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Approximate solution of zero point problem involving H-accretive maps in Banach spaces and applications

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Abstract

In this manuscript, we introduce two iterative methods for finding the common zeros of two H-accretive mappings in uniformly smooth and uniformly convex Banach spaces. The proposed iterative methods are based on Mann and Halpern iterative methods and viscosity approximation method. Strong convergence results are established for iterative algorithms. Applications based on convex minimization problem, variational inequality problem and equilibrium problem are derived from the main result. Numerical implementation of the main results and application are demonstrated by some examples. Our results extend, generalize, and unify the previously known results given in literature.

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1 Introduction

For a real Banach space $\mathcal{B}$ with a nonempty, convex, and closed subset $\mathcal{C}$ and for a set-valued mapping $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ whose domain, range, and graph are defined as

$$\begin{aligned}
&\mathtt{dom}(\Phi) = \{y \in \mathcal{B} : \Phi(y) \neq \emptyset\}, \\
&\mathtt{range}(\Phi) = \bigcup \{\Phi(y) : y \in \mathtt{dom}(\Phi)\} \quad \text{and} \\
&\mathtt{graph}(\Phi) = \{(y,z) \in \mathcal{B} \times \mathcal{B} : y \in \mathtt{dom}(\Phi), z \in \Phi(y)\},
\end{aligned}$$

respectively, the *inclusion problem* or *zero point problem* is to search a point $\tilde{y} \in \mathcal{B}$ such that

$$0 \in \Phi(\tilde{y}). \tag{1.1}$$

The set of zeros of Φ is defined as

$$\Phi^{-1}(0) = \{y \in \mathtt{dom}(\Phi) : 0 \in \Phi(y)\}.$$

A very effective and convenient approach to solve a large number of problems in optimization theory and nonlinear analysis is their formulation in the zero point problem. In problem (1.1), different mathematicians defined Φ in various ways in various frameworks, such as maximal monotone in Hilbert spaces and m-accretive in Banach spaces, and they introduced iterative schemes to solve (1.1).

In 2004, Fang et al. [12] introduced the notion of H-accretive mappings in Banach spaces. It is noted that H-accretive mapping is a generalization of m-accretive mapping. They defined the resolvent operator affiliated to H-accretive mapping and showed that it is single-valued and Lipschitz continuous.

One of the important methods to solve (1.1) for maximal monotone operator Φ in a Hilbert space $\mathcal{H}$ is proximal point algorithm which was introduced by Martinet [20] and defined in the following manner:

$$\begin{cases} x_1 \in \mathcal{H}, \\ x_{n+1} = J_{\mu_n,\Phi}(x_n), \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $I : \mathcal{H} \to \mathcal{H}$ is the identity mapping, $J_{\mu_n,\Phi}(x_n) = (I + \mu_n \Phi)^{-1}(x_n)$, and $\{\mu_n\} \subset \mathbb{R}_+$.

In 1976, Rockafellar [28] posed an open question regarding the strong convergence of the sequence obtained from (1.2); indeed an example was given by Güler [13] discussing that Rockafellar's proximal point algorithm does not strongly converge. Furthermore, Bauschke et al. [3] showed by an example that proximal point algorithm only converges weakly but not in norm.

In 2005, a modification of Mann iteration method was presented by Kim et al. [17] for finding the solution of (1.1) with an m-accretive mapping Φ in a uniformly smooth Banach space and described in the following manner:

$$\begin{cases} x_1 \in \mathcal{B}, \\ y_n = J_{\mu_n,\Phi}(x_n), \\ x_{n+1} = \rho_n l + (1 - \rho_n) y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $l \in \mathcal{B}$, $\{\rho_n\} \subset (0, 1)$, and $\{\mu_n\} \subset \mathbb{R}_+$ with certain conditions. They established an outcome of the strong convergence of the sequence $\{x_n\}$ acquired from (1.3).

In 2007, the following iteration scheme:

$$\begin{cases} x_1 \in \mathcal{C}, \\ y_n = \kappa_n x_n + (1 - \kappa_n) J_{\mu_n,\Phi}(x_n), \\ x_{n+1} = \rho_n l + (1 - \rho_n) y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $l \in \mathcal{C}$ and the sequences $\{\kappa_n\}$, $\{\rho_n\} \subset (0, 1)$, was described by Qin et al. [22] in a uniformly smooth Banach space. The demonstration of the strong convergence of the sequence $\{x_n\}$ acquired from (1.4) to a zero of an m-accretive mapping Φ was also given.

Based on viscosity approximation method, in 2008, Chen et al. [9] developed an iterative scheme in the following manner:

$$\begin{cases} x_1 \in \mathcal{C}, \\ x_{n+1} = \rho_n g(x_n) + (1 - \rho_n) J_{\mu_n,\Phi}(x_n), \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $g : \mathcal{C} \to \mathcal{C}$ is a contraction mapping. They established an outcome which ensures the strong convergence of the sequence $\{x_n\}$ acquired from (1.5) to a zero of an m-accretive mapping Φ in a uniformly smooth Banach space. For detailed reviews regarding iteration methods to various inclusion problems, see [1, 6, 7, 10, 29, 30] and the references therein.

In this manuscript, we consider the problem of searching common zeros of two H-accretive mappings in a uniformly smooth and uniformly convex Banach space, i.e., search $\tilde{y} \in \mathcal{B}$ such that

$$\tilde{y} \in \Omega = \Phi^{-1}(0) \cap \Psi^{-1}(0), \tag{1.6}$$

where $\Phi, \Psi : \mathcal{B} \to 2^{\mathcal{B}}$ are H-accretive mappings.

In 2005, Bauschke et al. [2] studied problem (1.6) for two maximal monotone mappings Φ and Ψ in a Hilbert space and developed the following iterative method:

$$\begin{cases} x_0 \in \mathcal{H}, \\ x_{2n+1} = J_{\mu,\Phi}(x_{2n}), \quad n \in \mathbb{N} \cup \{0\}, \\ x_{2n} = J_{\mu,\Psi}(x_{2n-1}), \quad n \in \mathbb{N}. \end{cases} \tag{1.7}$$

They proved that the sequence $\{x_n\}$ obtained from (1.7) is weakly convergent to a fixed point of $J_{\mu,\Phi}J_{\mu,\Psi}$. This iterative method is based on alternating resolvents method which is an extension of alternating projections method introduced and studied by von Neumann [33] and Bregmann [5]. For other modification of von Neumann's alternating projections method, see [8].

Further, in 2011, Boikanyo et al. [4] constructed an iterative scheme using alternating resolvents method in the following manner:

$$\begin{cases} x_0 \in \mathcal{H}, \\ x_{2n+1} = J_{\mu_n,\Phi}(\rho_n l + (1-\rho_n)x_{2n} + e_n), \quad n \in \mathbb{N} \cup \{0\}, \\ x_{2n} = J_{\nu_n,\Psi}(x_{2n-1} + e'_n), \quad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $l \in \mathcal{H}$, $\{\rho_n\} \subset (0,1)$, and $\{e_n\}$, $\{e'_n\}$ are error sequences. Under appropriate restrictions on control sequences, they established the strong convergence of the sequence obtained from (1.8) to a common zero point of maximal monotone mappings Φ and Ψ in a Hilbert space.

Motivated by the above discussed work, we introduce two iterative methods which are based on Mann and Halpern iteration methods and viscosity approximation method. We call it Mann–Halpern type iterative method and Mann type viscosity iterative method. Our proposed iteration methods include alternating resolvents method as a particular case for suitable choice of control sequence. Strong convergence results are established for both iterative methods. Some consequences and applications are derived from the main result. We present numerical examples to show the implementation of our main result and its application. Our results extend, generalize, and unify the results given by Bauschke et al. [2], Kim et al. [17], Qin et al. [22], Chen et al. [9], and Boikanyo et al. [4].

2 Preliminaries

For the whole of this manuscript, let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$ over the field of real numbers $\mathbb{R}$, and let $\mathcal{B}^*$ be its dual space. Let $\langle\cdot,\cdot\rangle$ be the duality pairing between $\mathcal{B}$ and $\mathcal{B}^*$. We use $y_n \to y$, $y_n \rightharpoonup y$, and $y_n \xrightharpoonup{*} y$ as symbolic representations for strong, weak, and weak* convergence of the sequence $\{y_n\}$ to y in $\mathcal{B}$ and in $\mathcal{B}^*$, respectively. Let $\mathcal{F}(\mathcal{T})$ be the set of all fixed points of a mapping $\mathcal{T}$, i.e., $\mathcal{F}(\mathcal{T}) = \{y \in \mathtt{dom}(\mathcal{T}) : \mathcal{T}(y) = y\}$. Let $\mathbb{N}$ and $\mathbb{R}_+$ be the set of all natural numbers and all positive real numbers, respectively.

We take $\mathcal{J} : \mathcal{B} \to 2^{\mathcal{B}^*}$ to symbolize the normalized duality mapping, given by

$$\mathcal{J}(y) = \{\phi \in \mathcal{B}^* : \langle y, \phi\rangle = \|y\|^2 = \|\phi\|^2\}, \quad \forall y \in \mathcal{B}.$$

Clearly, if $\mathcal{B} = \mathcal{H}$, a Hilbert space, then $\mathcal{J} = I$, where $I : \mathcal{B} \to \mathcal{B}$ is the identity mapping. When $\mathcal{B}$ is smooth, we know from [11] that $\mathcal{J}$ is single-valued. Now we recollect the following definitions and results which are needed for establishing our main results.

Definition 2.1 Let $U_{\mathcal{B}} = \{y \in \mathcal{B} : \|y\| = 1\}$. The modulus of convexity $\delta_{\mathcal{B}} : (0, 2] \to [0, 1]$ and the modulus of smoothness $\rho_{\mathcal{B}} : [0, +\infty) \to [0, +\infty)$ of $\mathcal{B}$ are defined by

$$\delta_{\mathcal{B}}(\epsilon) := \inf\left\{\frac{2 - \|y + z\|}{2} : y, z \in U_{\mathcal{B}}, \|y - z\| \geq \epsilon\right\} \quad \text{and}$$
$$\rho_{\mathcal{B}}(h) := \sup\left\{\frac{\|y + z\| + \|y - z\|}{2} - 1 : y \in U_{\mathcal{B}}, \|z\| \leq h\right\},$$

respectively. Then the Banach space $\mathcal{B}$ is termed

- *uniformly convex* if $\delta_{\mathcal{B}}(\epsilon) > 0$, $\forall 0 < \epsilon \leq 2$;
- *uniformly smooth* if $\frac{\rho_{\mathcal{B}}(h)}{h} \to 0$ as $h \to 0$.

Definition 2.2 A mapping $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is termed

- ϑ_1-*Lipschitz continuous* if there exists a constant $\vartheta_1 \in \mathbb{R}_+$ such that

  $$\|\mathcal{T}(y) - \mathcal{T}(z)\| \leq \vartheta_1\|y - z\|, \quad \forall y, z \in \mathcal{B}.$$

  If $0 < \vartheta_1 < 1$, then $\mathcal{T}$ is termed ϑ_1-*contraction* and if $\vartheta_1 = 1$, then $\mathcal{T}$ is termed *nonexpansive*.
- *accretive* if there exists $j(y - z) \in \mathcal{J}(y - z)$ such that

  $$\langle\mathcal{T}(y) - \mathcal{T}(z), j(y - z)\rangle \geq 0, \quad \forall y, z \in \mathcal{B};$$

- *strictly accretive* if there exists $j(y - z) \in \mathcal{J}(y - z)$ such that

  $$\langle\mathcal{T}(y) - \mathcal{T}(z), j(y - z)\rangle \geq 0, \quad \forall y, z \in \mathcal{B}$$

  and equality holds if and only if $y = z$;
- ϑ_2-*strongly accretive* if there exists a constant $\vartheta_2 \in \mathbb{R}_+$ and $j(y - z) \in \mathcal{J}(y - z)$ such that

  $$\langle\mathcal{T}(y) - \mathcal{T}(z), j(y - z)\rangle \geq \vartheta_2\|y - z\|^2, \quad \forall y, z \in \mathcal{B}.$$

Definition 2.3 A set-valued mapping $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ is termed

- *accretive* if there exists $j(y-z) \in \mathcal{J}(y-z)$ such that

$$\langle v - w, j(y-z)\rangle \geq 0, \quad \forall y, z \in \mathcal{B}, v \in \Phi(y), w \in \Phi(z);$$

- *m-accretive* if Φ is accretive and $\mathtt{range}(I + \mu\Phi) = \mathcal{B}$, $\forall \mu \in \mathbb{R}_+$.

