10, 12]) *Suppose that operators A and B satisfy all the conditions of Lemma* 2.2. *Then the equation*

$$\lambda A(u,u) + \lambda B(u,u) = u$$

has a unique solution u_λ *in* P_e *for all* $\lambda > 0$, *which satisfies*:

(i) *If there exists* $r \in (0,1)$ *such that*

$$\varphi(\mu) \geq \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),$$

then u_λ *is continuous with respect to* $\lambda \in (0, +\infty)$. *That is, for any* $\lambda_0 \in (0, +\infty)$,

$$\|u_\lambda - u_{\lambda_0}\| \to 0, \quad \text{as } \lambda \to \lambda_0.$$

(ii) *If*

$$\varphi(\mu) \geq \frac{\mu^{\frac{1}{2}} - \mu}{\kappa} + \mu^{\frac{1}{2}}, \quad \forall \mu \in (0,1),$$

then $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1} < u_{\lambda_2}$.

(iii) *If there exists* $r \in (0, \frac{1}{2})$ *such that*

$$\varphi(\mu) \geq \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),$$

then

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0, \qquad \lim_{\lambda \to +\infty} \|u_\lambda\| = +\infty.$$

Definition 2.4 ([5]) Let $\alpha > 0$. The Riemann–Liouville fractional integral of order α of a function $z : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.5 ([5]) Let $\alpha > 0$. The Riemann–Liouville fractional derivative of order α of a continuous function $z : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} z(t) = \left(\frac{d}{dt}\right)^n I_{0^+}^{n-\alpha} z(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{z(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.6 ([6]) *Let $z \in C(0,1) \cap L^1(0,1)$. Then the fractional differential equation*

$$D_{0^+}^{\alpha} z(t) = 0$$

has a unique solution

$$z(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, i = 1, 2, \ldots, n,$$

where n is the smallest integer greater than or equal to α.

Lemma 2.7 ([6]) *Let $z \in C(0,1) \cap L^1(0,1)$ and $D_{0^+}^{\alpha} z \in C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} z(t) = z(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, i = 1, 2, \ldots, n,$$

where n is the smallest integer greater than or equal to α.

Lemma 2.8 ([5]) *Suppose that $z \in C(0,1) \cap L^1(0,1)$, then*
(i) $I_{0^+}^{\alpha} I_{0^+}^{\beta} z(t) = I_{0^+}^{\alpha+\beta} z(t)$ *for* $\alpha, \beta > 0$;
(ii) $D_{0^+}^{\beta} I_{0^+}^{\alpha} z(t) = I_{0^+}^{\alpha-\beta} z(t)$ *for* $\alpha \ge \beta > 0$.

Lemma 2.9 *Suppose that BVP (1.1) has a solution $z \in C[0,1]$, then $u = D_{0^+}^{\nu_{n-2}} z$ is a solution of the following boundary value problem*:

$$\begin{cases} D_{0^+}^{\gamma-\nu_{n-2}} u(t) + \lambda f(t, I_{0^+}^{\nu_{n-2}} u(t), I_{0^+}^{\nu_{n-2}-\nu_1} u(t), \ldots, I_{0^+}^{\nu_{n-2}-\nu_{n-3}} u(t), u(t)) = 0, \\ D_{0^+}^{q_{n-2}-\nu_{n-2}} u(0) = 0, \quad 0 < t < 1, \\ D_{0^+}^{\gamma_0-\nu_{n-2}} u(1) = \sum_{i=1}^{p} a_i \int_{I_i} w_i(s) D_{0^+}^{\alpha_i-\nu_{n-2}} u(s)\,dA_i(s) \\ \qquad\qquad + \sum_{j=1}^{\infty} b_j D_{0^+}^{\beta_j-\nu_{n-2}} u(\xi_j), \end{cases} \tag{2.1}$$

where $1 < \gamma - \nu_{n-2} \le 2$. On the other hand, if we assume BVP (2.1) has a solution $u \in C[0,1]$, then BVP (1.1) has a solution $z = I_{0^+}^{\nu_{n-2}} u$.

Proof Suppose that $z \in C[0,1]$ and satisfies BVP (1.1). Let

$$u(t) = D_{0^+}^{\nu_{n-2}} z(t), \quad t \in [0,1]. \tag{2.2}$$

It follows from Lemma 2.7 that

$$I_{0^+}^{\nu_{n-2}} u(t) = I_{0^+}^{\nu_{n-2}} D_{0^+}^{\nu_{n-2}} z(t) = z(t) + c_1 t^{\nu_{n-2}-1} + \cdots + c_{n-2} t^{\nu_{n-2}-(n-2)},$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, \dots, n-2$). Hence,

$$z(t) = I_{0^+}^{\nu_{n-2}} u(t) - c_1 t^{\nu_{n-2}-1} - \cdots - c_{n-2} t^{\nu_{n-2}-(n-2)}.$$

Since $z(0) = 0, \dots, D_{0^+}^{q_{n-3}} z(0) = 0$, we immediately obtain that $c_1 = \cdots = c_{n-2} = 0$. Thus,

$$z(t) = I_{0^+}^{\nu_{n-2}} u(t), \quad t \in [0,1]. \tag{2.3}$$

By using (ii) in Lemma 2.8, we have

$$D_{0^+}^{\nu_i} z(t) = D_{0^+}^{\nu_i} I_{0^+}^{\nu_{n-2}} u(t) = I_{0^+}^{\nu_{n-2}-\nu_i} u(t), \quad i = 1, 2, \dots, n-3, \tag{2.4}$$

and

$$\begin{aligned} D_{0^+}^{\gamma} z(t) &= D_{0^+}^{\gamma} I_{0^+}^{\nu_{n-2}} u(t) = \frac{d^n}{dt^n} I_{0^+}^{n-\gamma} I_{0^+}^{\nu_{n-2}} u(t) = \frac{d^n}{dt^n} I_{0^+}^{n-(\gamma-\nu_{n-2})} u(t) \\ &= \frac{d^2}{dt^2} \frac{d^{n-2}}{dt^{n-2}} I_{0^+}^{n-2} I_{0^+}^{2-(\gamma-\nu_{n-2})} u(t) \\ &= D_{0^+}^{\gamma-\nu_{n-2}} u(t). \end{aligned} \tag{2.5}$$

Similar to (2.5), we have

$$D_{0^+}^{\gamma_0} z(t) = D_{0^+}^{\gamma_0-\nu_{n-2}} u(t), \tag{2.6}$$

$$D_{0^+}^{q_{n-2}} z(t) = D_{0^+}^{q_{n-2}-\nu_{n-2}} u(t), \tag{2.7}$$

$$D_{0^+}^{\alpha_i} z(t) = D_{0^+}^{\alpha_i-\nu_{n-2}} u(t), \quad i = 1, 2, \dots, p, \tag{2.8}$$

$$D_{0^+}^{\beta_j} z(t) = D_{0^+}^{\beta_j-\nu_{n-2}} u(t), \quad j = 1, 2, \dots. \tag{2.9}$$

It follows from (2.2)–(2.5) that

$$\begin{aligned} &D_{0^+}^{\gamma-\nu_{n-2}} u(t) + \lambda f\bigl(t, I_{0+}^{\nu_{n-2}} u(t), I_{0+}^{\nu_{n-2}-\nu_1} u(t), \dots, I_{0+}^{\nu_{n-2}-\nu_{n-3}} u(t), u(t)\bigr) \\ &\quad = D_{0_+}^{\gamma} z(t) + \lambda f\bigl(t, z(t), D_{0+}^{\nu_1} z(t), \dots, D_{0+}^{\nu_{n-3}} z(t), D_{0+}^{\nu_{n-2}} z(t)\bigr) \\ &\quad = 0. \end{aligned} \tag{2.10}$$

On the basis of (2.6), (2.8), and (2.9), we have

$$\begin{aligned} D_{0+}^{\gamma_0-\nu_{n-2}} u(1) &= D_{0^+}^{\gamma_0} z(1) \\ &= \sum_{i=1}^{p} a_i \int_{I_i} w_i(s) D_{0+}^{\alpha_i} z(s)\, dA_i(s) + \sum_{j=1}^{\infty} b_j D_{0+}^{\beta_j} z(\xi_j) \\ &= \sum_{i=1}^{p} a_i \int_{I_i} w_i(s) D_{0+}^{\alpha_i-\nu_{n-2}} u(s)\, dA_i(s) + \sum_{j=1}^{\infty} b_j D_{0+}^{\beta_j-\nu_{n-2}} u(\xi_j). \end{aligned} \tag{2.11}$$

From (2.7), we have

$$D_{0^+}^{q_{n-2}-\nu_{n-2}} u(0) = D_{0^+}^{q_{n-2}} z(0) = 0. \tag{2.12}$$

Combining (2.10)–(2.12), we deduce that $u = D_{0^+}^{\nu_{n-2}} z$ is a solution of BVP (2.1).

On the other hand, we consider the case that BVP (2.1) has a solution $u \in C[0,1]$. Let $z(t) = I_{0^+}^{\nu_{n-2}} u(t)$, $t \in [0,1]$. Then $z = I_{0^+}^{\nu_{n-2}} u$ is a solution of BVP (1.1). The proof is similar to Lemma 3 in [31]. So, we omit details. □

Remark 2.10 With the analysis of Lemma 2.9, it is enough to show that the work on searching solutions of BVP (1.1) is equivalent to finding solutions of BVP (2.1). Accordingly, we will focus on seeking the solutions of BVP (2.1) in the rest of this paper.

Lemma 2.11 *Let* $x \in C(0,1) \cap L^1(0,1)$. *Then the boundary value problem*

$$\begin{cases} D_{0+}^{\gamma-\nu_{n-2}} u(t) + x(t) = 0, \quad 0 < t < 1, 1 < \gamma - \nu_{n-2} \leq 2, \\ D_{0^+}^{q_{n-2}-\nu_{n-2}} u(0) = 0, \\ D_{0+}^{\gamma_0-\nu_{n-2}} u(1) = \sum_{i=1}^{p} a_i \int_{I_i} w_i(s) D_{0+}^{\alpha_i-\nu_{n-2}} u(s)\, dA_i(s) + \sum_{j=1}^{\infty} b_j D_{0+}^{\beta_j-\nu_{n-2}} u(\xi_j), \end{cases} \tag{2.13}$$

is equivalent to

$$u(t) = \int_0^1 K(t,s) x(s)\, ds, \tag{2.14}$$

where

$$\begin{aligned} K(t,s) = {}& K_0(t,s) + t^{\gamma-\nu_{n-2}-1} \sum_{i=1}^{p} \left(\int_{I_i} K_i(\tau,s) w_i(\tau)\, dA_i(\tau) \right) \\ & + t^{\gamma-\nu_{n-2}-1} \sum_{j=1}^{\infty} H_j(\xi_j, s), \end{aligned} \tag{2.15}$$

in which

$$K_0(t,s) = \frac{1}{\Gamma(\gamma-\nu_{n-2})} \begin{cases} t^{\gamma-\nu_{n-2}-1}(1-s)^{\gamma-\gamma_0-1} - (t-s)^{\gamma-\nu_{n-2}-1}, & 0 \leq s \leq t \leq 1, \\ t^{\gamma-\nu_{n-2}-1}(1-s)^{\gamma-\gamma_0-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$K_i(t,s) = \frac{a_i}{\sigma \Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\alpha_i)} \begin{cases} t^{\gamma-\alpha_i-1}(1-s)^{\gamma-\gamma_0-1} - (t-s)^{\gamma-\alpha_i-1}, & 0 \leq s \leq t \leq 1, \\ t^{\gamma-\alpha_i-1}(1-s)^{\gamma-\gamma_0-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$(i = 1, 2, \ldots, p)$,

$$H_j(t,s) = \frac{b_j}{\sigma \Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\beta_j)} \begin{cases} t^{\gamma-\beta_j-1}(1-s)^{\gamma-\gamma_0-1} - (t-s)^{\gamma-\beta_j-1}, & 0 \leq s \leq t \leq 1, \\ t^{\gamma-\beta_j-1}(1-s)^{\gamma-\gamma_0-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$(j = 1, 2, \ldots)$,

$$\sigma = \frac{1}{\Gamma(\gamma-\gamma_0)} - \sum_{i=1}^{p} \frac{a_i}{\Gamma(\gamma-\alpha_i)} \int_{I_i} s^{\gamma-\alpha_i-1} w_i(s)\, dA_i(s) - \sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)} \xi_j^{\gamma-\beta_j-1} \neq 0.$$

Obviously, $K(t,s)$ *is continuous on* $[0,1] \times [0,1]$.

Proof By using Lemma 2.7, we may express (2.13) as

$$u(t) = -\int_0^t \frac{(t-s)^{\gamma-\nu_{n-2}-1}}{\Gamma(\gamma-\nu_{n-2})}x(s)\,ds + c_1 t^{\gamma-\nu_{n-2}-1} + c_2 t^{\gamma-\nu_{n-2}-2}, \tag{2.16}$$

where $c_1, c_2 \in \mathbb{R}$. Since $D_{0^+}^{q_{n-2}-\nu_{n-2}}u(0) = 0$, we get $c_2 = 0$ and rewrite (2.16) as

$$u(t) = -\int_0^t \frac{(t-s)^{\gamma-\nu_{n-2}-1}}{\Gamma(\gamma-\nu_{n-2})}x(s)\,ds + c_1 t^{\gamma-\nu_{n-2}-1}. \tag{2.17}$$

With the help of (ii) in Lemma 2.8, we have

$$D_{0+}^{\gamma_0-\nu_{n-2}}u(t) = -\frac{1}{(\gamma-\gamma_0)}\int_0^t (t-s)^{\gamma-\gamma_0-1}x(s)\,ds + c_1\frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma-\gamma_0)}t^{\gamma-\gamma_0-1},$$
$$D_{0+}^{\alpha_i-\nu_{n-2}}u(t) = -\frac{1}{(\gamma-\alpha_i)}\int_0^t (t-s)^{\gamma-\alpha_i-1}x(s)\,ds + c_1\frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma-\alpha_i)}t^{\gamma-\alpha_i-1}, \quad i = 1,2,\ldots,p,$$

and

$$D_{0+}^{\beta_j-\nu_{n-2}}u(t) = -\frac{1}{(\gamma-\beta_j)}\int_0^t (t-s)^{\gamma-\beta_j-1}x(s)\,ds + c_1\frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma-\beta_j)}t^{\gamma-\beta_i-1}, \quad j = 1,2,\ldots,$$

which combined with the boundary condition

$$D_{0+}^{\gamma_0-\nu_{n-2}}u(1) = \sum_{i=1}^{p} a_i \int_{I_i} w_i(s) D_{0+}^{\alpha_i-\nu_{n-2}}u(s)\,dA_i(s) + \sum_{j=1}^{\infty} b_j D_{0+}^{\beta_j-\nu_{n-2}}u(\xi_j)$$

yields

$$\begin{aligned} c_1 = {} & \frac{1}{\sigma\Gamma(\gamma-\nu_{n-2})}\Bigg\{\frac{1}{\Gamma(\gamma-\gamma_0)}\int_0^1 (1-s)^{\gamma-\gamma_0-1}x(s)\,ds \\ & - \sum_{i=1}^{p}\frac{a_i}{\Gamma(\gamma-\alpha_i)}\int_{I_i} w_i(s)\biggl(\int_0^s (s-\tau)^{\gamma-\alpha_i-1}x(\tau)\,d\tau\biggr)\,dA_i(s) \\ & - \sum_{j=1}^{\infty}\frac{b_j}{\Gamma(\gamma-\beta_j)}\int_0^{\xi_j}(\xi_j-\tau)^{\gamma-\beta_j-1}x(\tau)\,d\tau\Bigg\}, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} \sigma = {} & \frac{1}{\Gamma(\gamma-\gamma_0)} - \sum_{i=1}^{p}\frac{a_i}{\Gamma(\gamma-\alpha_i)}\int_{I_i} s^{\gamma-\alpha_i-1}w_i(s)\,dA_i(s) - \sum_{j=1}^{\infty}\frac{b_j}{\Gamma(\gamma-\beta_j)}\xi_j^{\gamma-\beta_j-1} \\ & \neq 0. \end{aligned} \tag{2.19}$$

Applying (2.18) into (2.17), we can obtain

$$\begin{aligned} u(t) = {} & -\frac{1}{\Gamma(\gamma-\nu_{n-2})}\int_0^t (t-s)^{\gamma-\nu_{n-2}-1}x(s)\,ds \\ & + \frac{t^{\gamma-\nu_{n-2}-1}}{\sigma\Gamma(\gamma-\nu_{n-2})}\Bigg\{\frac{1}{\Gamma(\gamma-\gamma_0)}\int_0^1 (1-s)^{\gamma-\gamma_0-1}x(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{p} \frac{a_i}{\Gamma(\gamma-\alpha_i)} \int_{I_i} \left(\int_0^s (s-\tau)^{\gamma-\alpha_i-1} x(\tau)\,d\tau \right) w_i(s)\,dA_i(s) \\
&- \sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)} \int_0^{\xi_j} (\xi_j-\tau)^{\gamma-\beta_j-1} x(\tau)\,d\tau \Bigg\} \\
= &-\frac{1}{\Gamma(\gamma-\nu_{n-2})} \int_0^t (t-s)^{\gamma-\nu_{n-2}-1} x(s)\,ds \\
&+ \Bigg\{ \frac{1}{\Gamma(\gamma-\nu_{n-2})} + \frac{1}{\sigma\Gamma(\gamma-\nu_{n-2})} \Bigg(\sum_{i=1}^{p} \frac{a_i}{\Gamma(\gamma-\alpha_i)} \int_{I_i} w_i(s) s^{\gamma-\alpha_i-1}\,dA_i(s) \\
&+ \sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)} \xi_j^{\gamma-\beta_j-1} \Bigg) \Bigg\} t^{\gamma-\nu_{n-2}-1} \int_0^1 (1-s)^{\gamma-\gamma_0-1} x(s)\,ds \\
&- \frac{t^{\gamma-\nu_{n-2}-1}}{\sigma\Gamma(\gamma-\nu_{n-2})} \Bigg\{ \sum_{i=1}^{p} \frac{a_i}{\Gamma(\gamma-\alpha_i)} \int_{I_i} \left(\int_0^s (s-\tau)^{\gamma-\alpha_i-1} x(\tau)\,d\tau \right) w_i(s)\,dA_i(s) \\
&- \sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)} \int_0^{\xi_j} (\xi_j-\tau)^{\gamma-\beta_j-1} x(\tau)\,d\tau \Bigg\} \\
= &-\frac{1}{\Gamma(\gamma-\nu_{n-2})} \int_0^t (t-s)^{\gamma-\nu_{n-2}-1} x(s)\,ds + \frac{t^{\gamma-\nu_{n-2}-1}}{\Gamma(\gamma-\nu_{n-2})} \int_0^1 (1-s)^{\gamma-\gamma_0-1} x(s)\,ds \\
&+ \sum_{i=1}^{p} \frac{a_i t^{\gamma-\nu_{n-2}-1}}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\alpha_i)} \int_{I_i} \left(\int_0^1 s^{\gamma-\alpha_i-1} (1-\tau)^{\gamma-\gamma_0-1} x(\tau)\,d\tau \right) w_i(s)\,dA_i(s) \\
&- \sum_{i=1}^{p} \frac{a_i t^{\gamma-\nu_{n-2}-1}}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\alpha_i)} \int_{I_i} \left(\int_0^s (s-\tau)^{\gamma-\gamma_0-1} x(\tau)\,d\tau \right) w_i(s)\,dA_i(s) \\
&+ \sum_{j=1}^{\infty} \frac{b_j t^{\gamma-\nu_{n-2}-1}}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\beta_j)} \int_0^1 \xi_j^{\gamma-\beta_j-1} (1-s)^{\gamma-\gamma_0-1} x(s)\,ds \\
&- \sum_{j=1}^{\infty} \frac{b_j t^{\gamma-\nu_{n-2}-1}}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\beta_j)} \int_0^{\xi_j} (\xi_j-s)^{\gamma-\beta_j-1} x(s)\,ds \\
= &\int_0^1 K_0(t,s)x(s)\,ds + \sum_{i=1}^{p} \int_{I_i} t^{\gamma-\nu_{n-2}-1} \left(\int_0^1 K_i(s,\tau)x(\tau)\,d\tau \right) w_i(s)\,dA_i(s) \\
&+ \sum_{j=1}^{\infty} \int_0^1 t^{\gamma-\nu_{n-2}-1} H_j(\xi_j,s)x(s)\,ds \\
= &\int_0^1 K_0(t,s)x(s)\,ds + \int_0^1 t^{\gamma-\nu_{n-2}-1} \sum_{i=1}^{p} \left(\int_{I_i} K_i(\tau,s) w_i(\tau)\,dA_i(\tau) \right) x(s)\,ds \\
&+ \int_0^1 t^{\gamma-\nu_{n-2}-1} \sum_{j=1}^{\infty} H_j(\xi_j,s)x(s)\,ds \\
= &\int_0^1 K(t,s)x(s)\,ds.
\end{aligned}
$$