Definition 2.4 A mapping $J_{\mu,\Phi} : \mathtt{range}(I + \mu\Phi) \to \mathtt{dom}(\Phi)$ defined by $J_{\mu,\Phi}(y) = (I + \mu\Phi)^{-1}(y)$, $\mu \in \mathbb{R}_+$, is termed *resolvent* affiliated to an accretive mapping Φ.

Definition 2.5 ([12]) Let $H : \mathcal{B} \to \mathcal{B}$ be a single-valued mapping. Then a set-valued mapping $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ is termed *H-accretive* if the following are satisfied:

- Φ is accretive;
- $\mathtt{range}(H + \vartheta_3\Phi) = \mathcal{B}$, where $\vartheta_3 \in \mathbb{R}_+$.

For further generalizations of H-accretive mapping, see [15, 16].

Remark 2.1 From Definition 2.5, we have the following as special cases:

- If $\mathcal{B} = \mathcal{H}$, a Hilbert space and $H \equiv I$, then the H-accretive mapping becomes maximal monotone mapping.
- If $H \equiv I$, then the H-accretive mapping becomes m-accretive mapping.
- The identity mapping I is m-accretive, but it is not H-accretive mapping, where $H(y) = y^2$ (see [12]).
- If we take $\mathcal{B} = \mathbb{R}$, $H(y) = -y^3$, and

$$\mathcal{T}(y) = \operatorname{sgn}(y) = \begin{cases} 1, & y > 0, \\ 0, & y = 0, \\ -1, & y < 0, \end{cases}$$

for all $y \in \mathcal{B}$, then $\mathcal{T}$ is H-accretive, but it is not m-accretive (see [19]).

Lemma 2.1 ([12]) *Let $H : \mathcal{B} \to \mathcal{B}$ and $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ be strictly accretive and H-accretive mappings, respectively. Then the operator $(H + \mu\Phi)^{-1}$, where $\mu \in \mathbb{R}_+$, is single-valued.*

Definition 2.6 ([12]) Let $H : \mathcal{B} \to \mathcal{B}$ and $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ be strictly accretive and H-accretive mappings, respectively. The *resolvent operator* $J^H_{\mu,\Phi} : \mathcal{B} \to \mathcal{B}$ affiliated to H and Φ is defined by

$$J^H_{\mu,\Phi}(y) = (H + \mu\Phi)^{-1}(y), \quad \forall y \in \mathcal{B}, \mu \in \mathbb{R}_+.$$

Using Definition 2.6, one can easily prove that

$$\mathcal{F}\left(J^H_{\mu,\Phi}H\right) = \Phi^{-1}(0). \tag{2.1}$$

Lemma 2.2 ([12]) *Let $H : \mathcal{B} \to \mathcal{B}$ be a ξ_H-strongly accretive mapping, and let $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ be an H-accretive mapping. Then the resolvent operator $J^H_{\mu,\Phi} : \mathcal{B} \to \mathcal{B}$ is $\frac{1}{\xi_H}$-Lipschitz continuous.*

We give an example which legitimizes Lemma 2.2.

Example 2.1 Let $\mathcal{B} = \mathbb{R}$ with usual norm $\|y\| = |y|$. Let $H : \mathcal{B} \to \mathcal{B}$ and $\varPhi : \mathcal{B} \to 2^{\mathcal{B}}$ be defined by

$$H(y) = 3y \quad \text{and} \quad \varPhi(y) = \{6y\}, \quad \forall y \in \mathcal{B},$$

respectively. It can be conveniently verified that H is 3-strongly accretive, $\varPhi$ is accretive, and for $\mu = 2$, $\mathtt{range}(H + \mu\varPhi) = \mathcal{B}$. Hence, $\varPhi$ is H-accretive. Therefore the resolvent operator $J^H_{\mu,\varPhi} : \mathcal{B} \to \mathcal{B}$ affiliated with H and $\varPhi$ is given by

$$J^H_{\mu,\varPhi}(y) = \frac{y}{15}, \quad \forall y \in \mathcal{B}. \tag{2.2}$$

It is quite easy to verify that the resolvent operator defined in (2.2) is single-valued and $\frac{1}{3}$-Lipschitz continuous.

Definition 2.7 Let $\mathcal{C}$ and $\mathcal{D}$ be nonempty subsets of a Banach space $\mathcal{B}$ such that $\mathcal{C}$ is closed and convex and $\mathcal{D} \subset \mathcal{C}$. A mapping $Q : \mathcal{C} \to \mathcal{D}$ is termed

- *sunny* if $Q[Q(y) + h(I - Q)y] = Q(y)$, $\forall y \in \mathcal{C}$ and $h \geq 0$, whenever $Q(y) + h(I - Q)y \in \mathcal{C}$;
- *retraction* if $Q(y) = y$, $\forall y \in \mathcal{D}$;
- *sunny nonexpansive retraction* from $\mathcal{C}$ onto $\mathcal{D}$ if Q is a retraction from $\mathcal{C}$ onto $\mathcal{D}$, which is also sunny and nonexpansive.

Moreover, $\mathcal{D}$ is termed *sunny nonexpansive retract* of $\mathcal{C}$ if there exists a sunny nonexpansive retraction Q from $\mathcal{C}$ onto $\mathcal{D}$ (see [18]).

Proposition 2.3 ([23]) *Let $\mathcal{C}$ and $\mathcal{D}$ be nonempty subsets of a smooth Banach space $\mathcal{B}$ such that $\mathcal{C}$ is closed and convex and $\mathcal{D} \subset \mathcal{C}$. If $\mathcal{D}$ is a retract of $\mathcal{C}$ with a retraction $Q : \mathcal{C} \to \mathcal{D}$, then the following are equivalent*:

(i) *Q is sunny and nonexpansive*;
(ii) $\|Q(y) - Q(z)\|^2 \leq \langle y - z, \mathcal{J}(Q(y) - Q(z))\rangle$, $\forall y, z \in \mathcal{C}$;
(iii) $\langle y - Q(y), \mathcal{J}(z - Q(y))\rangle \leq 0$, $\forall y \in \mathcal{C}, z \in \mathcal{D}$.

Theorem 2.4 ([35]) *Let $\mathcal{C}$ be a nonempty, convex, and closed subset of a uniformly smooth Banach space $\mathcal{B}$, and let $W : \mathcal{C} \to \mathcal{C}$ be a nonexpansive mapping with $\mathcal{F}(W) \neq \emptyset$. Then the sequence $\{x_t\}$ defined by $x_t = tg(x_t) + (1 - t)W(x_t)$, where $g \in \varPi_{\mathcal{C}}$, the set of all contractions on $\mathcal{C}$ and $0 < t < 1$, converges strongly to a point in $\mathcal{F}(W)$ as $t \to 0$. If $Q_{\mathcal{F}(W)} : \varPi_{\mathcal{C}} \to \mathcal{F}(W)$ is defined as*

$$Q_{\mathcal{F}(W)}(g) := \lim_{t \to 0} x_t, \quad g \in \varPi_{\mathcal{C}}, \tag{2.3}$$

then $Q_{\mathcal{F}(W)}(g)$ solves the following variational inequality:

$$\bigl\langle (I - g)Q_{\mathcal{F}(W)}(g), \mathcal{J}\bigl(Q_{\mathcal{F}(W)}(g) - x\bigr)\bigr\rangle \leq 0, \quad g \in \varPi_{\mathcal{C}}, x \in \mathcal{F}(W).$$

In particular, if $g = l \in \mathcal{C}$ is a constant, then (2.3) is reduced to the sunny nonexpansive retraction of Reich [25] from $\mathcal{C}$ onto $\mathcal{F}(W)$ fulfilling

$$\bigl\langle Q_{\mathcal{F}(W)}(l) - l, \mathcal{J}\bigl(Q_{\mathcal{F}(W)}(l) - x\bigr)\bigr\rangle \leq 0, \quad l \in \mathcal{C}, x \in \mathcal{F}(W).$$

Proposition 2.5 ([19]) *Let $H : \mathcal{B} \to \mathcal{B}$ be a strongly accretive and Lipschitz continuous mapping with constant ξ_H; let $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ be an H-accretive mapping. Then the mappings $HJ^H_{\mu,\Phi} : \mathcal{B} \to \mathcal{B}$ and $J^H_{\mu,\Phi}H : \mathcal{B} \to \mathcal{B}$ are nonexpansive.*

Lemma 2.6 ([21]) *Let $\mathcal{B}$ be a Banach space. Then, for all $y, z \in \mathcal{B}$ and $j(y + z) \in \mathcal{J}(y + z)$,*

$$\|y + z\|^2 \le \|y\|^2 + 2\langle z, j(y + z)\rangle.$$

Lemma 2.7 ([31]) *Let $\{x_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space $\mathcal{B}$, and let $\{\sigma_n\} \subset [0, 1]$ with $0 < \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n < 1$. Assume that $x_{n+1} = (1 - \sigma_n)w_n + \sigma_n x_n$, $n \in \mathbb{N}$, and $\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|w_n - x_n\| = 0$.*

Lemma 2.8 ([35]) *Assume that $\{x_n\}$ is a sequence in $\mathbb{R}_+ \cup \{0\}$ such that*

$$x_{n+1} \le (1 - \theta_n)x_n + \eta_n, \quad n \in \mathbb{N},$$

where $\{\theta_n\} \subset (0, 1)$ and $\{\eta_n\} \subset \mathbb{R}$ with
(i) $\sum_{n=1}^{\infty} \theta_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\eta_n}{\theta_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\eta_n| < \infty$.
Then $\lim_{n\to\infty} x_n = 0$.

Lemma 2.9 ([34]) *Let $\mathcal{B}$ be a uniformly convex Banach space. Then there exists a function $f : [0, +\infty) \to [0, +\infty)$ which is continuous, strictly increasing, and convex with $f(0) = 0$ such that*

$$\bigl\|hy + (1 - h)z\bigr\|^2 \le h\|y\|^2 + (1 - h)\|z\|^2 - h(1 - h)f\bigl(\|y - z\|\bigr)$$

for all $y, z \in U_{\mathcal{B}}[0, r] := \{v \in \mathcal{B} : \|v\| \le r, r \in \mathbb{R}_+\}$ and $0 \le h \le 1$.

Lemma 2.10 ([14, 24]) *Let $\mathcal{B}$ be a real uniformly convex and smooth Banach space. Then there exists a function $f : [0, 2r] \to \mathbb{R}$ which is continuous, strictly increasing, and convex with $f(0) = 0$ such that*

$$f\bigl(\|y - z\|\bigr) \le \|y\|^2 - 2\langle y, \mathcal{J}(z)\rangle + \|z\|^2, \quad \forall y, z \in U_{\mathcal{B}}[0, r].$$

3 Main results

In this section, we demonstrate strong convergence of the sequences acquired from the proposed iterative methods for finding a common zero of two H-accretive mappings. First, we prove the following technical lemma.