The proof is complete. □

Lemma 2.12 *Let $\sigma > 0$ (defined in (2.19) of Lemma 2.11), $\int_{I_i} s^{\gamma-\alpha_i-1} w_i(s)\,dA_i(s) \geq 0$ ($i = 1,2,\ldots,p$), and $0 < \sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)} \xi_j^{\gamma-\beta_j-1} < \infty$. Then the functions $K_0(t,s)$, $K_i(t,s)$ ($i = 1,2,\ldots,p$) and $H_j(t,s)$ ($j = 1,2,\ldots$) given in Lemma 2.11 have the following properties:*

(i) $t^{\gamma-\nu_{n-2}-1} k_0(s) \leq K_0(t,s) \leq \frac{1}{\Gamma(\gamma-\nu_{n-2})} t^{\gamma-\nu_{n-2}-1}$, *where*

$$k_0(s) = \frac{1}{\Gamma(\gamma-\nu_{n-2})}(1-s)^{\gamma-\gamma_0-1}\bigl(1-(1-s)^{\gamma_0-\nu_{n-2}}\bigr).$$

(ii) $t^{\gamma-\alpha_i-1} k_i(s) \leq K_i(t,s) \leq \frac{a_i}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\alpha_i)} t^{\gamma-\alpha_i-1}$ ($i = 1,2,\ldots,p$), *where*

$$k_i(s) = \frac{a_i}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\alpha_i)}(1-s)^{\gamma-\gamma_0-1}\bigl(1-(1-s)^{\gamma_0-\alpha_i}\bigr).$$

(iii) $t^{\gamma-\beta_j-1} h_j(s) \leq H_j(t,s) \leq \frac{b_j}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\beta_j)} t^{\gamma-\beta_j-1}$ ($j = 1,2,\ldots$), *where*

$$h_j(s) = \frac{b_j}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\beta_j)}(1-s)^{\gamma-\gamma_0-1}\bigl(1-(1-s)^{\gamma_0-\beta_j}\bigr).$$

Proof (i) For $s \leq t$,

$$\begin{aligned}
K_0(t,s) &= \frac{1}{\Gamma(\gamma-\nu_{n-2})}\bigl(t^{\gamma-\nu_{n-2}-1}(1-s)^{\gamma-\gamma_0-1} - (t-s)^{\gamma-\nu_{n-2}-1}\bigr) \\
&\geq \frac{t^{\gamma-\nu_{n-2}-1}}{\Gamma(\gamma-\nu_{n-2})}\bigl((1-s)^{\gamma-\gamma_0-1} - (1-s)^{\gamma-\nu_{n-2}-1}\bigr) \\
&= \frac{t^{\gamma-\nu_{n-2}-1}(1-s)^{\gamma-\gamma_0-1}}{\Gamma(\gamma-\nu_{n-2})}\bigl(1-(1-s)^{\gamma_0-\nu_{n-2}}\bigr) \\
&= t^{\gamma-\nu_{n-2}-1} k_0(s), \\
K_0(t,s) &\leq \frac{t^{\gamma-\nu_{n-2}-1}}{\Gamma(\gamma-\nu_{n-2})}.
\end{aligned}$$

For $t \leq s$,

$$\begin{aligned}
K_0(t,s) &= \frac{t^{\gamma-\nu_{n-2}-1}}{\Gamma(\gamma-\nu_{n-2})}(1-s)^{\gamma-\gamma_0-1} \\
&\geq \frac{t^{\gamma-\nu_{n-2}-1}(1-s)^{\gamma-\gamma_0-1}}{\Gamma(\gamma-\nu_{n-2})}\bigl(1-(1-s)^{\gamma_0-\nu_{n-2}}\bigr) \\
&= t^{\gamma-\nu_{n-2}-1} k_0(s), \\
K_0(t,s) &\leq \frac{t^{\gamma-\nu_{n-2}-1}}{\Gamma(\gamma-\nu_{n-2})}.
\end{aligned}$$

Using the same argument again, it is straightforward to infer (ii) and (iii). The proof is complete. □

Lemma 2.13 *Let $\sigma > 0$ (defined in (2.19) of Lemma 2.11), $\int_{I_i} s^{\gamma-\alpha_i-1} w_i(s)\,dA_i(s) \geq 0$ ($i = 1,2,\ldots,p$), and $0 < \sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)} \xi_j^{\gamma-\beta_j-1} < \infty$. Then the Green's function $K(t,s)$ defined in Lemma 2.11 satisfies:*

(i) $K(t,s) \leq Q_1 e(t)$ *for* $t,s \in [0,1]$, *where* $e(t) = t^{\gamma-\nu_{n-2}-1}$,

$$Q_1 = \frac{1}{\Gamma(\gamma-\nu_{n-2})} + \sum_{i=1}^{p} \frac{a_i}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\alpha_i)} \int_{I_i} \tau^{\gamma-\alpha_i-1} w_i(\tau)\,dA_i(\tau) + \sum_{j=1}^{\infty} \frac{b_j}{\sigma\Gamma(\gamma-\nu_{n-2})\Gamma(\gamma-\beta_j)} \xi_j^{\gamma-\beta_j-1}.$$

(ii) $K(t,s) \geq Q_2(s)e(t)$ *for* $t,s \in [0,1]$, *where*

$$Q_2(s) = k_0(s) + \sum_{i=1}^{p} k_i(s) \int_{I_i} \tau^{\gamma-\alpha_i-1} w_i(\tau)\,dA_i(\tau) + \sum_{j=1}^{\infty} H_j(\xi_j,s).$$

(iii) $K(t,s) > 0$ *for* $t,s \in (0,1)$.

Proof The conclusion can be easily given by Lemma 2.12. So we omit it. □

In this paper, we equip $E = C[0,1]$ with the norm $\|u\| = \sup_{0\leq t\leq 1} |u(t)|$. Then $(E, \|\cdot\|)$ is a Banach space. Let $P = \{u \in E : u(t) \geq 0, t \in [0,1]\}$ be a cone in E. Let us define a nonlinear operator $T_\lambda : P \to P$ by

$$(T_\lambda u)(t) = \lambda \int_0^1 K(t,s) f\bigl(s, I_{0+}^{\nu_{n-2}} u(s), I_{0+}^{\nu_{n-2}-\nu_1} u(s), \ldots, I_{0+}^{\nu_{n-2}-\nu_{n-3}} u(s), u(s)\bigr)\,ds, \quad t \in [0,1]. \tag{2.20}$$

It is easy to check that BVP (2.1) has a solution if and only if the operator T_λ has a fixed point.

3 Main results

Theorem 3.1 *Suppose that* $f \in C((0,1) \times (0,+\infty)^{n-1}, \mathbb{R}_+)$ *satisfies*:

(H_1) *There exist two functions* $f_1, f_2 \in C((0,1) \times (0,+\infty)^{2(n-1)}, \mathbb{R}_+)$ *such that*

$$f(t,x_1,x_2,\ldots,x_{n-1}) = f_1(t,x_1,x_2,\ldots,x_{n-1},x_1,x_2,\ldots,x_{n-1}) + f_2(t,x_1,x_2,\ldots,x_{n-1},x_1,x_2,\ldots,x_{n-1}).$$

(H_2) *For all* $t \in (0,1)$ *and* $(y_1,y_2,\ldots,y_{n-1}) \in (0,+\infty)^{n-1}$, $f_1(t,x_1,x_2,\ldots,x_{n-1},y_1,y_2,\ldots,y_{n-1})$, $f_2(t,x_1,x_2,\ldots,x_{n-1},y_1,y_2,\ldots,y_{n-1})$ *are increasing in* $(x_1,x_2,\ldots,x_{n-1}) \in (0,+\infty)^{n-1}$; *for all* $t \in (0,1)$ *and* $(x_1,x_2,\ldots,x_{n-1}) \in (0,+\infty)^{n-1}$, $f_1(t,x_1,x_2,\ldots,x_{n-1},y_1,y_2,\ldots,y_{n-1})$, $f_2(t,x_1,x_2,\ldots,x_{n-1},y_1,y_2,\ldots,y_{n-1})$ *are decreasing in* $(y_1,y_2,\ldots,y_{n-1}) \in (0,+\infty)^{n-1}$.

(H_3) *For all* $\mu \in (0,1)$, *there exists* $\varphi(\mu) \in (\mu,1]$ *such that, for all* $t \in (0,1)$ *and* $(x_1,x_2,\ldots,x_{n-1}), (y_1,y_2,\ldots,y_{n-1}) \in (0,+\infty)^{n-1}$,

$$f_1\bigl(t,\mu x_1,\mu x_2,\ldots,\mu x_{n-1},\mu^{-1}y_1,\mu^{-1}y_2,\ldots,\mu^{-1}y_{n-1}\bigr) \geq \varphi(\mu) f_1(t,x_1,x_2,\ldots,x_{n-1},y_1,y_2,\ldots,y_{n-1}),$$

$$f_2\big(t, \mu x_1, \mu x_2, \dots, \mu x_{n-1}, \mu^{-1} y_1, \mu^{-1} y_2, \dots, \mu^{-1} y_{n-1}\big)$$
$$\geq \mu f_2(t, x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_{n-1}).$$

(H_4) *There exists a constant $\kappa > 0$ such that, for all $t \in (0,1)$ and $(x_1, x_2, \dots, x_{n-1})$, $(y_1, y_2, \dots, y_{n-1}) \in (0, +\infty)^{n-1}$,*

$$f_1(t, x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_{n-1}) \geq \kappa f_2(t, x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_{n-1}).$$

(H_5) *The functions f_1 and f_2 satisfy*

$$0 < \int_0^1 f_1\big(s, 1, 1, \dots, 1, s^{\gamma-1}, s^{\gamma-1}, \dots, s^{\gamma-1}\big)\, ds < +\infty,$$
$$0 < \int_0^1 f_2\big(s, 1, 1, \dots, 1, s^{\gamma-1}, s^{\gamma-1}, \dots, s^{\gamma-1}\big)\, ds < +\infty.$$

Then BVP (1.1) *has a unique solution z_λ^* in P, and there exists a constant $\eta_\lambda \in (0,1)$ such that*

$$\frac{\eta_\lambda \Gamma(\gamma - \nu_{n-2})}{\Gamma(\gamma)} t^{\gamma-1} \leq z_\lambda^*(t) \leq \frac{\Gamma(\gamma - \nu_{n-2})}{\eta_\lambda \Gamma(\gamma)} t^{\gamma-1}, \quad t \in [0,1].$$

And at the same time, z_λ^* satisfies:

(i) *If there exists $r \in (0,1)$ such that*

$$\varphi(\mu) \geq \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),$$

then z_λ^* is continuous with respect to $\lambda \in (0, +\infty)$, i.e., for $\forall\, \lambda_0 \in (0, +\infty)$,

$$\big\| z_\lambda^* - z_{\lambda_0}^* \big\| \to 0, \quad \text{as } \lambda \to \lambda_0.$$

(ii) *If*

$$\varphi(\mu) \geq \frac{\mu^{\frac{1}{2}} - \mu}{\kappa} + \mu^{\frac{1}{2}}, \quad \forall \mu \in (0,1),$$

then $0 < \lambda_1 < \lambda_2$ implies $z_{\lambda_1}^* < z_{\lambda_2}^*$.

(iii) *If there exists $r \in (0, \frac{1}{2})$ such that*

$$\varphi(\mu) \geq \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),$$

then

$$\lim_{\lambda \to 0^+} \big\| z_\lambda^* \big\| = 0, \qquad \lim_{\lambda \to +\infty} \big\| z_\lambda^* \big\| = +\infty.$$

Moreover, for any initial values z_0, $\tilde{z}_0 \in P_e$, by constructing successively the sequences as follows:

$$\begin{aligned}
z_n(t) = I_{0^+}^{\nu_{n-2}} \bigg\{ &\lambda \int_0^1 K(t,s) f_1\big(s, I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s), \\
& I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s)\big)\,ds \\
& + \lambda \int_0^1 K(t,s) f_2\big(s, I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s), \\
& I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s)\big)\,ds \bigg\}, \\
\tilde{z}_n(t) = I_{0^+}^{\nu_{n-2}} \bigg\{ &\lambda \int_0^1 K(t,s) f_1\big(s, I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s), \\
& I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s)\big)\,ds \\
& + \lambda \int_0^1 K(t,s) f_2\big(s, I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s), \\
& I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s)\big)\,ds \bigg\}, \\
& n = 1, 2, \ldots,
\end{aligned}$$

we have $z_n \to z_\lambda^*$ and $\tilde{z}_n \to z_\lambda^*$ in E, as $n \to \infty$.

Proof Let $P_e = \{u \in E : u \sim e\}$, where $e(t) = t^{\gamma-\nu_{n-2}-1}$. Then P_e is a component of P. Now, we define two operators $A_\lambda, B_\lambda : P_e \times P_e \to P$ by

$$\begin{aligned}
A_\lambda(u,v)(t) = \lambda \int_0^1 & K(t,s) f_1\big(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \ldots, u(s), \\
& I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \ldots, v(s)\big)\,ds, \\
B_\lambda(u,v)(t) = \lambda \int_0^1 & K(t,s) f_2\big(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \ldots, u(s), \\
& I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \ldots, v(s)\big)\,ds.
\end{aligned}$$

Combining the definition of T_λ in (2.20) and (H_1), we have

$$\begin{aligned}
(T_\lambda u)(t) &= \lambda \int_0^1 K(t,s) f\big(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \ldots, I_{0^+}^{\nu_{n-2}-\nu_{n-3}} u(s), u(s)\big)\,ds \\
&= \lambda \int_0^1 K(t,s) f_1\big(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \ldots, u(s), \\
&\quad I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \ldots, u(s)\big)\,ds \\
&\quad + \lambda \int_0^1 K(t,s) f_2\big(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \ldots, u(s), \\
&\quad I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \ldots, u(s)\big)\,ds \\
&= A_\lambda(u,u)(t) + B_\lambda(u,u)(t), \quad t \in [0,1]. \qquad (3.1)
\end{aligned}$$

Then we can conclude that u is the solution of BVP (2.1) if u satisfies $u = A_\lambda(u,u) + B_\lambda(u,u)$.

We prove that $A_\lambda, B_\lambda : P_e \times P_e \to P$ are well defined at first. For any $u, v \in P_e$, there exists a constant $\omega \in (0,1)$ such that $\omega e(t) \le u(t) \le \frac{1}{\omega} e(t), \omega e(t) \le v(t) \le \frac{1}{\omega} e(t), t \in [0,1]$. Moreover, by the definition of fractional integral and $e(t) \le 1$, for all $t \in [0,1]$,

$$\begin{aligned} I_{0^+}^{\nu_{n-2}} e(t) &= \frac{1}{\Gamma(\nu_{n-2})} \int_0^t (t-s)^{\nu_{n-2}-1} s^{\gamma-\nu_{n-2}-1}\, ds \\ &= \frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma)} t^{\gamma-1} \le 1 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} I_{0^+}^{\nu_{n-2}-\nu_i} e(t) &= \frac{1}{\Gamma(\nu_{n-2}-\nu_i)} \int_0^t (t-s)^{\nu_{n-2}-\nu_i-1} s^{\gamma-\nu_{n-2}-1}\, ds \\ &= \frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma-\nu_i)} t^{\gamma-\nu_i-1} \le 1, \quad i = 1,2,\dots,n-3. \end{aligned} \tag{3.3}$$

Thus, by (H_1), (H_2), (H_3), (H_5), (3.2), and (3.3), we know that, for all $t \in [0,1]$,

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_0^1 K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \dots, u(s), \\ &\quad I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \dots, v(s)\bigr)\, ds \\ &\le \lambda \int_0^1 K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} \omega^{-1} e(s), I_{0^+}^{\nu_{n-2}-\nu_1} \omega^{-1} e(s), \dots, \omega^{-1} e(s), \\ &\quad I_{0^+}^{\nu_{n-2}} \omega e(s), I_{0^+}^{\nu_{n-2}-\nu_1} \omega e(s), \dots, \omega e(s)\bigr)\, ds \\ &\le \lambda \int_0^1 K(t,s) f_1\biggl(s, \omega^{-1}, \omega^{-1}, \dots, \omega^{-1}, \\ &\quad \frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma)} \omega s^{\gamma-1}, \frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma-\nu_1)} \omega s^{\gamma-\nu_1-1}, \dots, \omega s^{\gamma-\nu_{n-2}-1}\biggr)\, ds \\ &\le \lambda \int_0^1 K(t,s) f_1\bigl(s, (\rho\omega)^{-1}, (\rho\omega)^{-1}, \dots, (\rho\omega)^{-1}, \\ &\quad \rho\omega s^{\gamma-1}, \rho\omega s^{\gamma-\nu_1-1}, \dots, \omega s^{\gamma-\nu_{n-3}-1}\bigr)\, ds \\ &\le \lambda \frac{Q_1}{\varphi(\rho\omega)} e(t) \int_0^1 f_1\bigl(s, 1, 1, \dots, 1, s^{\gamma-1}, s^{\gamma-1}, \dots, s^{\gamma-1}\bigr)\, ds \\ &< +\infty, \end{aligned} \tag{3.4}$$

where

$$\rho = \min\biggl\{ \frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma)}, \frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma-\nu_1)}, \dots, \frac{\Gamma(\gamma-\nu_{n-2})}{\Gamma(\gamma-\nu_{n-3})}, 1 \biggr\} > 0.$$

Similarly, for all $t \in [0,1]$,

$$B_\lambda(u,v)(t) \le \lambda \frac{Q_1}{\rho\omega} e(t) \int_0^1 f_2\bigl(s, 1, 1, \dots, 1, s^{\gamma-1}, s^{\gamma-1}, \dots, s^{\gamma-1}\bigr)\, ds < +\infty. \tag{3.5}$$

So, $A_\lambda, B_\lambda : P_e \times P_e \to P$ are well defined.