Lemma 3.1 *Let $H : \mathcal{B} \to \mathcal{B}$ and $\Phi : \mathcal{B} \to 2^{\mathcal{B}}$ be strictly accretive and H-accretive mappings, respectively. Then, for $\mu, \nu \in \mathbb{R}_+$ and $y \in \mathcal{B}$,*

$$J^H_{\mu,\Phi}(y) = J^H_{\nu,\Phi}\left(\frac{\nu}{\mu}y + \left(1 - \frac{\nu}{\mu}\right)HJ^H_{\mu,\Phi}(y)\right).$$

Proof For $y \in \mathcal{B}$, let $\tilde{y} = J^H_{\mu,\Phi}(y)$. This implies that

$$y = H(\tilde{y}) + \mu\tilde{z} \quad \text{for some } \tilde{z} \in \Phi(\tilde{y}).$$

Then

$$\begin{aligned}\frac{\nu}{\mu}y + \left(1 - \frac{\nu}{\mu}\right)HJ^H_{\mu,\Phi}(y) &= \frac{\nu}{\mu}\big(H(\tilde{y}) + \mu\tilde{z}\big) + \left(1 - \frac{\nu}{\mu}\right)H(\tilde{y}) \\ &\in (H + \nu\Phi)(\tilde{y}),\end{aligned}$$

which implies that

$$J^H_{\nu,\Phi}\left(\frac{\nu}{\mu}y + \left(1 - \frac{\nu}{\mu}\right)HJ^H_{\mu,\Phi}(y)\right) = \tilde{y} = J^H_{\mu,\Phi}(y). \qquad \square$$

Theorem 3.2 *Let $\mathcal{B}$ be a uniformly convex and uniformly smooth Banach space. Let $H_1, H_2 : \mathcal{B} \to \mathcal{B}$ be single-valued mappings such that H_1 is strongly accretive and Lipschitz continuous with constant ξ_{H_1} and H_2 is strongly accretive and Lipschitz continuous with constant ζ_{H_2}. Let $\Phi, \Psi : \mathcal{B} \to 2^{\mathcal{B}}$ be H_1-accretive and H_2-accretive mappings, respectively. Assume that $\Omega := \Phi^{-1}(0) \cap \Psi^{-1}(0) \neq \emptyset$. Let the sequences $\{y_n\}$ and $\{x_n\}$ be generated by the following iterative scheme*:

$$\begin{cases} x_1 \in \mathcal{B}, \\ y_n = \kappa_n x_n + (1 - \kappa_n)J^{H_1}_{\mu_n,\Phi}H_1(x_n), \\ x_{n+1} = \rho_n l + \sigma_n x_n + \tau_n J^{H_2}_{\nu_n,\Psi}H_2(y_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $l \in \mathcal{B}$ is an arbitrary element, sequences $\{\mu_n\}, \{\nu_n\}$ are in $\mathbb{R}_+$, and sequences $\{\kappa_n\}, \{\rho_n\}, \{\sigma_n\}, \{\tau_n\}$ are in $[0,1]$ with $\rho_n + \sigma_n + \tau_n = 1$, $n \in \mathbb{N}$. Assume that the following conditions are fulfilled:

(C_1) $\liminf_{n\to\infty} \kappa_n\tau_n > 0$, $\lim_{n\to\infty} |\kappa_{n+1} - \kappa_n| = 0$;

(C_2) $\lim_{n\to\infty} \rho_n = 0$, $\sum_{n=1}^{\infty} \rho_n = \infty$;

(C_3) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;

(C_4) *For some $\epsilon \in \mathbb{R}_+$ and for all $n \in \mathbb{N}$, $\mu_n \geq \epsilon$, $\nu_n \geq \epsilon$ and* $\lim_{n\to\infty} |\mu_{n+1} - \mu_n| = 0$ *and* $\lim_{n\to\infty} |\nu_{n+1} - \nu_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $Q_\Omega l$, where $Q_\Omega : \mathcal{B} \to \Omega$ is a sunny nonexpansive retraction from $\mathcal{B}$ onto Ω.

Proof The proof will be divided into six steps.

Step 1. We show that $\{x_n\}$ and $\{y_n\}$ are bounded. Assume that $w \in \Omega$. By utilizing (2.1), we find that

$$J^{H_1}_{\mu_n,\Phi}H_1(w) = w \in \Phi^{-1}(0) \quad \text{and} \quad J^{H_2}_{\nu_n,\Psi}H_2(w) = w \in \Psi^{-1}(0). \tag{3.2}$$

Then, from (3.1), (3.2), and Proposition 2.5, we have

$$\begin{aligned}\|x_{n+1} - w\| &= \left\| \rho_n l + \sigma_n x_n + \tau_n J^{H_2}_{\nu_n,\Psi} H_2(y_n) - w \right\| \\ &\leq \rho_n \|l - w\| + \sigma_n \|x_n - w\| + \tau_n \left\| J^{H_2}_{\nu_n,\Psi} H_2(y_n) - w \right\| \\ &\leq \rho_n \|l - w\| + \sigma_n \|x_n - w\| + \tau_n \|y_n - w\|. \end{aligned} \tag{3.3}$$

Now we calculate $\|y_n - w\|$.

$$\begin{aligned}\|y_n - w\| &= \left\| \kappa_n x_n + (1-\kappa_n) J^{H_1}_{\mu_n,\Phi} H_1(x_n) - w \right\| \\ &\leq \kappa_n \|x_n - w\| + (1-\kappa_n) \left\| J^{H_1}_{\mu_n,\Phi} H_1(x_n) - w \right\| \\ &\leq \kappa_n \|x_n - w\| + (1-\kappa_n) \|x_n - w\|, \\ \Rightarrow \quad \|y_n - w\| &\leq \|x_n - w\|. \end{aligned} \tag{3.4}$$

Using (3.4) and the relation $\rho_n + \sigma_n + \tau_n = 1$ in (3.3), we obtain by induction

$$\begin{aligned}\|x_{n+1} - w\| &\leq \rho_n \|l - w\| + \sigma_n \|x_n - w\| + \tau_n \|x_n - w\| \\ &= \rho_n \|l - w\| + (1-\rho_n) \|x_n - w\| \\ &\leq \max\left\{ \|l - w\|, \|x_n - w\| \right\} \\ &\;\;\vdots \\ &\leq \max\left\{ \|l - w\|, \|x_1 - w\| \right\}. \end{aligned}$$

Hence $\{x_n\}$ is bounded. Therefore $\{y_n\}$, $\{J^{H_1}_{\mu_n,\Phi} H_1(x_n)\}$, and $\{J^{H_2}_{\nu_n,\Psi} H_2(y_n)\}$ are also bounded.

Step 2. We claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$.

Setting $x_{n+1} = (1-\sigma_n) w_n + \sigma_n x_n$, we see that $w_n = \frac{x_{n+1} - \sigma_n x_n}{1-\sigma_n}$. Then

$$\begin{aligned}\|w_{n+1} - w_n\| &= \left\| \frac{x_{n+2} - \sigma_{n+1} x_{n+1}}{1-\sigma_{n+1}} - \frac{x_{n+1} - \sigma_n x_n}{1-\sigma_n} \right\| \\ &= \left\| \frac{\rho_{n+1} l + \tau_{n+1} J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1})}{1-\sigma_{n+1}} - \frac{\rho_n l + \tau_n J^{H_2}_{\nu_n,\Psi} H_2(y_n)}{1-\sigma_n} \right\| \\ &= \left\| \frac{\rho_{n+1} l + \tau_{n+1} J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1})}{1-\sigma_{n+1}} - \frac{\tau_n J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1})}{1-\sigma_n} \right. \\ &\quad \left. + \frac{\tau_n J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1})}{1-\sigma_n} - \frac{\rho_n l + \tau_n J^{H_2}_{\nu_n,\Psi} H_2(y_n)}{1-\sigma_n} \right\| \\ &\leq \left| \frac{\rho_{n+1}}{1-\sigma_{n+1}} - \frac{\rho_n}{1-\sigma_n} \right| \|l\| + \left| \frac{\tau_{n+1}}{1-\sigma_{n+1}} - \frac{\tau_n}{1-\sigma_n} \right| \left\| J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1}) \right\| \\ &\quad + \left(\frac{\tau_n}{1-\sigma_n} \right) \left\| J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1}) - J^{H_2}_{\nu_n,\Psi} H_2(y_n) \right\| \\ &\leq \left| \frac{\rho_{n+1}}{1-\sigma_{n+1}} - \frac{\rho_n}{1-\sigma_n} \right| L_1 + \left\| J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1}) - J^{H_2}_{\nu_n,\Psi} H_2(y_n) \right\|, \end{aligned} \tag{3.5}$$

where $L_1 = \sup_n \{\|l\| + \|J^{H_2}_{\nu_{n+1},\Psi} H_2(y_{n+1})\|\}$. Now, two cases arise.

Case I. When $\nu_{n+1} \geq \nu_n$, by utilizing Lemmas 3.1 and 2.2, we acquire

$$\begin{aligned}
&\bigl\|J^{H_2}_{\nu_{n+1},\Psi}H_2(y_{n+1}) - J^{H_2}_{\nu_n,\Psi}H_2(y_n)\bigr\| \\
&\quad = \left\| J^{H_2}_{\nu_n,\Psi}\left(\frac{\nu_n}{\nu_{n+1}}H_2(y_{n+1}) + \left(1 - \frac{\nu_n}{\nu_{n+1}}\right)H_2 J^{H_2}_{\nu_{n+1},\Psi}H_2(y_{n+1})\right) - J^{H_2}_{\nu_n,\Psi}H_2(y_n)\right\| \\
&\quad \leq \frac{1}{\zeta_{H_2}}\left\{\bigl\|H_2(y_{n+1}) - H_2(y_n)\bigr\| + \left|\frac{\nu_{n+1} - \nu_n}{\nu_{n+1}}\right| \bigl\|H_2 J^{H_2}_{\nu_{n+1},\Psi}H_2(y_{n+1}) - H_2(y_{n+1})\bigr\|\right\} \\
&\quad \leq \|y_{n+1} - y_n\| + |\nu_{n+1} - \nu_n| L_2,
\end{aligned} \tag{3.6}$$

where $L_2 = \frac{1}{\epsilon}[\sup_n\{\|y_n\| + \|J^{H_2}_{\nu_n,\Psi}H_2(y_n)\|\}]$.

Case II. When $\nu_{n+1} < \nu_n$, by utilizing Lemmas 3.1 and 2.2, we acquire

$$\begin{aligned}
&\bigl\|J^{H_2}_{\nu_{n+1},\Psi}H_2(y_{n+1}) - J^{H_2}_{\nu_n,\Psi}H_2(y_n)\bigr\| \\
&\quad = \left\| J^{H_2}_{\nu_{n+1},\Psi}H_2(y_{n+1}) - J^{H_2}_{\nu_{n+1},\Psi}\left(\frac{\nu_{n+1}}{\nu_n}H_2(y_n) + \left(1 - \frac{\nu_{n+1}}{\nu_n}\right)H_2 J^{H_2}_{\nu_n,\Psi}H_2(y_n)\right)\right\| \\
&\quad \leq \frac{1}{\zeta_{H_2}}\left\{\bigl\|H_2(y_{n+1}) - H_2(y_n)\bigr\| + \left|\frac{\nu_n - \nu_{n+1}}{\nu_n}\right| \bigl\|H_2(y_n) - H_2 J^{H_2}_{\nu_n,\Psi}H_2(y_n)\bigr\|\right\} \\
&\quad \leq \|y_{n+1} - y_n\| + |\nu_{n+1} - \nu_n| L_2.
\end{aligned} \tag{3.7}$$

From (3.5), (3.6), and (3.7), we obtain

$$\begin{aligned}
&\|w_{n+1} - w_n\| \\
&\quad \leq \left|\frac{\rho_{n+1}}{1 - \sigma_{n+1}} - \frac{\rho_n}{1 - \sigma_n}\right| L_1 + \|y_{n+1} - y_n\| + |\nu_{n+1} - \nu_n| L_2.
\end{aligned} \tag{3.8}$$