Now, we prove that $A_\lambda, B_\lambda : P_e \times P_e \to P_e$. Taking a constant $W > 1$ such that

$$\begin{aligned} W > \max\biggl\{ & \frac{\lambda Q_1}{\varphi(\rho\omega)} \int_0^1 f_1\bigl(s,1,1,\ldots,1,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1}\bigr)\,ds, \\ & \frac{\lambda Q_1}{\rho\omega} \int_0^1 f_2\bigl(s,1,1,\ldots,1,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1}\bigr)\,ds, \\ & \biggl(\lambda\varphi(\rho\omega) \int_0^1 Q_2(s) f_1\bigl(s,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1},1,1,\ldots,1\bigr)\,ds\biggr)^{-1}, \\ & \biggl(\lambda\rho\omega \int_0^1 Q_2(s) f_2\bigl(s,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1},1,1,\ldots,1\bigr)\,ds\biggr)^{-1}\biggr\}. \end{aligned} \tag{3.6}$$

Then from (H_1), (H_2), and (H_3), for all $t \in [0,1]$,

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_0^1 K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s),\ldots,u(s), \\ &\qquad I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s),\ldots,v(s)\bigr)\,ds \\ &\geq \lambda \int_0^1 K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} \omega e(s), I_{0^+}^{\nu_{n-2}-\nu_1} \omega e(s),\ldots,\omega e(s), \\ &\qquad I_{0^+}^{\nu_{n-2}} \omega^{-1} e(s), I_{0^+}^{\nu_{n-2}-\nu_1} \omega^{-1} e(s),\ldots,\omega^{-1} e(s)\bigr)\,ds \\ &\geq \lambda \int_0^1 K(t,s) f_1\bigl(s, \rho\omega s^{\gamma-1}, \rho\omega s^{\gamma-\nu_1-1},\ldots,\omega s^{\gamma-\nu-1}, \\ &\qquad (\rho\omega)^{-1},(\rho\omega)^{-1},\ldots,(\rho\omega)^{-1}\bigr)\,ds \\ &\geq \lambda\varphi(\rho\omega) e(t) \int_0^1 Q_2(s) f_1\bigl(s,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1},1,1,\ldots,1\bigr)\,ds \\ &\geq W^{-1} e(t) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} B_\lambda(u,v)(t) &\geq \lambda\rho\omega e(t) \int_0^1 Q_2(s) f_2\bigl(s,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1},1,1,\ldots,1\bigr)\,ds \\ &\geq W^{-1} e(t). \end{aligned} \tag{3.8}$$

On the other hand, from (3.4) and (3.5), we know, for all $u, v \in P_e$, $t \in [0,1]$,

$$\begin{aligned} A_\lambda(u,v)(t) &\leq \frac{\lambda Q_1}{\varphi(\rho\omega)} e(t) \int_0^1 f_1\bigl(s,1,1,\ldots,1,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1}\bigr)\,ds \\ &\leq W e(t) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} B_\lambda(u,v)(t) &\leq \frac{\lambda Q_1}{\rho\omega} e(t) \int_0^1 f_2\bigl(s,1,1,\ldots,1,s^{\gamma-1},s^{\gamma-1},\ldots,s^{\gamma-1}\bigr)\,ds \\ &\leq W e(t). \end{aligned} \tag{3.10}$$

So, $A_\lambda, B_\lambda : P_e \times P_e \to P_e$.

In fact, from (H_2), it is easy to check that A_λ, B_λ are mixed monotone operators. Furthermore, it follows from (H_3) that, for all $\mu \in (0,1)$, there exists $\varphi(\mu) \in (\mu, 1]$ such that, for any $u, v \in P_e$, $t \in [0,1]$,

$$\begin{aligned} A_\lambda\bigl(\mu u, \mu^{-1}v\bigr)(t) = {}& \lambda \int_0^1 K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} \mu u(s), I_{0^+}^{\nu_{n-2}-\nu_1} \mu u(s), \dots, \mu u(s), \\ & I_{0^+}^{\nu_{n-2}} \mu^{-1} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} \mu^{-1} v(s), \dots, \mu^{-1} v(s)\bigr)\, ds \\ = {}& \lambda \int_0^1 K(t,s) f_1\bigl(s, \mu I_{0^+}^{\nu_{n-2}} u(s), \mu I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \dots, \mu u(s), \\ & \mu^{-1} I_{0^+}^{\nu_{n-2}} v(s), \mu^{-1} I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \dots, \mu^{-1} v(s)\bigr)\, ds \\ \geq {}& \lambda \varphi(\mu) \int_0^1 K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \dots, u(s), \\ & I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \dots, v(s)\bigr)\, ds \\ = {}& \varphi(\mu) A_\lambda(u,v)(t) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} B_\lambda\bigl(\mu u, \mu^{-1}v\bigr)(t) = {}& \lambda \int_0^1 K(t,s) f_2\bigl(s, I_{0^+}^{\nu_{n-2}} \mu u(s), I_{0^+}^{\nu_{n-2}-\nu_1} \mu u(s), \dots, \mu u(s), \\ & I_{0^+}^{\nu_{n-2}} \mu^{-1} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} \mu^{-1} v(s), \dots, \mu^{-1} v(s)\bigr)\, ds \\ \geq {}& \lambda \mu \int_0^1 K(t,s) f_2\bigl(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \dots, u(s), \\ & I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \dots, v(s)\bigr)\, ds \\ = {}& \mu B_\lambda(u,v)(t). \end{aligned} \tag{3.12}$$

By (H_4), we infer that there exists $\kappa > 0$ such that, for all u, $v \in P_e$, $t \in [0,1]$,

$$\begin{aligned} A_\lambda(u,v)(t) = {}& \lambda \int_0^1 K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} u(s), \dots, u(s), \\ & I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \dots, v(s)\bigr)\, ds \\ \geq {}& \kappa \lambda \int_0^1 K(t,s) f_2\bigl(s, I_{0^+}^{\nu_{n-2}} u(s), I_{0^+}^{\nu_{n-2}-\nu_1} z(s), \dots, u(s), \\ & I_{0^+}^{\nu_{n-2}} v(s), I_{0^+}^{\nu_{n-2}-\nu_1} v(s), \dots, v(s)\bigr)\, ds \\ = {}& \kappa B_\lambda(u,v)(t). \end{aligned} \tag{3.13}$$

Combining (3.11)–(3.13) and using Lemma 2.2, we infer that there exists a unique fixed point $u_\lambda^* \in P_e$ such that

$$A_\lambda\bigl(u_\lambda^*, u_\lambda^*\bigr) + B_\lambda\bigl(u_\lambda^*, u_\lambda^*\bigr) = u_\lambda^*.$$

That is, BVP (2.1) has a unique solution $u_\lambda^* \in P_e$. Since $u_\lambda^* \in P_e$, there exists a constant $\eta_\lambda \in (0,1)$ such that

$$\eta_\lambda t^{\gamma-\nu_{n-2}-1} \le u_\lambda^*(t) \le \frac{1}{\eta_\lambda} t^{\gamma-\nu_{n-2}-1}, \quad t \in [0,1]. \tag{3.14}$$

Moreover, for any $\lambda > 0$, let $\tilde{A} = (\lambda^{-1}A_\lambda)$ and $\tilde{B} = (\lambda^{-1}B_\lambda)$. Obviously, $\tilde{A}$ and $\tilde{B}$ satisfy all the conditions of Lemma 2.3. With the preceding proof, we can infer that u_λ^* is the unique positive solution of the following equation:

$$\lambda\tilde{A}(u,u) + \lambda\tilde{B}(u,u) = A(u,u) + B(u,u) = u.$$

By means of Lemma 2.3, we know that u_λ^* satisfies:

(1) If there exists $r \in (0,1)$ such that

$$\varphi(\mu) \ge \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),$$

then u_λ^* is continuous with respect to $\lambda \in (0,+\infty)$. That is, for $\forall\ \lambda_0 \in (0,+\infty)$,

$$\left\| u_\lambda^* - u_{\lambda_0}^* \right\| \to 0, \quad \text{as } \lambda \to \lambda_0.$$

(2) If

$$\varphi(\mu) \ge \frac{\mu^{\frac{1}{2}} - \mu}{\kappa} + \mu^{\frac{1}{2}}, \quad \forall \mu \in (0,1),$$

then $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* < u_{\lambda_2}^*$.