By utilizing (3.1) and Lemma 3.1, we acquire

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad = \bigl\|\kappa_{n+1}x_{n+1} + (1 - \kappa_{n+1})J^{H_1}_{\mu_{n+1},\Phi}H_1(x_{n+1}) - \kappa_n x_n - (1 - \kappa_n)J^{H_1}_{\mu_n,\Phi}H_1(x_n)\bigr\| \\
&\quad \leq \kappa_{n+1}\|x_{n+1} - x_n\| + |\kappa_{n+1} - \kappa_n|\bigl(\|x_n\| + \bigl\|J^{H_1}_{\mu_n,\Phi}H_1(x_n)\bigr\|\bigr) \\
&\qquad + (1 - \kappa_{n+1})\bigl\|J^{H_1}_{\mu_{n+1},\Phi}H_1(x_{n+1}) - J^{H_1}_{\mu_n,\Phi}H_1(x_n)\bigr\| \\
&\quad = \kappa_{n+1}\|x_{n+1} - x_n\| + |\kappa_{n+1} - \kappa_n|\bigl(\|x_n\| + \bigl\|J^{H_1}_{\mu_n,\Phi}H_1(x_n)\bigr\|\bigr) + (1 - \kappa_{n+1}) \\
&\qquad \cdot \left\| J^{H_1}_{\mu_n,\Phi}\left(\frac{\mu_n}{\mu_{n+1}}H_1(x_{n+1}) + \left(1 - \frac{\mu_n}{\mu_{n+1}}\right)H_1 J^{H_1}_{\mu_{n+1},\Phi}H_1(x_{n+1})\right) - J^{H_1}_{\mu_n,\Phi}H_1(x_n)\right\| \\
&\quad \leq \kappa_{n+1}\|x_{n+1} - x_n\| + |\kappa_{n+1} - \kappa_n|\bigl(\|x_n\| + \bigl\|J^{H_1}_{\mu_n,\Phi}H_1(x_n)\bigr\|\bigr) \\
&\qquad + \frac{(1 - \kappa_{n+1})}{\xi_{H_1}}\Bigl\{\bigl\|H_1(x_{n+1}) - H_1(x_n)\bigr\| \\
&\qquad + \left|\frac{\mu_{n+1} - \mu_n}{\mu_{n+1}}\right| \bigl\|H_1 J^{H_1}_{\mu_{n+1},\Phi}H_1(x_{n+1}) - H_1(x_{n+1})\bigr\|\Bigr\} \\
&\quad \leq \|x_{n+1} - x_n\| + |\kappa_{n+1} - \kappa_n| L_3 + |\mu_{n+1} - \mu_n| L_4,
\end{aligned} \tag{3.9}$$

where $L_3 = \sup_n\{\|x_n\| + \|J^{H_1}_{\mu_n,\Phi}H_1(x_n)\|\}$ and $L_4 = \frac{1}{\epsilon}[\sup_n\{\|J^{H_1}_{\mu_n,\Phi}H_1(x_n)\| + \|x_n\|\}]$.

Combining (3.8) and (3.9), we acquire

$$\begin{aligned}
&\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \left|\frac{\rho_{n+1}}{1-\sigma_{n+1}}-\frac{\rho_n}{1-\sigma_n}\right|L_1+|\nu_{n+1}-\nu_n|L_2+|\kappa_{n+1}-\kappa_n|L_3+|\mu_{n+1}-\mu_n|L_4.
\end{aligned}\tag{3.10}$$

From (C_1), (C_2), (C_4), and (3.10), we acquire

$$\limsup_{n\to\infty}\bigl(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|\bigr)\le 0.$$

It follows from Lemma 2.7 that $\lim_{n\to\infty}\|w_n-x_n\|=0$, and hence

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\sigma_n)\|w_n-x_n\|=0.\tag{3.11}$$

From (C_1), (C_4), (3.9), and (3.11), we acquire

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0.$$

Step 3. Our claim is $\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$. From (3.1), (3.4), and Lemma 2.10, we have

$$\begin{aligned}
&\|y_n-w\|^2 \\
&\quad=\bigl\langle\kappa_n(x_n-w)+(1-\kappa_n)\bigl(J^{H_1}_{\mu_n,\Phi}H_1(x_n)-w\bigr),\mathcal{J}(y_n-w)\bigr\rangle,\quad w\in\Omega \\
&\quad\le\kappa_n\bigl\langle x_n-w,\mathcal{J}(y_n-w)\bigr\rangle+(1-\kappa_n)\bigl\|J^{H_1}_{\mu_n,\Phi}H_1(x_n)-w\bigr\|\|y_n-w\| \\
&\quad\le\frac{\kappa_n}{2}\bigl[\|x_n-w\|^2-f\bigl(\bigl\|(x_n-w)-(y_n-w)\bigr\|\bigr)+\|y_n-w\|^2\bigr] \\
&\qquad+(1-\kappa_n)\|x_n-w\|\|y_n-w\| \\
&\quad\le\frac{\kappa_n}{2}\bigl[\|x_n-w\|^2-f\bigl(\|x_n-y_n\|\bigr)+\|y_n-w\|^2\bigr]+(1-\kappa_n)\|x_n-w\|^2 \\
&\quad=\left(\frac{2-\kappa_n}{2}\right)\|x_n-w\|^2+\frac{\kappa_n}{2}\bigl[\|y_n-w\|^2-f\bigl(\|x_n-y_n\|\bigr)\bigr],
\end{aligned}$$

which implies that

$$\|y_n-w\|^2\le\|x_n-w\|^2-\left(\frac{\kappa_n}{2-\kappa_n}\right)f\bigl(\|x_n-y_n\|\bigr).\tag{3.12}$$

Then, by using (3.1) and (3.12), we acquire

$$\begin{aligned}
&\|x_{n+1}-w\|^2 \\
&\quad\le\rho_n\|l-w\|^2+\sigma_n\|x_n-w\|^2+\tau_n\bigl\|J^{H_2}_{\nu_n,\Psi}H_2(y_n)-w\bigr\|^2 \\
&\quad\le\rho_n\|l-w\|^2+\sigma_n\|x_n-w\|^2+\tau_n\|y_n-w\|^2 \\
&\quad\le\rho_n\|l-w\|^2+\sigma_n\|x_n-w\|^2+\tau_n\left[\|x_n-w\|^2-\left(\frac{\kappa_n}{2-\kappa_n}\right)f\bigl(\|x_n-y_n\|\bigr)\right] \\
&\quad\le\rho_n\|l-w\|^2+\|x_n-w\|^2-\tau_n\left(\frac{\kappa_n}{2-\kappa_n}\right)f\bigl(\|x_n-y_n\|\bigr),
\end{aligned}$$

which implies that

$$\left(\frac{\kappa_n\tau_n}{2-\kappa_n}\right)f\left(\|x_n-y_n\|\right) \leq \rho_n\|l-w\|^2+\|x_n-w\|^2-\|x_{n+1}-w\|^2$$
$$\leq \rho_n\|l-w\|^2+\|x_{n+1}-x_n\|\left(\|x_n-w\|+\|x_{n+1}-w\|\right). \tag{3.13}$$

By utilizing (3.11), conditions (C_1)–(C_2), and the property of f in (3.13), we obtain

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.14}$$

Since $\|x_{n+1}-y_n\| \leq \|x_{n+1}-x_n\|+\|x_n-y_n\|$, so from (3.11) and (3.14), we acquire that $\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$.

Step 4. We claim that $\lim_{n\to\infty}\|W(x_n)-x_n\|=0$, where $W=\frac{1}{2}(J^{H_1}_{\vartheta,\Phi}H_1+J^{H_2}_{\vartheta,\Psi}H_2)$ and $0<\vartheta<\epsilon$.

$$\|W(x_n)-x_n\| \leq \frac{1}{2}\left(\left\|x_n-J^{H_1}_{\vartheta,\Phi}H_1(x_n)\right\|+\left\|x_n-J^{H_2}_{\vartheta,\Psi}H_2(x_n)\right\|\right). \tag{3.15}$$

First, we compute $\|x_n-J^{H_1}_{\vartheta,\Phi}H_1(x_n)\|$. From Lemma 3.1 and (3.1), we have

$$\begin{aligned}
&\left\|x_n-J^{H_1}_{\vartheta,\Phi}H_1(x_n)\right\| \\
&\quad\leq \|x_n-y_n\|+\left\|y_n-J^{H_1}_{\vartheta,\Phi}H_1(y_n)\right\|+\left\|J^{H_1}_{\vartheta,\Phi}H_1(x_n)-J^{H_1}_{\vartheta,\Phi}H_1(y_n)\right\| \\
&\quad\leq 2\|x_n-y_n\|+\kappa_n\left\|x_n-J^{H_1}_{\mu_n,\Phi}H_1(x_n)\right\|+\left\|J^{H_1}_{\mu_n,\Phi}H_1(x_n)-J^{H_1}_{\vartheta,\Phi}H_1(y_n)\right\| \\
&\quad= 2\|x_n-y_n\|+\kappa_n\left\|x_n-J^{H_1}_{\mu_n,\Phi}H_1(x_n)\right\| \\
&\qquad+\left\|J^{H_1}_{\vartheta,\Phi}\left(\frac{\vartheta}{\mu_n}H_1(x_n)+\left(1-\frac{\vartheta}{\mu_n}\right)H_1J^{H_1}_{\mu_n,\Phi}H_1(x_n)\right)-J^{H_1}_{\vartheta,\Phi}H_1(y_n)\right\| \\
&\quad\leq 2\|x_n-y_n\|+\kappa_n\left\|x_n-J^{H_1}_{\mu_n,\Phi}H_1(x_n)\right\| \\
&\qquad+\frac{1}{\xi_{H_1}}\left\{\left\|H_1(x_n)-H_1(y_n)\right\|+\left|\frac{\mu_n-\vartheta}{\mu_n}\right|\left\|H_1J^{H_1}_{\mu_n,\Phi}H_1(x_n)-H_1(x_n)\right\|\right\} \\
&\quad\leq 3\|x_n-y_n\|+\left(\frac{\mu_n-\vartheta}{\mu_n}+\kappa_n\right)\left\|J^{H_1}_{\mu_n,\Phi}H_1(x_n)-x_n\right\| \\
&\quad=\left\{3+\frac{1}{(1-\kappa_n)}\left(\frac{\mu_n-\vartheta}{\mu_n}+\kappa_n\right)\right\}\|x_n-y_n\|.
\end{aligned}$$

Since $\|x_n-y_n\|\to 0$ as $n\to\infty$ (from (3.14)), we acquire that $\lim_{n\to\infty}\|x_n-J^{H_1}_{\vartheta,\Phi}H_1(x_n)\|=0$.

Next, we compute $\|x_n-J^{H_2}_{\vartheta,\Psi}H_2(x_n)\|$.

$$\begin{aligned}
&\left\|x_n-J^{H_2}_{\vartheta,\Psi}H_2(x_n)\right\| \\
&\quad\leq \left\|x_n-J^{H_2}_{\nu_n,\Psi}H_2(x_n)\right\|+\left\|J^{H_2}_{\nu_n,\Psi}H_2(x_n)-J^{H_2}_{\vartheta,\Psi}H_2(x_{n+1})\right\| \\
&\qquad+\left\|J^{H_2}_{\vartheta,\Psi}H_2(x_{n+1})-J^{H_2}_{\vartheta,\Psi}H_2(x_n)\right\| \\
&\quad\leq \|x_{n+1}-x_n\|+\left\|x_n-J^{H_2}_{\nu_n,\Psi}H_2(x_n)\right\| \\
&\qquad+\left\|J^{H_2}_{\vartheta,\Psi}\left(\frac{\vartheta}{\nu_n}H_2(x_n)+\left(1-\frac{\vartheta}{\nu_n}\right)H_2J^{H_2}_{\nu_n,\Psi}H_2(x_n)\right)-J^{H_2}_{\vartheta,\Psi}H_2(x_{n+1})\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq 2\|x_{n+1}-x_n\| + \left(2-\frac{\vartheta}{\nu_n}\right)\left\|x_n - J^{H_2}_{\nu_n,\Psi}H_2(x_n)\right\| \\
&\leq 2\|x_{n+1}-x_n\| \\
&\quad + \left(2-\frac{\vartheta}{\nu_n}\right)\left(\left\|x_n - J^{H_2}_{\nu_n,\Psi}H_2(y_n)\right\| + \left\|J^{H_2}_{\nu_n,\Psi}H_2(x_n) - J^{H_2}_{\nu_n,\Psi}H_2(y_n)\right\|\right) \\
&\leq 2\|x_{n+1}-x_n\| + \left(2-\frac{\vartheta}{\nu_n}\right)\left\{\left\|x_n - \left(\frac{x_{n+1}-\rho_n l - \sigma_n x_n}{\tau_n}\right)\right\| + \|x_n - y_n\|\right\} \\
&\leq 2\|x_{n+1}-x_n\| + \left(2-\frac{\vartheta}{\nu_n}\right)\left\{\frac{1}{\tau_n}\left(\|x_{n+1}-x_n\| + \rho_n\|x_n - l\|\right) + \|x_n - y_n\|\right\}.
\end{aligned}$$

Since $\|x_{n+1}-x_n\| \to 0$ (from (3.11)), $\|x_n - y_n\| \to 0$ (from (3.14)), and $\rho_n \to 0$ (from (C_2)) as $n \to \infty$, we acquire that $\lim_{n\to\infty} \|x_n - J^{H_2}_{\vartheta,\Psi}H_2(x_n)\| = 0$. Hence it follows from (3.15) that $\lim_{n\to\infty} \|W(x_n) - x_n\| = 0$.

Step 5. Our claim is $\limsup_{n\to\infty}\langle (I-Q_\Omega)l, \mathcal{J}(x_n - Q_\Omega l)\rangle \leq 0$. Define a sequence $\{x_t\}$ by $x_t = tl + (1-t)W(x_t)$, $t \in (0,1)$. Then Theorem 2.4 ensures the strong convergence of $\{x_t\}$ to $Q_\Omega l \in \mathcal{F}(W) = \Omega$. Now

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&\quad = \left\|tl + (1-t)W(x_t) - x_n\right\|^2 \\
&\quad = \left\|t(l-x_n) + (1-t)\left(W(x_t)-x_n\right)\right\|^2 \\
&\quad = \left\langle t(l-x_n) + (1-t)\left(W(x_t)-x_n\right), \mathcal{J}(x_t - x_n)\right\rangle \\
&\quad = t\left\langle l - x_t, \mathcal{J}(x_t-x_n)\right\rangle + t\left\langle x_t - x_n, \mathcal{J}(x_t-x_n)\right\rangle \\
&\qquad + (1-t)\left\{\left\langle W(x_t) - W(x_n), \mathcal{J}(x_t-x_n)\right\rangle + \left\langle W(x_n) - x_n, \mathcal{J}(x_t-x_n)\right\rangle\right\} \\
&\quad \leq t\left\langle l - x_t, \mathcal{J}(x_t-x_n)\right\rangle + t\|x_t - x_n\|^2 \\
&\qquad + (1-t)\left\{\left\|W(x_t) - W(x_n)\right\|\|x_t-x_n\| + \left\|W(x_n)-x_n\right\|\|x_t - x_n\|\right\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&t\left\langle l - x_t, \mathcal{J}(x_n - x_t)\right\rangle \\
&\quad \leq (1-t)\left\{\left\|W(x_t) - W(x_n)\right\|\|x_t-x_n\| - \|x_t-x_n\|^2 + \left\|W(x_n)-x_n\right\|\|x_t - x_n\|\right\} \\
&\quad \leq (1-t)\left\{\|x_t-x_n\|^2 - \|x_t-x_n\|^2 + \left\|W(x_n)-x_n\right\|\|x_t - x_n\|\right\} \\
&\quad \leq \left\|W(x_n)-x_n\right\|\|x_t - x_n\|.
\end{aligned}$$

Let $L_5 = \sup\{\|x_t - x_n\| : t \in (0,1), n \in \mathbb{N}\}$. Then

$$\left\langle l - x_t, \mathcal{J}(x_n - x_t)\right\rangle \leq \frac{L_5}{t}\left\|W(x_n) - x_n\right\|.$$

Since $\|W(x_n) - x_n\| \to 0$ as $n \to \infty$, we acquire

$$\limsup_{n\to\infty}\left\langle l - x_t, \mathcal{J}(x_n - x_t)\right\rangle \leq 0. \tag{3.16}$$

By utilizing the fact that $x_t \to Q_\Omega l$ as $t \to 0^+$ and $\mathcal{J}$ is norm-to-weak* uniformly continuous on bounded subsets of $\mathcal{B}$, we acquire

$$\begin{aligned}
&\bigl|\bigl\langle (I-Q_\Omega)l, \mathcal{J}(x_n - Q_\Omega l)\bigr\rangle - \bigl\langle l - x_t, \mathcal{J}(x_n - x_t)\bigr\rangle\bigr| \\
&\quad = \bigl|\bigl\langle l - Q_\Omega l, \mathcal{J}(x_n - Q_\Omega l) - \mathcal{J}(x_n - x_t)\bigr\rangle + \bigl\langle x_t - Q_\Omega l, \mathcal{J}(x_n - x_t)\bigr\rangle\bigr| \\
&\quad \le \bigl|\bigl\langle l - Q_\Omega l, \mathcal{J}(x_n - Q_\Omega l) - \mathcal{J}(x_n - x_t)\bigr\rangle\bigr| + L_5\|x_t - Q_\Omega l\| \\
&\quad \longrightarrow 0 \quad \text{as } h \to 0^+.
\end{aligned}$$

For $\varepsilon > 0$, there is $0 < \delta < 1$ such that

$$\bigl\langle (I-Q_\Omega)l, \mathcal{J}(x_n - Q_\Omega l)\bigr\rangle < \bigl\langle l - x_t, \mathcal{J}(x_n - x_t)\bigr\rangle + \varepsilon, \quad \forall 0 < t < \delta.$$

From (3.16), we acquire

$$\limsup_{n\to\infty}\bigl\langle (I-Q_\Omega)l, \mathcal{J}(x_n - Q_\Omega l)\bigr\rangle \le \limsup_{n\to\infty}\bigl\langle l - x_t, \mathcal{J}(x_n - x_t)\bigr\rangle + \varepsilon \le \varepsilon.$$

Since ε is arbitrary, we acquire that $\limsup_{n\to\infty}\langle (I-Q_\Omega)l, \mathcal{J}(x_n - Q_\Omega l)\rangle \le 0$.

Step 6. Finally, our claim is $x_n \to Q_\Omega l$ as $n \to \infty$. From (3.1), (3.4), Lemmas 2.6 and 2.9, it follows that

$$\begin{aligned}
&\|x_{n+1} - Q_\Omega l\|^2 \\
&\quad = \bigl\|\rho_n l + \sigma_n x_n + \tau_n J^{H_2}_{\nu_n,\Phi} H_2(y_n) - Q_\Omega l\bigr\|^2 \\
&\quad \le \bigl\|\sigma_n(x_n - Q_\Omega l) + \tau_n\bigl(J^{H_2}_{\nu_n,\Psi} H_2(y_n) - Q_\Omega l\bigr)\bigr\|^2 + 2\rho_n\bigl\langle l - Q_\Omega l, \mathcal{J}(x_{n+1} - Q_\Omega l)\bigr\rangle \\
&\quad \le (1-\sigma_n)\left\|\frac{\tau_n}{(1-\sigma_n)}\bigl(J^{H_2}_{\nu_n,\Psi} H_2(y_n) - Q_\Omega l\bigr)\right\|^2 + \sigma_n\|x_n - Q_\Omega l\|^2 \\
&\qquad + 2\rho_n\bigl\langle l - Q_\Omega l, \mathcal{J}(x_{n+1} - Q_\Omega l)\bigr\rangle \\
&\quad \le \frac{\tau_n^2}{(1-\sigma_n)}\|y_n - Q_\Omega l\|^2 + \sigma_n\|x_n - Q_\Omega l\|^2 + 2\rho_n\bigl\langle l - Q_\Omega l, \mathcal{J}(x_{n+1} - Q_\Omega l)\bigr\rangle \\
&\quad \le \left\{\frac{\tau_n^2}{(1-\sigma_n)} + \sigma_n\right\}\|x_n - Q_\Omega l\|^2 + 2\rho_n\bigl\langle l - Q_\Omega l, \mathcal{J}(x_{n+1} - Q_\Omega l)\bigr\rangle \\
&\quad = \left\{(1-\rho_n) + \frac{\rho_n^2}{(1-\sigma_n)} - \rho_n\right\}\|x_n - Q_\Omega l\|^2 + 2\rho_n\bigl\langle l - Q_\Omega l, \mathcal{J}(x_{n+1} - Q_\Omega l)\bigr\rangle \\
&\quad = (1-\rho_n)\|x_n - Q_\Omega l\|^2 \\
&\qquad + \rho_n\left\{\left(\frac{\rho_n}{1-\sigma_n} - 1\right)\|x_n - Q_\Omega l\|^2 + 2\bigl\langle l - Q_\Omega l, \mathcal{J}(x_{n+1} - Q_\Omega l)\bigr\rangle\right\} \\
&\quad = (1-\rho_n)\|x_n - Q_\Omega l\|^2 + \eta_n.
\end{aligned}$$

Evidently, $\sum_{n=1}^{\infty}\rho_n = \infty$, $\{\rho_n\} \subset (0,1)$ and $\limsup_{n\to\infty}\frac{\eta_n}{\rho_n} \le 0$. Hence, by Lemma 2.8, we acquire that $x_n \to Q_\Omega l$ as $n \to \infty$. Thus, the proof is completed. □

In the next theorem, we prove the strong convergence of the sequence generated by the following Mann type viscosity approximation method:

$$\begin{cases} u_1 \in \mathcal{B}, \\ v_n = \kappa_n u_n + (1-\kappa_n) J^{H_1}_{\mu_n,\Phi} H_1(u_n), \\ u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n J^{H_2}_{\nu_n,\Psi} H_2(v_n), \quad n \in \mathbb{N}. \end{cases} \tag{3.17}$$

Theorem 3.3 *Let $\mathcal{B}$ be a uniformly convex and uniformly smooth Banach space. Let $H_1, H_2 : \mathcal{B} \to \mathcal{B}$ be single-valued mappings such that H_1 is strongly accretive and Lipschitz continuous with constant ξ_{H_1} and H_2 is strongly accretive and Lipschitz continuous with constant ζ_{H_2}. Let $\Phi, \Psi : \mathcal{B} \to 2^{\mathcal{B}}$ be H_1-accretive and H_2-accretive mappings, respectively; let $g : \mathcal{B} \to \mathcal{B}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Assume that $\Omega := \Phi^{-1}(0) \cap \Psi^{-1}(0) \neq \emptyset$. Let the sequences $\{v_n\}$ and $\{u_n\}$ be generated by (3.17). Assume that conditions (C_1)–(C_4) of Theorem 3.2 are fulfilled. Then the sequence $\{u_n\}$ converges strongly to $\tilde{u} = Q_\Omega g(\tilde{u}) \in \Omega$.*

Proof Assume that $\tilde{u}$ is a unique fixed point of $Q_\Omega g$. Then $Q_\Omega g(\tilde{u}) = \tilde{u}$. If we put $g(\tilde{u})$ in place of l in (3.1), then Theorem 3.2 ensures the strong convergence of $\{x_n\}$ to $Q_\Omega g(\tilde{u}) = \tilde{u}$, i.e., $\lim_{n\to\infty} \|x_n - \tilde{u}\| = 0$.

First, we demonstrate that $\lim_{n\to\infty} \|u_n - x_n\| = 0$. We are assuming on the contrary that

$$\limsup_{n\to\infty} \|u_n - x_n\| > 0.$$

Then we can pick ε such that $0 < \varepsilon < \limsup_{n\to\infty} \|u_n - x_n\|$. As $\{x_n\}$ is strongly convergent to $\tilde{u}$, so there exists $m' \in \mathbb{N}$ such that

$$\|x_n - \tilde{u}\| < \left(\frac{1-\vartheta}{\vartheta}\right)\varepsilon, \quad \forall n \geq m'.$$

Now two possibilities arise.

(P1) There exists $m \in \mathbb{N}$ with $m \geq m'$ and $\|u_m - x_m\| \leq \varepsilon$;

(P2) $\|u_n - x_n\| > \varepsilon$, $\forall n \geq m'$.