(3) If there exists $r \in (0, \frac{1}{2})$ such that

$$\varphi(\mu) \ge \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),$$

then

$$\lim_{\lambda \to 0^+} \left\| u_\lambda^* \right\| = 0, \qquad \lim_{\lambda \to +\infty} \left\| u_\lambda^* \right\| = +\infty.$$

Furthermore, by Lemma 2.2, we can infer that, for any initial values $u_0, v_0 \in P_e$, by constructing successively the sequences as follows:

$$\begin{aligned} u_n(t) = \lambda \int_0^1 & K(t,s) f_1\big(s, I_{0^+}^{\nu_{n-2}} u_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} u_{n-1}(s), \ldots, u_{n-1}(s), \\ & I_{0^+}^{\nu_{n-2}} v_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} v_{n-1}(s), \ldots, v_{n-1}(s)\big)\, ds \\ & + \lambda \int_0^1 K(t,s) f_2\big(s, I_{0^+}^{\nu_{n-2}} u_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} u_{n-1}(s), \ldots, u_{n-1}(s), \\ & I_{0^+}^{\nu_{n-2}} v_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} v_{n-1}(s), \ldots, v_{n-1}(s)\big)\, ds, \end{aligned}$$

$$
\begin{aligned}
v_n(t) = \lambda \int_0^1 & K(t,s) f_1\bigl(s, I_{0^+}^{\nu_{n-2}} v_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} v_{n-1}(s), \ldots, v_{n-1}(s), \\
& I_{0^+}^{\nu_{n-2}} u_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} u_{n-1}(s), \ldots, u_{n-1}(s)\bigr)\, ds \\
& + \lambda \int_0^1 K(t,s) f_2\bigl(s, I_{0^+}^{\nu_{n-2}} v_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} v_{n-1}(s), \ldots, v_{n-1}(s), \\
& I_{0^+}^{\nu_{n-2}} u_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} u_{n-1}(s), \ldots, u_{n-1}(s)\bigr)\, ds, \\
& t \in [0,1], \quad n = 1, 2, \ldots,
\end{aligned}
$$

we have

$$
u_n \to u_\lambda^*, \qquad v_n \to u_\lambda^*, \quad \text{in } E, \text{ as } n \to \infty.
$$

Finally, by what we have proved in Lemma 2.9, we know $z_\lambda^* = I^{\nu_{n-2}} u_\lambda^*$ is the unique positive solution of BVP (1.1). From (3.14), we know that z^* satisfies

$$
\frac{\eta_\lambda \Gamma(\gamma - \nu_{n-2})}{\Gamma(\gamma)} t^{\gamma-1} \le z_\lambda^*(t) \le \frac{\Gamma(\gamma - \nu_{n-2})}{\eta_\lambda \Gamma(\gamma)} t^{\gamma-1}, \quad t \in [0,1]. \tag{3.15}
$$

And from the monotonicity and continuity of fractional integral, we get:

(i) If there exists $r \in (0,1)$ such that

$$
\varphi(\mu) \ge \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),
$$

then for $\forall\, \lambda_0 \in (0, +\infty)$,

$$
\|z_\lambda^* - z_{\lambda_0}^*\| \to 0, \quad \text{as } \lambda \to \lambda_0.
$$

(ii) If

$$
\varphi(\mu) \ge \frac{\mu^{\frac{1}{2}} - \mu}{\kappa} + \mu^{\frac{1}{2}}, \quad \forall \mu \in (0,1),
$$

then $0 < \lambda_1 < \lambda_2$ implies $z_{\lambda_1}^* < z_{\lambda_2}^*$.

(iii) If there exists $r \in (0, \frac{1}{2})$ such that

$$
\varphi(\mu) \ge \frac{\mu^r - \mu}{\kappa} + \mu^r, \quad \forall \mu \in (0,1),
$$

then

$$
\lim_{\lambda \to 0^+} \|z_\lambda^*\| = 0, \qquad \lim_{\lambda \to +\infty} \|z_\lambda^*\| = +\infty.
$$

Furthermore, for any initial values $z_0, \tilde{z}_0 \in P_e$, by constructing successively the sequences as follows:

$$\begin{aligned} z_n(t) = I_{0^+}^{\nu_{n-2}} \bigg\{ &\lambda \int_0^1 K(t,s) f_1\big(s, I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s), \\ & I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s)\big)\,ds \\ & + \lambda \int_0^1 K(t,s) f_2\big(s, I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s), \\ & I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s)\big)\,ds \bigg\}, \\ \tilde{z}_n(t) = I_{0^+}^{\nu_{n-2}} \bigg\{ &\lambda \int_0^1 K(t,s) f_1\big(s, I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s), \\ & I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s)\big)\,ds \\ & + \lambda \int_0^1 K(t,s) f_2\big(s, I_{0^+}^{\nu_{n-2}} \tilde{z}_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} \tilde{z}_{n-1}(s), \ldots, \tilde{z}_{n-1}(s), \\ & I_{0^+}^{\nu_{n-2}} z_{n-1}(s), I_{0^+}^{\nu_{n-2}-\nu_1} z_{n-1}(s), \ldots, z_{n-1}(s)\big)\,ds \bigg\}, \\ & t \in [0,1], \quad n = 1, 2, \ldots, \end{aligned}$$

we have

$$z_n \to z_\lambda^*, \qquad \tilde{z}_n \to z_\lambda^*, \quad \text{in } E, \text{ as } n \to \infty.$$

The proof of Theorem 3.1 is completed. □

4 Examples

Example 4.1 We consider the following problem:

$$\begin{cases} D_{0+}^{\frac{5}{2}} z(t) + \lambda(6t^{-\frac{1}{3}} u^{\frac{1}{8}} (D_{0+}^{\frac{3}{5}} z(t))^{-\frac{1}{8}} + 5u^{\frac{1}{6}}(1-t^2)^{-\frac{1}{2}} + (1+t^2)^{-1} \\ \quad \times ((6D_{0+}^{\frac{3}{5}} z(t))^{\frac{1}{3}} + (D_{0+}^{\frac{3}{5}} z(t))^{\frac{1}{4}} + 1) + (tu)^{-\frac{1}{5}}(7 + (u+1)^{-\frac{4}{5}})) = 0, \quad 0 < t < 1, \\ z(0) = D_{0+}^{\frac{2}{3}} z(0) = 0, \\ D_{0+}^{\frac{3}{2}} z(1) = 2\int_0^1 s^{\frac{3}{4}} (1-s)^2 D_{0+}^{\frac{5}{4}} z(s)\,dA_1(s) + \frac{1}{2}\int_0^{\frac{2}{3}} s^{\frac{7}{8}} (1+s^2)^{-1} D_{0+}^{\frac{11}{8}} z(s)\,dA_2(s) \\ \qquad + \sum_{j=1}^{\infty} (5j-4)^{-1}(5j+1)^{-1} D_{0+}^{\frac{3}{2}-\frac{1}{2(7+j)}} u((28+2j)^{-1}), \end{cases} \tag{4.1}$$

where $\lambda > 0$ is a parameter, and

$$A_1(t) = \begin{cases} \frac{1}{7}, & t \in [0, \frac{1}{2}), \\ \frac{8}{7}, & t \in [\frac{1}{2}, 1], \end{cases} \qquad A_2(t) = \begin{cases} \frac{1}{9}, & t \in [0, \frac{1}{2}), \\ \frac{10}{9}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Let

$$\begin{aligned} f(t,u,v) = {}& 6t^{-\frac{1}{3}} u^{\frac{1}{8}} v^{-\frac{1}{8}} + 5u^{\frac{1}{6}}\big(1-t^2\big)^{-\frac{1}{2}} + \big(6v^{\frac{1}{3}} + v^{\frac{1}{4}} + 1\big)\big(1+t^2\big)^{-1} \\ & + (tu)^{-\frac{1}{5}}\big(7 + (u+1)^{-\frac{4}{5}}\big), \end{aligned}$$

$n = 3$, $\gamma = \frac{5}{2}$, $\gamma_0 = \frac{3}{2}$, $\nu_1 = \frac{3}{5}$, $q_1 = \frac{2}{3}$, $a_1 = 2$, $a_2 = \frac{1}{2}$, $\alpha_1 = \frac{5}{4}$, $\alpha_2 = \frac{11}{8}$, $I_1 = [0,1]$, $I_2 = [0,\frac{2}{3}]$, $b_j = (5j-4)^{-1}(5j+1)^{-1}$ $(j = 1,2,\ldots)$, $\beta_j = \frac{3}{2} - \frac{1}{2^{(7+j)}}$ $(j = 1,2,\ldots)$, $\xi_j = (28+2j)^{-1}$ $(j = 1,2,\ldots)$, $w_1(t) = t^{\frac{3}{4}}(1-t)^2$, $w_2(t) = t^{\frac{7}{8}}(1+t^2)^{-1}$. Then problem (4.1) can be transformed into BVP (1.1) for $\lambda > 0$.

By simple computation, we have a rough estimate

$$\begin{aligned}
&\int_{I_1} \tau^{\gamma-\alpha_1-1} w_1(\tau)\,dA_1(\tau) = \int_0^1 \tau(1-\tau)^2\,dA_1(\tau) = 0.125 > 0,\\
&\int_{I_2} \tau^{\gamma-\alpha_2-1} w_2(\tau)\,dA_2(\tau) = \int_0^{\frac{2}{3}} \tau\left(1+\tau^2\right)^{-1}\,dA_2(\tau) = 0.4 > 0,\\
&\sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)}\xi_j^{\gamma-\beta_j-1} = \sum_{j=1}^{\infty} \frac{1}{\Gamma(1+2^{-(7+j)})}(5j-4)^{-1}(5j+1)^{-1}(28+2j)^{-\frac{1}{2^{(7+j)}}}\\
&\qquad \le \sum_{j=1}^{\infty}(5j-4)^{-1}(5j+1)^{-1} = 0.2,
\end{aligned}$$

and

$$\begin{aligned}
\sigma &= \frac{1}{\Gamma(\gamma-\gamma_0)} - \sum_{i=1}^{p} \frac{a_i}{\Gamma(\gamma-\alpha_i)} \int_{I_i} s^{\gamma-\alpha_i-1} w_i(s)\,dA_i(s) - \sum_{j=1}^{\infty} \frac{b_j}{\Gamma(\gamma-\beta_j)}\xi_j^{\gamma-\beta_j-1}\\
&\approx 1 - \frac{2}{\Gamma(\frac{5}{4})}\int_0^1 s^{\frac{1}{4}} s^{\frac{3}{4}}(1-s)^2\,ds - \frac{1}{2\Gamma(\frac{9}{8})}\int_0^{\frac{2}{3}} s^{\frac{1}{8}} s^{\frac{7}{8}}\left(1+s^2\right)^{-1}\,ds\\
&\quad - \sum_{j=1}^{\infty} \frac{1}{\Gamma(1+2^{-(7+j)})}(5j-4)^{-1}(5j+1)^{-1}(28+2j)^{-\frac{1}{2^{(7+j)}}}\\
&\ge 1 - 0.25 - 0.2 - 0.2 = 0.35 > 0,
\end{aligned}$$

which means the properties of Green's function in Lemma 2.13 are achieved. Let

$$f_1(t,u,v,w,z) = 5t^{-\frac{1}{3}} u^{\frac{1}{8}} z^{-\frac{1}{8}} + 4u^{\frac{1}{6}}\left(1-t^2\right)^{-\frac{1}{2}} + \left(6v^{\frac{1}{3}}+1\right)\left(1+t^2\right)^{-1} + 7(tw)^{-\frac{1}{5}}$$

and

$$f_2(t,u,v,w,z) = t^{-\frac{1}{3}} u^{\frac{1}{8}} z^{-\frac{1}{8}} + u^{\frac{1}{6}}\left(1-t^2\right)^{-\frac{1}{2}} + v^{\frac{1}{4}}\left(1+t^2\right)^{-1} + (tw)^{-\frac{1}{5}}(w+1)^{-\frac{4}{5}}.$$

Then

$$f(t,u,v) = f_1(t,u,v,u,v) + f_2(t,u,v,u,v).$$

It is easy to check the following conditions:

(1) For all $t \in (0,1)$ and $(w,z) \in (0,+\infty)^2$, $f_1(t,u,v,w,z)$, $f_2(t,u,v,w,z)$ are increasing in $(u,v) \in (0,+\infty)^2$; for all $t \in (0,1)$ and $(u,v) \in (0,+\infty)^2$, $f_1(t,u,v,w,z)$, $f_2(t,u,v,w,z)$ are decreasing in $(w,z) \in (0,+\infty)^2$.

(2) Let $\varphi(\mu) = \mu^{\frac{1}{3}}$. Then, for $\mu \in (0,1)$, $t \in (0,1)$, and $(u,v,w,z) \in (0,+\infty)^4$,

$$\begin{aligned} f_1\bigl(t,\mu u,\mu v,\mu^{-1}w,\mu^{-1}z\bigr) &= 5\mu^{\frac{1}{4}}t^{-\frac{1}{3}}u^{\frac{1}{8}}z^{-\frac{1}{8}} + 4\mu^{\frac{1}{6}}u^{\frac{1}{6}}\bigl(1-t^2\bigr)^{-\frac{1}{2}} \\ &\quad + \bigl(6\mu^{\frac{1}{3}}v^{\frac{1}{3}}+1\bigr)\bigl(1+t^2\bigr)^{-1} + 7\mu^{\frac{1}{5}}(tw)^{-\frac{1}{5}} \\ &\geq \mu^{\frac{1}{3}}f_1(t,u,v,w,z), \\ f_2\bigl(t,\mu u,\mu v,\mu^{-1}w,\mu^{-1}z\bigr) &= \mu^{\frac{1}{4}}t^{-\frac{1}{3}}u^{\frac{1}{8}}z^{-\frac{1}{8}} + \mu^{\frac{1}{6}}u^{\frac{1}{6}}\bigl(1-t^2\bigr)^{-\frac{1}{2}} \\ &\quad + \mu^{\frac{1}{4}}v^{\frac{1}{4}}\bigl(1+t^2\bigr)^{-1} + \mu^{\frac{4}{15}}(tw)^{-\frac{1}{5}}(w+\mu)^{-\frac{4}{5}} \\ &\geq \mu f_2(t,u,v,w,z). \end{aligned}$$

(3) Let $\kappa = 4$. Then, for all $(u,v,w,z) \in (0,+\infty)^4$,

$$f_1(t,u,v,w,z) \geq \kappa f_2(t,u,v,w,z).$$

(4) The functions f_1 and f_2 satisfy

$$\begin{aligned} 0 &< \int_0^1 f_1\bigl(s,1,1,s^{\gamma-1},s^{\gamma-1}\bigr)\,ds \\ &= \int_0^1 \bigl(5s^{-\frac{25}{48}} + 4\bigl(1-s^2\bigr)^{-\frac{1}{2}} + 7\bigl(1+s^2\bigr)^{-1} + 7s^{-\frac{1}{2}}\bigr) < +\infty, \\ 0 &< \int_0^1 f_2\bigl(s,1,1,s^{\gamma-1},s^{\gamma-1}\bigr)\,ds \leq \int_0^1 \bigl(s^{-\frac{25}{48}} + \bigl(1-s^2\bigr)^{-\frac{1}{2}} + \bigl(1+s^2\bigr)^{-1} + s^{-\frac{1}{2}}\bigr) < +\infty. \end{aligned}$$

Let $r = \frac{7}{15} < \frac{1}{2}$. It is easy to check that

(i)

$$\varphi(\mu) = \mu^{\frac{1}{3}} \geq \frac{\mu^r - \mu}{\kappa} + \mu^r = \frac{5}{4}\mu^{\frac{7}{15}} - \frac{1}{4}\mu, \quad \forall \mu \in (0,1).$$

(ii)

$$\varphi(\mu) = \mu^{\frac{1}{3}} \geq \frac{\mu^{\frac{1}{2}} - \mu}{\kappa} + \mu^{\frac{1}{2}} = \frac{5}{4}\mu^{\frac{1}{2}} - \frac{1}{4}\mu, \quad \forall \mu \in (0,1).$$

Therefore, the assumptions of Theorem 3.1 are satisfied. Then BVP (4.1) has a unique solution z_λ^* in P, and there exists a constant $\eta_\lambda \in (0,1)$ such that

$$\frac{6\Gamma(\frac{9}{10})\eta_\lambda}{5\Gamma(\frac{1}{2})}t^{\frac{3}{2}} \leq z_\lambda^*(t) \leq \frac{6\Gamma(\frac{9}{10})}{5\Gamma(\frac{1}{2})\eta_\lambda}t^{\frac{3}{2}}, \quad t \in [0,1].$$

And at the same time, z_λ^* satisfies:

(i) z_λ^* is continuous with respect to $\lambda \in (0,+\infty)$, i.e., for $\forall\ \lambda_0 \in (0,+\infty)$,

$$\bigl\|z_\lambda^* - z_{\lambda_0}^*\bigr\| \to 0, \quad \text{as } \lambda \to \lambda_0.$$

(ii) $0 < \lambda_1 < \lambda_2$ implies $z_{\lambda_1}^* < z_{\lambda_2}^*$.

(iii)

$$\lim_{\lambda\to 0^+}\|z_\lambda^*\| = 0, \qquad \lim_{\lambda\to +\infty}\|z_\lambda^*\| = +\infty.$$

Moreover, for any initial values $z_0, \tilde{z}_0 \in P_e$, by constructing successively the sequences as follows:

$$\begin{aligned} z_n(t) &= I_{0^+}^{\frac{3}{5}}\biggl\{\lambda\int_0^1 K(t,s)f_1\bigl(s, I_{0^+}^{\frac{3}{5}}z_{n-1}(s), z_{n-1}(s), I_{0^+}^{\frac{3}{5}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s)\bigr)\,ds \\ &\quad + \lambda\int_0^1 K(t,s)f_2\bigl(s, I_{0^+}^{\frac{3}{5}}z_{n-1}(s), z_{n-1}(s), I_{0^+}^{\frac{3}{5}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s)\bigr)\,ds\biggr\}, \\ \tilde{z}_n(t) &= I_{0^+}^{\frac{3}{5}}\biggl\{\lambda\int_0^1 K(t,s)f_1\bigl(s, I_{0^+}^{\frac{3}{5}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s), I_{0^+}^{\frac{3}{5}}z_{n-1}(s), z_{n-1}(s)\bigr)\,ds \\ &\quad + \lambda\int_0^1 K(t,s)f_2(\bigl(s, I_{0^+}^{\frac{3}{5}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s), I_{0^+}^{\frac{3}{5}}z_{n-1}(s), z_{n-1}(s)\bigr)\,ds\biggr\}, \\ &\quad n = 1, 2, \ldots, \end{aligned}$$

we have $z_n \to z_\lambda^*$ and $\tilde{z}_n \to z_\lambda^*$ in E, as $n \to \infty$.