In possibility (P1),

$$\begin{aligned} &\|u_{m+1} - x_{m+1}\| \\ &\quad = \left\| \rho_m g(v_m) + \sigma_m u_m + \tau_m J^{H_2}_{\nu_m,\Psi} H_2(v_m) - \rho_m g(\tilde{u}) - \sigma_m x_m - \tau_m J^{H_2}_{\nu_m,\Psi} H_2(y_m) \right\| \\ &\quad \leq \rho_m \vartheta \|v_m - \tilde{u}\| + \sigma_m \|u_m - x_m\| + \tau_m \|v_m - y_m\| \\ &\quad = \rho_m \vartheta \left\| \kappa_m u_m + (1-\kappa_m) J^{H_1}_{\mu_m,\Phi} H_1(u_m) - \tilde{u} \right\| + \sigma_m \|u_m - x_m\| \\ &\qquad + \tau_m \left\| \kappa_m u_m + (1-\kappa_m) J^{H_1}_{\mu_m,\Phi} H_1(u_m) - \kappa_m x_m - (1-\kappa_m) J^{H_1}_{\mu_m,\Phi} H_1(x_m) \right\| \\ &\quad \leq \rho_m \vartheta \|u_m - \tilde{u}\| + (\sigma_m + \tau_m) \|u_m - x_m\| \\ &\quad \leq \rho_m \vartheta \|x_m - \tilde{u}\| + \bigl(1 - \rho_m(1-\vartheta)\bigr) \|u_m - x_m\| \\ &\quad \leq \rho_m (1-\vartheta)\varepsilon + \bigl(1 - \rho_m(1-\vartheta)\bigr)\varepsilon = \varepsilon. \end{aligned}$$

By induction,

$$\|u_{n+1} - x_{n+1}\| \le \varepsilon, \quad \forall n \ge m,$$

which is a contradiction to $\varepsilon < \limsup_{n\to\infty} \|u_n - x_n\|$.

In possibility (P2), for all $n \ge m'$, we acquire

$$\begin{aligned}
&\|u_{n+1} - x_{n+1}\| \\
&\quad \le \rho_n \vartheta \|v_n - \tilde{u}\| + \sigma_n \|u_n - x_n\| + \tau_n \|v_n - y_n\| \\
&\quad = \rho_n \vartheta \bigl\|\kappa_n u_n + (1-\kappa_n) J^{H_1}_{\mu_n,\Phi} H_1(u_n) - \tilde{u}\bigr\| + \sigma_n \|u_n - x_n\| \\
&\qquad + \tau_n \bigl\|\kappa_n u_n + (1-\kappa_n) J^{H_1}_{\mu_n,\Phi} H_1(u_n) - \kappa_n x_n - (1-\kappa_n) J^{H_1}_{\mu_n,\Phi} H_1(x_n)\bigr\| \\
&\quad \le \rho_n \vartheta \|u_n - \tilde{u}\| + (\sigma_n + \tau_n) \|u_n - x_n\| \\
&\quad \le \bigl(1 - \rho_n(1-\vartheta)\bigr) \|u_n - x_n\| + \rho_n \vartheta \|x_n - \tilde{u}\|.
\end{aligned}$$

By Lemma 2.8, we acquire that $\lim_{n\to\infty} \|u_n - x_n\| = 0$, which is a contradiction. Therefore $\lim_{n\to\infty} \|u_n - x_n\| = 0$, and hence

$$\lim_{n\to\infty} \|u_n - \tilde{u}\| \le \lim_{n\to\infty} \|u_n - x_n\| + \lim_{n\to\infty} \|x_n - \tilde{u}\| = 0.$$

Thus, the proof is completed. □

4 Consequences

In this section, we deduce some consequences from our main results.

Corollary 4.1 *Let $\mathcal{B}$ be a uniformly convex and uniformly smooth Banach space. Let $\Phi, \Psi : \mathcal{B} \to 2^{\mathcal{B}}$ be m-accretive mappings, and let $g : \mathcal{B} \to \mathcal{B}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Assume that $\Omega := \Phi^{-1}(0) \cap \Psi^{-1}(0) \neq \emptyset$. Let the sequences $\{v_n\}$ and $\{u_n\}$ be generated by the following iterative scheme*:

$$\begin{cases} u_1 \in \mathcal{B}, \\ v_n = \kappa_n u_n + (1-\kappa_n) J_{\mu_n,\Phi}(u_n), \\ u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n J_{v_n,\Psi}(v_n), \quad n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $J_{\mu_n,\Phi}(u_n) = (I + \mu_n \Phi)^{-1}(u_n)$, $J_{v_n,\Psi}(v_n) = (I + v_n \Psi)^{-1}(v_n)$, sequences $\{\mu_n\}$, $\{v_n\}$ are in $\mathbb{R}_+$ and $\{\kappa_n\}$, $\{\rho_n\}$, $\{\sigma_n\}$, $\{\tau_n\}$ are in $[0,1]$ with $\rho_n + \sigma_n + \tau_n = 1$, $n \in \mathbb{N}$. Assume that conditions (C_1)–(C_4) *of Theorem 3.2 are fulfilled. Then the sequence $\{u_n\}$ converges strongly to $\tilde{u} = Q_\Omega g(\tilde{u}) \in \Omega$.*

Proof If we take $H_1 \equiv H_2 \equiv I$ in Theorem 3.3, then the conclusion holds. □

In the framework of Hilbert spaces, we have the following outcomes.

Corollary 4.2 *Let $\mathcal{H}$ be a Hilbert space. Let $H_1, H_2 : \mathcal{B} \to \mathcal{B}$ be single-valued mappings such that H_1 is strongly monotone and Lipschitz continuous with constant ξ_{H_1} and H_2 is* *strongly monotone and Lipschitz continuous with constant ζ_{H_2}. Let $\varPhi, \varPsi : \mathcal{H} \to 2^{\mathcal{H}}$ be H_1-monotone and H_2-monotone mappings, respectively, and let $g : \mathcal{H} \to \mathcal{H}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Assume that $\varOmega := \varPhi^{-1}(0) \cap \varPsi^{-1}(0) \neq \emptyset$. Let the sequences $\{v_n\}$ and $\{u_n\}$ be generated by the iterative scheme (3.17). Assume that conditions (C_1)–(C_4) of Theorem 3.2 are fulfilled. Then the sequence $\{u_n\}$ converges strongly to $\tilde{u} = P_{\varOmega} g(\tilde{u}) \in \varOmega$, where $P_{\varOmega} : \mathcal{H} \to \varOmega$ is a metric projection from $\mathcal{H}$ onto $\varOmega$.*

For $H \equiv I$, the H-monotone mapping becomes maximal monotone mapping in Hilbert spaces. Hence we have the following outcome.

Corollary 4.3 *Let $\mathcal{H}$ be a Hilbert space. Let $\varPhi, \varPsi : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone mappings, and let $g : \mathcal{H} \to \mathcal{H}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Assume that $\varOmega := \varPhi^{-1}(0) \cap \varPsi^{-1}(0) \neq \emptyset$. Let the sequences $\{v_n\}$ and $\{u_n\}$ be generated by the iterative scheme (4.1). Assume that conditions (C_1)–(C_4) of Theorem 3.2 are fulfilled. Then the sequence $\{u_n\}$ converges strongly to $\tilde{u} = P_{\varOmega} g(\tilde{u}) \in \varOmega$.*

5 Applications

In this section, firstly, we examine a few of applications based on convex minimization problem.

Theorem 5.1 *Let $\mathcal{H}$ be a Hilbert space. Let $f_1, f_2 : \mathcal{H} \to (-\infty, +\infty]$ be proper mappings with convexity and lower semi-continuity, and let $g : \mathcal{H} \to \mathcal{H}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Assume that $\varOmega := (\partial f_1)^{-1}(0) \cap (\partial f_2)^{-1}(0) \neq \emptyset$. Let the sequences $\{v_n\}$, $\{w_n\}$, and $\{u_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} u_1 \in \mathcal{H}, \\ v_n = \kappa_n u_n + (1 - \kappa_n) \operatorname{argmin}_{y \in \mathcal{H}} \{f_1(y) + \frac{1}{2\mu_n} \|u_n - y\|^2\}, \\ w_n = \operatorname{argmin}_{y \in \mathcal{H}} \{f_2(y) + \frac{1}{2\nu_n} \|v_n - y\|^2\}, \\ u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n w_n, \quad n \in \mathbb{N}, \end{cases} \tag{5.1}$$

where sequences $\{\mu_n\}$, $\{\nu_n\}$ are in $\mathbb{R}_+$ and sequences $\{\kappa_n\}$, $\{\rho_n\}$, $\{\sigma_n\}$, $\{\tau_n\}$ are in $[0,1]$ with $\rho_n + \sigma_n + \tau_n = 1$, $n \in \mathbb{N}$. Assume that conditions (C_1)–(C_4) of Theorem 3.2 are fulfilled. Then the sequence $\{u_n\}$ converges strongly to $\tilde{u} = P_{\varOmega} g(\tilde{u}) \in \varOmega$.

Proof From (5.1), we can write

$$\frac{v_n - \kappa_n u_n}{(1 - \kappa_n)} = \operatorname*{argmin}_{y \in \mathcal{H}} \left\{ f_1(y) + \frac{1}{2\mu_n} \|u_n - y\|^2 \right\}. \tag{5.2}$$

For a proper mapping with convexity and lower semi-continuity, the subdifferential mapping is maximal monotone (see [26]). Thus ∂f_1 and ∂f_2 are maximal monotone in $\mathcal{H}$. Hence (5.2) is equivalent to

$$u_n \in \left(\frac{v_n - \kappa_n u_n}{1 - \kappa_n} \right) + \mu_n \partial f_1 \left(\frac{v_n - \kappa_n u_n}{1 - \kappa_n} \right),$$

that is,

$$v_n = \kappa_n u_n + (1-\kappa_n)J_{\mu_n,\partial f_1}(u_n), \quad \text{with } J_{\mu_n,\partial f_1}(u_n) = (I+\mu_n\partial f_1)^{-1}(u_n),$$

and

$$w_n = \operatorname*{argmin}_{y\in\mathcal{H}}\left\{f_2(y) + \frac{1}{2\nu_n}\|v_n - y\|^2\right\}$$

is equivalent to

$$v_n \in w_n + \nu_n\partial f_2(w_n), \quad \text{i.e., } w_n = J_{\nu_n,\partial f_2}(v_n), \text{with } J_{\nu_n,\partial f_2}(v_n) = (I+\nu_n\partial f_2)^{-1}(v_n).$$

Therefore,

$$u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n J_{\nu_n,\partial f_2}(v_n).$$

With the assistance of Corollary 4.3, including $\Phi = \partial f_1$ and $\Psi = \partial f_2$, we get the strong convergence of $\{u_n\}$ to a point $\tilde{u} \in \Omega$, which is the conclusion. □

Corollary 5.2 *Let $\mathcal{H}$ be a Hilbert space. Let $f : \mathcal{H} \to (-\infty,+\infty]$ be a proper mapping with convexity and lower semi-continuity, and let $g : \mathcal{H} \to \mathcal{H}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Assume that $\Omega := (\partial f)^{-1}(0) \neq \emptyset$. Let the sequences $\{v_n\}$ and $\{u_n\}$ be generated by the following iterative scheme*:

$$\begin{cases} u_1 \in \mathcal{H}, \\ v_n = \kappa_n u_n + (1-\kappa_n)\operatorname{argmin}_{y\in\mathcal{H}}\{f(y) + \frac{1}{2\mu_n}\|u_n - y\|^2\}, \\ u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n v_n, \quad n \in \mathbb{N}, \end{cases} \tag{5.3}$$

where $\{\mu_n\} \subset \mathbb{R}_+$ and sequences $\{\kappa_n\}$, $\{\rho_n\}$, $\{\sigma_n\}$, $\{\tau_n\}$ are in $[0,1]$ with $\rho_n + \sigma_n + \tau_n = 1$, $n \in \mathbb{N}$. Assume that conditions (C_1)–(C_4) of Theorem 3.2 are fulfilled. Then the sequence $\{u_n\}$ converges strongly to $\tilde{u} = P_\Omega g(\tilde{u}) \in \Omega$.

For a nonempty, convex, and closed subset $\mathcal{C}$ of a Hilbert space $\mathcal{H}$, the indicator function symbolized by $\mathcal{I}_\mathcal{C}$ is defined by

$$\mathcal{I}_\mathcal{C}(y) = \begin{cases} 0, & \text{if } y \in \mathcal{C}, \\ +\infty, & \text{if } y \notin \mathcal{C}, \end{cases}$$

and the normal cone for $\mathcal{C}$ symbolized by $\mathcal{N}_\mathcal{C}$ at $y \in \mathcal{C}$ is defined by

$$\mathcal{N}_\mathcal{C}(y) = \{x \in \mathcal{H} : \langle z - y, x\rangle \leq 0, \forall z \in \mathcal{C}\}.$$

Theorem 5.3 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be nonempty, convex, and closed subsets of a Hilbert space $\mathcal{H}$. Let $g : \mathcal{H} \to \mathcal{H}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Let $P_{\mathcal{C}_1} : \mathcal{H} \to \mathcal{C}_1$ and*

$P_{\mathcal{C}_2}: \mathcal{H} \to \mathcal{C}_2$ *be projections. Assume that* $\Omega := \mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$. *Let the sequences* $\{v_n\}$ *and* $\{u_n\}$ *be generated by the following iterative scheme*:

$$\begin{cases} u_1 \in \mathcal{H}, \\ v_n = \kappa_n u_n + (1-\kappa_n) P_{\mathcal{C}_1}(u_n), \\ u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n P_{\mathcal{C}_2}(v_n), \quad n \in \mathbb{N}, \end{cases} \tag{5.4}$$

where sequences $\{\kappa_n\}$, $\{\rho_n\}$, $\{\sigma_n\}$, *and* $\{\tau_n\}$ *are in* $[0,1]$ *with* $\rho_n + \sigma_n + \tau_n = 1$, $n \in \mathbb{N}$. *Assume that conditions* (C_1)–(C_3) *of Theorem* 3.2 *are fulfilled. Then the sequence* $\{u_n\}$ *converges strongly to* $\tilde{u} = P_\Omega g(\tilde{u}) \in \Omega$.

Proof Since indicator functions $\mathcal{I}_{\mathcal{C}_1}$ and $\mathcal{I}_{\mathcal{C}_2}$ of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, are proper mappings with convexity and lower semi-continuity and $\operatorname{argmin} \mathcal{I}_{\mathcal{C}_1} = \mathcal{C}_1$ and $\operatorname{argmin} \mathcal{I}_{\mathcal{C}_2} = \mathcal{C}_2$, therefore

$$\Omega = \mathcal{C}_1 \cap \mathcal{C}_2 = \operatorname{argmin} \mathcal{I}_{\mathcal{C}_1} \cap \operatorname{argmin} \mathcal{I}_{\mathcal{C}_2}.$$

With the assistance of Theorem 5.1 including $f_1 = \mathcal{I}_{\mathcal{C}_1}$ and $f_2 = \mathcal{I}_{\mathcal{C}_2}$ and for all convex closed subset $\mathcal{C}$ in $\mathcal{H}$ and for all $\mu \in \mathbb{R}_+$, $P_{\mathcal{C}} = (I + \mu \partial \mathcal{I}_{\mathcal{C}})^{-1} = (I + \mu \mathcal{N}_{\mathcal{C}})^{-1}$, we secure the conclusion. □

Next, we evaluate a common solution of the variational inequality problem and convex minimization problem.

For a nonempty, convex, and closed subset $\mathcal{C}$ of a Hilbert space $\mathcal{H}$ and for a single-valued monotone and hemi-continuous mapping $\Phi: \mathcal{C} \to \mathcal{H}$, the variational inequality problem is to seek a point $y \in \mathcal{C}$ such that the following inequality is fulfilled:

$$\langle z - y, \Phi(y) \rangle \geq 0, \quad \forall z \in \mathcal{C}. \tag{5.5}$$

$\mathtt{VI}(\mathcal{C}, \Phi)$ stands for the solution set of the variational inequality problem (5.5).

Theorem 5.4 *Let* $\mathcal{C}$ *be a nonempty, convex, and closed subset of a Hilbert space* $\mathcal{H}$. *Let* $f: \mathcal{H} \to (-\infty, +\infty]$ *be a proper mapping with convexity and lower semi-continuity, and let* $\Phi: \mathcal{C} \to \mathcal{H}$ *be a monotone and hemi-continuous mapping. Assume that* $\Omega := \partial f^{-1}(0) \cap \mathtt{VI}(\mathcal{C}, \Phi) \neq \emptyset$. *Let the sequences* $\{v_n\}$, $\{w_n\}$, *and* $\{u_n\}$ *be generated by the following iterative scheme*:

$$\begin{cases} u_1 \in \mathcal{H}, \\ v_n = \kappa_n u_n + (1-\kappa_n) \operatorname{argmin}_{y \in \mathcal{H}} \{f_1(y) + \frac{1}{2\mu_n} \|u_n - y\|^2\}, \\ w_n = \mathtt{VI}(\mathcal{C}, \nu_n \Phi + I - v_n), \\ u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n w_n, \quad n \in \mathbb{N}, \end{cases} \tag{5.6}$$

where sequences $\{\mu_n\}$, $\{\nu_n\}$ *are in* $\mathbb{R}_+$ *and sequences* $\{\kappa_n\}$, $\{\rho_n\}$, $\{\sigma_n\}$, $\{\tau_n\}$ *are in* $[0,1]$ *with* $\rho_n + \sigma_n + \tau_n = 1$, $n \in \mathbb{N}$. *Assume that conditions* (C_1)–(C_4) *of Theorem* 3.2 *are fulfilled. Then the sequence* $\{u_n\}$ *converges strongly to* $\tilde{u} = P_\Omega g(\tilde{u}) \in \Omega$.

Proof Define a mapping $\mathcal{T} \subset \mathcal{H} \times \mathcal{H}$ by

$$\mathcal{T}(y) = \begin{cases} \Phi(y) + \mathcal{N}_{\mathcal{C}}(y), & \text{if } y \in \mathcal{C}, \\ \emptyset, & \text{if } y \notin \mathcal{C}. \end{cases}$$

From [27], we know that $\mathcal{T}$ is maximal monotone, and $\mathcal{T}^{-1}(0) = \mathtt{VI}(\mathcal{C}, \Phi)$.

From (5.5) and (5.6), we can easily observe that

$$w_n = \mathtt{VI}(\mathcal{C}, \nu_n \Phi + I - \nu_n) \quad \Leftrightarrow \quad \langle z - w_n, \nu_n \Phi(w_n) + w_n - \nu_n \rangle \geq 0, \quad \forall z \in \mathcal{C},$$

which gives

$$-\nu_n \Phi(w_n) - w_n + \nu_n \in \nu_n \mathcal{N}_{\mathcal{C}}(w_n), \quad \text{i.e., } w_n = (I + \nu_n \mathcal{T})^{-1}(\nu_n) = J_{\nu_n, \mathcal{T}}(\nu_n).$$

With the assistance of Corollary 4.3 and the proof of Theorem 5.1, we obtain the conclusion. □

Finally, we derive an application to equilibrium problem which is as follows:

Let $\mathcal{C}$ be a nonempty, convex, and closed subset of a Hilbert space $\mathcal{H}$. Let $\Theta : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bi-function. The equilibrium problem is to seek a point $y \in \mathcal{C}$ such that the following inequality is fulfilled:

$$\Theta(y, z) \geq 0, \quad \forall z \in \mathcal{C}. \tag{5.7}$$

Let the solution set of equilibrium problem be identified by $\mathtt{EP}(\Theta)$. To study problem (5.7), suppose that the following conditions are fulfilled by Θ:

(E_1) For all $z \in \mathcal{C}$, $\Theta(z, z) = 0$;

(E_2) Θ is monotone, i.e., for all $y, z \in \mathcal{C}$, $\Theta(y, z) + \Theta(z, y) \leq 0$;

(E_3) $\lim_{t \downarrow 0} \Theta((1 - t)y + tw, z) \leq \Theta(y, z)$ for each $y, z, w \in \mathcal{C}$;

(E_4) For each $y \in \mathcal{C}$, $z \longmapsto \Theta(y, z)$ is convex and lower semi-continuous.

Lemma 5.5 ([32]) *Let $\mathcal{C}$ be a nonempty, convex, and closed subset of a Hilbert space $\mathcal{H}$. Let $\Theta : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bi-function which fulfills* (E_1)–(E_4). *Then a set-valued mapping $A_\Theta : \mathcal{H} \to 2^{\mathcal{H}}$ defined by*

$$A_\Theta(y) = \begin{cases} \{w \in \mathcal{H} : \langle z - y, w \rangle \leq \Theta(y, z), \forall z \in \mathcal{C}\}, & y \in \mathcal{C}, \\ \emptyset, & y \notin \mathcal{C} \end{cases}$$

is maximal monotone with $\mathtt{dom}(A_\Theta) \subset \mathcal{C}$, $\mathtt{EP}(\Theta) = A_\Theta^{-1}(0)$, *and the resolvent* $\mathcal{T}_{\mu, A_\Theta} = (I + \mu A_\Theta)^{-1}$ *affiliated with A_Θ is defined by*

$$\mathcal{T}_{\mu, A_\Theta}(y) = \left\{ w \in \mathcal{C} : \Theta(w, z) + \frac{1}{\mu} \langle z - w, w - y \rangle \geq 0, \forall z \in \mathcal{C} \right\}, \quad \forall y \in \mathcal{H}.$$

Theorem 5.6 *Let $\mathcal{C}$ be a nonempty, convex, and closed subset of a Hilbert space $\mathcal{H}$. Let $\Theta_1, \Theta_2 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be bi-functions which fulfill* (E_1)–(E_4), *and let $g : \mathcal{H} \to \mathcal{H}$ be a ϑ-contraction mapping, where $0 < \vartheta < 1$. Assume that $\Omega := \mathtt{EP}(\Theta_1) \cap \mathtt{EP}(\Theta_2) \neq \emptyset$. Let the* *sequences* $\{v_n\}$ *and* $\{u_n\}$ *be generated by the following iterative scheme*:

$$\begin{cases} u_1 \in \mathcal{H}, \\ v_n = \kappa_n u_n + (1-\kappa_n)\mathcal{T}_{\mu_n,A_{\Theta_1}}(u_n), \\ u_{n+1} = \rho_n g(v_n) + \sigma_n u_n + \tau_n \mathcal{T}_{\nu_n,A_{\Theta_2}}(v_n), \quad n \in \mathbb{N}, \end{cases} \tag{5.8}$$

where sequences $\{\mu_n\}$, $\{\nu_n\}$ *are in* $\mathbb{R}_+$ *and sequences* $\{\kappa_n\}$, $\{\rho_n\}$, $\{\sigma_n\}$, $\{\tau_n\}$ *are in* $[0,1]$ *with* $\rho_n + \sigma_n + \tau_n = 1$, $n \in \mathbb{N}$. *Assume that conditions* (C_1)–(C_4) *of Theorem* 3.2 *are fulfilled. Then the sequence* $\{u_n\}$ *converges strongly to* $\tilde{u} = P_\Omega g(\tilde{u}) \in \Omega$.

Proof With the assistance of Corollary 4.3 and Lemma 5.5, we obtain the conclusion. □

6 Numerical implementation

Firstly, we discuss a numerical example to demonstrate the implementation of iterative schemes (3.1) and (3.17) in Theorem 3.2 and Theorem 3.3, respectively. Matlab R2012a is used for writing all the codes, and it is running on Lenovo Core(TM) i5-2320 CPU @ 3.00 GHz with 4 GB RAM.