Example 4.2 We consider the following problem:

$$\begin{cases} D_{0+}^{\frac{8}{3}}z(t) + z^{\frac{1}{4}}(t)(D_{0+}^{\frac{2}{3}}z(t))^{-\frac{1}{4}}(t^{-\frac{1}{3}} + t^{-\frac{1}{4}}) + (2D_{0+}^{\frac{2}{3}}z(t))^{\frac{1}{3}}(1-t^2)^{-\frac{1}{2}} \\ \quad + (tz(t))^{-\frac{1}{4}}(1 + (z(t)+1)^{-\frac{3}{4}}) = 0, \quad 0 < t < 1, \\ z(0) = D_{0+}^{\frac{3}{4}}z(0) = 0, \\ D_{0+}^{\frac{3}{2}}z(1) = 2\int_0^1 s^{\frac{3}{4}}(1-s)^2 D_{0+}^{\frac{5}{4}}z(s)\,dA_1(s) + \frac{1}{2}\int_0^{\frac{2}{3}} s^{\frac{7}{8}}(1+s^2)^{-1}D_{0+}^{\frac{11}{8}}z(s)\,dA_2(s) \\ \qquad + \sum_{j=1}^{\infty}(5j-4)^{-1}(5j+1)^{-1}D_{0+}^{\frac{5}{3}-2^{-(7+j)}}u((28+2j)^{-1}), \end{cases} \tag{4.2}$$

where

$$A_1(t) = \begin{cases} \frac{1}{11}, & t \in [0, \frac{1}{2}), \\ \frac{12}{11}, & t \in [\frac{1}{2}, 1], \end{cases} \qquad A_2(t) = \begin{cases} \frac{1}{13}, & t \in [0, \frac{1}{2}), \\ \frac{14}{13}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Let

$$f(t,u,v) = u^{\frac{1}{4}}v^{-\frac{1}{4}}\bigl(t^{-\frac{1}{3}} + t^{-\frac{1}{4}}\bigr) + 2v^{\frac{1}{3}}\bigl(1-t^2\bigr)^{-\frac{1}{2}} + (tu)^{-\frac{1}{4}}\bigl(1 + (u+1)^{-\frac{3}{4}}\bigr),$$

$\gamma = \frac{8}{3}$ $(n = 3)$, $\gamma_0 = \frac{3}{2}$, $\nu_1 = \frac{2}{3}$, $q_1 = \frac{3}{4}$, $a_1 = 2$, $a_2 = \frac{1}{2}$, $\alpha_1 = \frac{5}{4}$, $\alpha_2 = \frac{11}{8}$, $I_1 = [0,1]$, $I_2 = [0, \frac{2}{3}]$, $b_j = (5j-4)^{-1}(5j+1)^{-1}$ $(j = 1, 2, \ldots)$, $\beta_j = \frac{5}{3} - 2^{-(7+j)}$ $(j = 1, 2, \ldots)$, $\xi_j = (28+2j)^{-1}$ $(j = 1, 2, \ldots)$, $w_1(t) = t^{\frac{3}{4}}(1-t)^2$, $w_2(t) = t^{\frac{7}{8}}(1+t^2)^{-1}$. Then problem (4.2) can be transformed into BVP (1.1) for $\lambda = 1$. By simple computation, we have a rough estimate:

$$\begin{aligned} \int_{I_1}\tau^{\gamma-\alpha_1-1}w_1(\tau)\,dA_1(\tau) &= \int_0^1 \tau(1-\tau)^2\,dA_1(\tau) = 0.125 > 0, \\ \int_{I_2}\tau^{\gamma-\alpha_2-1}w_2(\tau)\,dA_2(\tau) &= \int_0^{\frac{2}{3}} \tau\bigl(1+\tau^2\bigr)^{-1}\,dA_2(\tau) = 0.4 > 0, \end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{b_j}{\Gamma(\gamma-\beta_j)}\xi_j^{\gamma-\beta_j-1} &= \sum_{j=1}^{\infty}\frac{1}{\Gamma(1+2^{-(7+j)})}(5j-4)^{-1}(5j+1)^{-1}(28+2j)^{-\frac{1}{2^{(7+j)}}} \\ &\le \sum_{j=1}^{\infty}(5j-4)^{-1}(5j+1)^{-1} = 0.2,\end{aligned}$$

and

$$\begin{aligned}\sigma &= \frac{1}{\Gamma(\gamma-\gamma_0)} - \sum_{i=1}^{p}\frac{a_i}{\Gamma(\gamma-\alpha_i)}\int_{I_i}s^{\gamma-\alpha_i-1}w_i(s)\,dA_i(s) - \sum_{j=1}^{\infty}\frac{b_j}{\Gamma(\gamma-\beta_j)}\xi_j^{\gamma-\beta_j-1} \\ &\approx 1 - \frac{2}{\Gamma(\frac{5}{4})}\int_0^1 s^{\frac{1}{4}}s^{\frac{3}{4}}(1-s)^2\,ds - \frac{1}{2\Gamma(\frac{9}{8})}\int_0^{\frac{2}{3}} s^{\frac{1}{8}}s^{\frac{7}{8}}\left(1+s^2\right)^{-1}ds \\ &\quad - \sum_{j=1}^{\infty}\frac{1}{\Gamma(1+2^{-(7+j)})}(5j-4)^{-1}(5j+1)^{-1}(28+2j)^{-\frac{1}{2^{(7+j)}}} \\ &\ge 1-0.25-0.2-0.2 = 0.35 > 0,\end{aligned}$$

which means the properties of Green's function in Lemma 2.13 are achieved. Let

$$f_1(t,u,v,w,z) = t^{-\frac{1}{3}}u^{\frac{1}{4}}z^{-\frac{1}{4}} + v^{\frac{1}{3}}\left(1-t^2\right)^{-\frac{1}{2}} + (tw)^{-\frac{1}{4}}$$

and

$$f_2(t,u,v,w,z) = t^{-\frac{1}{4}}u^{\frac{1}{4}}z^{-\frac{1}{4}} + v^{\frac{1}{3}}\left(1-t^2\right)^{-\frac{1}{2}} + (tw)^{-\frac{1}{4}}(w+1)^{-\frac{3}{4}}.$$

Then

$$f(t,u,v) = f_1(t,u,v,u,v) + f_2(t,u,v,u,v).$$

It is easy to check the following conditions:

(1) For all $t\in(0,1)$ and $(w,z)\in(0,+\infty)^2$, $f_1(t,u,v,w,z)$, $f_2(t,u,v,w,z)$ are increasing in $(u,v)\in(0,+\infty)^2$; for all $t\in(0,1)$ and $(u,v)\in(0,+\infty)^2$, $f_1(t,u,v,w,z)$, $f_2(t,u,v,w,z)$ are decreasing in $(w,z)\in(0,+\infty)^2$.

(2) Let $\varphi(\mu)=\mu^{\frac{1}{2}}$. Then, for $\mu\in(0,1)$, $t\in(0,1)$, and $(u,v,w,z)\in(0,+\infty)^4$,

$$\begin{aligned}&f_1\left(t,\mu u,\mu v,\mu^{-1}w,\mu^{-1}z\right) \\ &\quad = t^{-\frac{1}{3}}(\mu u)^{\frac{1}{4}}\left(\mu^{-1}z\right)^{-\frac{1}{4}} + (\mu v)^{\frac{1}{3}}\left(1-t^2\right)^{-\frac{1}{2}} + t^{-\frac{1}{4}}\left(\mu^{-1}w\right)^{-\frac{1}{4}} \\ &\quad \ge \mu^{\frac{1}{2}}f_1(t,u,v,w,z), \\ &f_2\left(t,\mu u,\mu v,\mu^{-1}w,\mu^{-1}z\right) \\ &\quad = t^{-\frac{1}{4}}(\mu u)^{\frac{1}{4}}\left(\mu^{-1}z\right)^{-\frac{1}{4}} + (\mu v)^{\frac{1}{3}}\left(1-t^2\right)^{-\frac{1}{2}} + \mu(tw)^{-\frac{1}{4}}(w+\mu)^{-\frac{3}{4}} \\ &\quad \ge \mu f_2(t,u,v,w,z).\end{aligned}$$

(3) Let $\kappa=1$. Then, for all $(u,v,w,z)\in(0,+\infty)^4$,

$$f_1(t,u,v,w,z) \ge f_2(t,u,v,w,z).$$

(4) The functions f_1 and f_2 satisfy

$$0 < \int_0^1 f_1\bigl(s,1,1,s^{\gamma-1},s^{\gamma-1}\bigr)\,ds = \int_0^1 \bigl(s^{-\frac{3}{4}} + \bigl(1-s^2\bigr)^{-\frac{1}{2}} + s^{-\frac{2}{3}}\bigr) < +\infty,$$
$$0 < \int_0^1 f_2\bigl(s,1,1,s^{\gamma-1},s^{\gamma-1}\bigr)\,ds \le \int_0^1 \bigl(s^{-\frac{2}{3}} + \bigl(1-s^2\bigr)^{-\frac{1}{2}} + s^{-\frac{2}{3}}\bigr) < +\infty.$$

Thus BVP (4.2) has a unique positive solution z_1^*. Then BVP (1.1) has a unique solution z_1^* in P, and there exists a constant $\eta_1 \in (0,1)$ such that

$$\frac{9\eta_1}{10\Gamma(\frac{2}{3})}t^{\frac{5}{3}} \le z_1^*(t) \le \frac{9}{10\Gamma(\frac{2}{3})\eta_1}t^{\frac{5}{3}}, \quad t \in [0,1].$$

Moreover, for any initial values $z_0, \tilde{z}_0 \in P_e$, by constructing successively the sequences as follows:

$$\begin{aligned} z_n(t) &= I_{0^+}^{\frac{2}{3}}\biggl\{\int_0^1 K(t,s)f_1\bigl(s, I_{0^+}^{\frac{2}{3}}z_{n-1}(s), z_{n-1}(s), I_{0^+}^{\frac{2}{3}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s)\bigr)\,ds \\ &\quad + \int_0^1 K(t,s)f_2\bigl(s, I_{0^+}^{\frac{2}{3}}z_{n-1}(s), z_{n-1}(s), I_{0^+}^{\frac{2}{3}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s)\bigr)\,ds\biggr\}, \\ \tilde{z}_n(t) &= I_{0^+}^{\frac{2}{3}}\biggl\{\int_0^1 K(t,s)f_1\bigl(s, I_{0^+}^{\frac{2}{3}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s), I_{0^+}^{\frac{2}{3}}z_{n-1}(s), z_{n-1}(s)\bigr)\,ds \\ &\quad + \int_0^1 K(t,s)f_2\bigl(s, I_{0^+}^{\frac{2}{3}}\tilde{z}_{n-1}(s), \tilde{z}_{n-1}(s), I_{0^+}^{\frac{2}{3}}z_{n-1}(s), z_{n-1}(s)\bigr)\,ds\biggr\}, \\ &\quad n = 1,2,\ldots, \end{aligned}$$

we have $z_n \to z_1^*$ and $\tilde{z}_n \to z_1^*$ in E, as $n \to \infty$.

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