Example 6.1 Let $\mathcal{B} = \mathbb{R}$, with the usual norm $|\cdot|$. Let us define $\Phi, \Psi : \mathbb{R} \to 2^{\mathbb{R}}$ by $\Phi(y) = \{3y\}$ and $\Psi(y) = \{4y\}$, and $H_1, H_2 : \mathbb{R} \to \mathbb{R}$ by $H_1(y) = 2y$ and $H_2(y) = \frac{y}{3}$ for all $y \in \mathbb{R}$. Easily, we can check that H_1 is strongly accretive and Lipschitz continuous with constant 2; H_2 is strongly accretive and Lipschitz continuous with constant $\frac{1}{3}$; Φ is H_1-accretive and Ψ is H_2-accretive. We observe that $\Omega = \Phi^{-1}(0) \cap \Psi^{-1}(0) = \{0\}$. On setting, for all $n \in \mathbb{N}$, $\kappa_n = \frac{3n+2}{6n}$, $\mu_n = \frac{2n+3}{n+1} = \nu_n$, $\rho_n = \frac{1}{n+1}$, $\sigma_n = \frac{1}{3}$, $\tau_n = \frac{2n-1}{3(n+1)}$, all conditions (C_1)–(C_4) of Theorem 3.2 are fulfilled. Now we consider two cases:

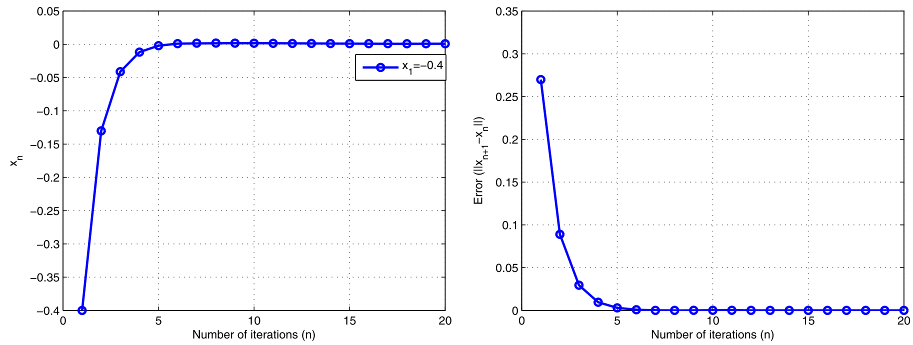
Case I. For the iterative scheme (3.1) of Theorem 3.2, let $l = 0.01$. Then the sequence $\{x_n\}$ obtained from (3.1) strongly converges to $Q_\Omega(0.01) = 0$. For $x_1 = -0.4$, the convergence of $\{x_n\}$ and behavior of error $\|x_{n+1} - x_n\|$ are numerically shown in Table 1 and graphically in Figures 1a and 1b, respectively.

Case II. For the iterative scheme (3.17) of Theorem 3.3, let $g(y) = \frac{y}{4}$ for all $y \in \mathbb{R}$. Then Table 1 and Fig. 2a ensures the strong convergence of the sequence $\{u_n\}$ obtained from (3.17) to $\tilde{u} = Q_\Omega(g(\tilde{u})) = 0$ for $u_1 = 0.5$. The graphical representation of error $\|u_{n+1} - u_n\|$ is displayed in Fig. 2b.

Next, we discuss another numerical example for the iterative scheme (5.1) in Theorem 5.1.

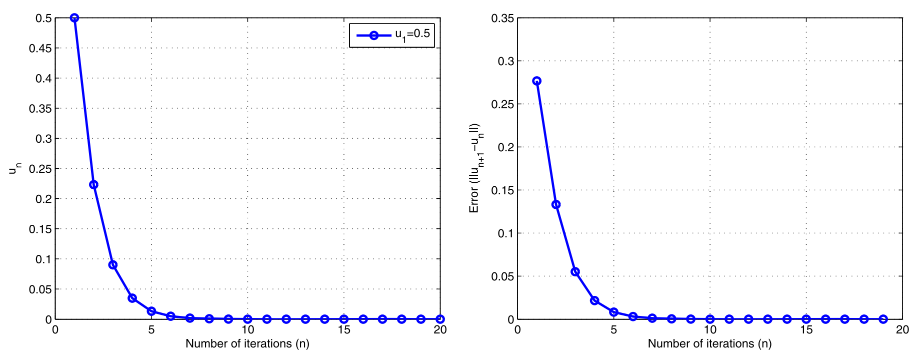
Table 1 Numerical outcomes of Example 6.1

Case I			Case II		
No. of iterat.	$x_1 = -0.4$		No. of iterat.	$u_1 = 0.5$	
	x_n	$\|x_{n+1} - x_n\|$		u_n	$\|u_{n+1} - u_n\|$
1	−0.4000	0.2698	1	0.5000	0.2767
4	−0.0117	0.0096	4	0.0349	0.0217
8	0.0019	0.0001	8	0.0007	0.0004
12	0.0013	0.0001	12	0.0000	7.4623e−06
16	0.0010	6.0789e−05	16	0.0000	1.2479e−07
20	0.0007	3.8738e−05	20	0.0000	2.0360e−09



(a) Convergence of $\{x_n\}$ **(b)** Error representation

Figure 1 Example 6.1, Case I with $x_1 = -0.4$



(a) Convergence of $\{u_n\}$ **(b)** Error representation

Figure 2 Example 6.1, Case II with $u_1 = 0.5$

Example 6.2 Let $\mathcal{H} = \mathbb{R}^3$, with the inner product defined by $\langle y, z\rangle = y_1z_1 + y_2z_2 + y_3z_3$, $\forall y = (y^1, y^2, y^3), z = (z^1, z^2, z^3) \in \mathbb{R}^3$ and induced Euclidean norm. For $k = 1, 2$, define $f_k : \mathbb{R}^3 \to (-\infty, +\infty]$ as follows:

$$f_k(y) = \langle M_k y, y\rangle + \frac{\langle N_k, y\rangle}{2} + C_k, \quad k = 1, 2, \tag{6.1}$$

with

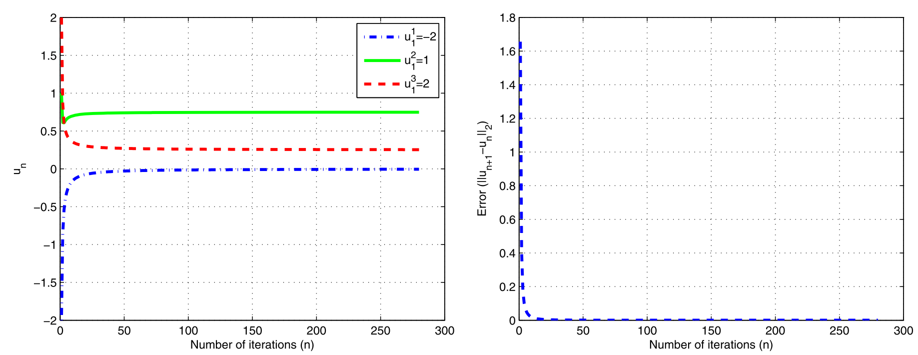
$$M_1 = \begin{pmatrix} 3 & -3 & -3 \\ -3 & 3 & 3 \\ -3 & 3 & 3 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 12 & -12 & -12 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & -12 & -12 \end{pmatrix}, \qquad C_1, C_2 \text{ are any constants.}$$

It can be easily demonstrated that f_1 and f_2 are proper mappings with convexity and continuity. In Theorem 5.1, we seek a point $\tilde{u} \in \Omega = (\partial f_1)^{-1}(0) \cap (\partial f_2)^{-1}(0)$, equivalently, $\tilde{u} \in \Omega = \operatorname{argmin}_{y\in\mathbb{R}^3} f_1(y) \cap \operatorname{argmin}_{y\in\mathbb{R}^3} f_2(y)$, where $f_1, f_2 : \mathbb{R}^3 \to (-\infty, +\infty]$ are proper map-

Table 2 Numerical outcomes of Example 6.2

Case I			Case II		
n	$u_1 = (-2, 1, 2)$		n	$u_1 = (0, 1, 2)$	
	(u_n^1, u_n^2, u_n^3)	$\|u_{n+1} - u_n\|_2$		(u_n^1, u_n^2, u_n^3)	$\|u_{n+1} - u_n\|_2$
1	(−2.0000, 1.0000, 2.0000)	1.6559	1	(0.0000, 1.0000, 2.0000)	1.5677
40	(−0.0363, 0.7380, 0.2755)	0.0011	40	(−0.0177, 0.4719, 0.5044)	0.0008
80	(−0.0181, 0.7440, 0.2627)	0.0003	80	(−0.0088, 0.4856, 0.5026)	0.0002
120	(−0.0120, 0.7460, 0.2585)	0.0001	120	(−0.0058, 0.4902, 0.5019)	9.2381e−05
160	(−0.0090, 0.7467, 0.2564)	7.1243e−05	160	(−0.0044, 0.4926, 0.5015)	5.2486e−05
200	(−0.0072, 0.7476, 0.2551)	4.5631e−05	200	(−0.0035, 0.4940, 0.5012)	3.3849e−05
240	(−0.0060, 0.7480, 0.2542)	3.1705e−05	240	(−0.0029, 0.4950, 0.5011)	2.3652e−05
280	(−0.0052, 0.7483, 0.2536)	2.3302e−05	280	(−0.0025, 0.4957, 0.5009)	1.7468e−05



(a) Convergence of $\{u_n\}$ **(b)** Error representation

Figure 3 Example 6.2, Case I with $u_1 = (-2, 1, 2)$

pings with convexity and lower semi-continuity. For f_k, $k = 1, 2$, as defined in (6.1),

$$\Omega = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 = 0, y_2 + y_3 = 1\}.$$

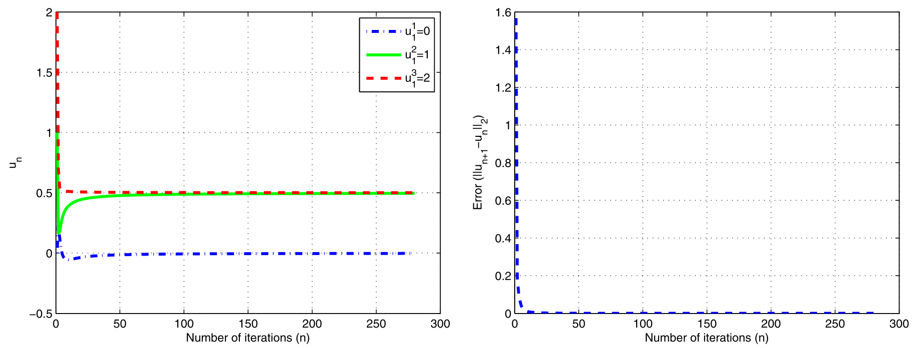
By taking, for all $n \in \mathbb{N}$, $\kappa_n = \frac{9n+8}{72n}$, $\mu_n = \frac{2n+3}{n+1} = \nu_n$, $\rho_n = \frac{1}{n+1}$, $\sigma_n = \frac{1}{9}$, $\tau_n = \frac{8n-1}{9(n+1)}$, it can be easily verified that all conditions (C_1)–(C_4) of Theorem 5.1 are fulfilled. Now two cases may occur:

Case I. If $g(y) = (-0.5, 1, 0.5)$ for all $y \in \mathbb{R}^3$, then the sequence $\{u_n\}$ obtained from (5.1) strongly converges to $\tilde{u} = P_\Omega(-0.5, 1, 0.5) = (0, 0.75, 0.25)$ for $u_1 = (-2, 1, 2)$. The numerical and graphical demonstration of convergence of the sequence $\{u_n\}$ and the behavior of error $\|u_{n+1} - u_n\|_2$ are displayed in Table 2 and Fig. 3, respectively.

Case II. If $g(y) = \frac{y+1}{16}$ for all $y \in \mathbb{R}^3$, then the sequence $\{u_n\}$ obtained from (5.1) strongly converges to $\tilde{u} = P_\Omega g(\tilde{u}) = (0, 0.5, 0.5)$ for $u_1 = (0, 1, 2)$. The convergence of $\{u_n\}$ and the behavior of error $\|u_{n+1} - u_n\|_2$ are presented numerically in Table 2 and graphically in Fig. 4.

7 Conclusion

In this manuscript, we have introduced two iterative methods for finding the common zeros of two H-accretive mappings in the framework of Banach spaces. These iterative methods are based on Mann and Halpern iterative methods and viscosity approximation method. It is easy to observe that for $\kappa_n = 0$, $n \in \mathbb{N}$, our proposed iterative methods consist



(a) Convergence of $\{u_n\}$ (b) Error representation

Figure 4 Example 6.2, Case II with $u_1 = (0, 1, 2)$

of method of alternating resolvents, and hence our work extends the methods developed by Bauschke et al. [2], Boikanyo et al. [4], and Liu et al. [19]. We have demonstrated the strong convergence results of the sequences generated by our proposed iterations. Further, we have examined some applications which are based on convex minimization problem, variational inequality problem, and equilibrium problem. Finally, we have presented some numerical examples for implementation of our main results and application.

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