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Certain integrals involving multivariate Mittag-Leffler function

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Abstract
The objective of this article is to present several new integral equalities involving the multivariate Mittag-Leffler functions which are associated with the Laguerre polynomials. To emphasize our main results, we also consider some important special cases. The main results of our paper are quite general in nature and yield a very large number of integral equalities involving polynomials occurring in problems of mathematical analysis and mathematical physics.

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1 Introduction and preliminaries

The function defined by the series representation

$$E_{\xi}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\xi n+1)}\quad(\xi>0,z\in\mathbb{C})\tag{1}$$

and its generalization

$$E_{\xi,\nu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\xi n+\nu)}\quad(\xi>0,\nu>0,z\in\mathbb{C})\tag{2}$$

were introduced and studied by Agarwal [1], Mittag-Leffler [2, 3], Humburt [4], Humbert and Agrawal [5], and Wiman [6, 7], where $\mathbb{C}$ is the set of complex numbers. The main properties of these functions are given in the book by Erdélyi et al. [8, Sect. 18.1], and a more extensive and detailed account on Mittag-Leffler functions is presented in Dzherbashyan [9, Chap. 2]. In particular, the functions (1) and (2) are entire functions of order $\rho=1/\xi$ and type $\sigma=1$; see, for example, [9, p. 118]. For a detailed account of various properties, generalizations, and applications of these functions, the reader may refer to an excellent work of Dzherbashyan [9], Kilbas and Saigo [10–13], Gorenflo and Mainardi [14], Gorenflo, Luchko and Rogosin [15], and Gorenflo, Kilbas and Rogosin [16].

The series representation of a generalization of (2) was introduced by Prabhaker [17] as:

$$E^{\delta}_{\xi,\nu}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\xi n+\nu)n!}z^n, \tag{3}$$

where $\xi,\nu,\delta\in\mathbb{C}$ $(\Re(\xi)>0)$. It is entire function of order $[\Re(\xi)]^{-1}$ (see [17, p. 7]) and $(\delta)_n$ denotes the Pochhammer symbol defined as:

$$(\delta)_n=\frac{\Gamma(\delta+n)}{\Gamma(\delta)}=\begin{cases}1, & n=0,\lambda\in\mathbb{C}/\{0\},\\ \delta(\delta+1)\cdots(\delta+n-1), & n\in\mathbb{C};\delta\in\mathbb{C}.\end{cases} \tag{4}$$

Srivastava and Tomoviski [18] studied and generalized the Mittag-Leffler-type function $E^{\delta}_{\xi,\nu}(z)$ as

$$E^{\delta;\kappa}_{\xi;\nu}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_{\kappa n}z^n}{\Gamma(\xi n+\nu)n!}, \tag{5}$$

where $\delta,\kappa,\xi,\nu,z\in\mathbb{C};\Re(\delta)>0,\Re(\kappa)>0,\Re(\xi)>0$.

A generalization of (5) was initiated by Salim and Faraj [19] as follows:

$$E^{\delta,\kappa,\rho}_{\xi,\nu,\varepsilon}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_{\rho n}}{\Gamma(\xi n+\nu)(\kappa)_{\varepsilon n}}\frac{z^n}{n!}, \tag{6}$$

where $\delta,\xi,\nu,\kappa,z\in\mathbb{C};\min(\Re(\delta),\Re(\xi),\Re(\nu),\Re(\kappa)>0);\rho,\varepsilon>0,\rho\leq\Re(\xi)+\varepsilon$.

Further, a multivariate generalization of Mittag-Leffler function $E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu;\varepsilon_i}(\cdot)$, which is a generalization of (6), was studied by Gujar et al. [20] in the following form:

$$E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu;\varepsilon_i}(z_1,\ldots,z_r)=\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r}}{\Gamma(\sum_{i=1}^{r}\xi_i m_i+\nu)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}}, \tag{7}$$

where $\delta_i,\xi_i,\nu,\kappa_i,z_i\in\mathbb{C};\min_{1\leq i\leq r}(\Re(\delta_i),\Re(\xi_i),\Re(\nu),\Re(\kappa_i))>0$ and $\rho_i,\varepsilon_i>0;\rho_i<\Re(\xi_i)+\varepsilon_i$ $(i=1,2,\ldots,r)$.

For a more detailed account of various properties, generalizations, and applications in terms of fractions of this function, the reader may refer to Mishra et al. [21], Purohit et al. [22], Saxena et al. [23] and Suthar et al. [24–26].

The polynomial $L^{(\mu,\tau)}_n(x)$ was defined by Prabhaker and Suman [27] as:

$$L^{(\mu,\tau)}_n(x)=\frac{\Gamma(\mu n+\tau+1)}{\Gamma(n+1)}\sum_{r=0}^{n}\frac{(-n)_r x^r}{r!\Gamma(\mu r+\tau+1)}, \tag{8}$$

where $\mu\in\mathbb{C}^+,\tau\in\mathbb{C}^+_{-1};n\in\mathbb{N}$. If $\mu=1$, then (8) reduces to

$$L^{(1,\tau)}_n(x)=\frac{\Gamma(n+\tau+1)}{\Gamma(n+1)}\sum_{r=0}^{\infty}\frac{(-n)_r x^r}{r!\Gamma(r+\tau+1)}=L^{\tau}_n(x), \tag{9}$$

where $L^{\tau}_n(x)$ is a well-known generalized Laguerre polynomial (see [28]).

In the sequel, the Konhauser polynomials of the second kind were defined by Srivastava [29] as:

$$\mathrm{Z}_n^{\tau}[x;r] = \frac{\Gamma(rn+\tau+1)}{\Gamma(n+1)}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{rj}}{r!\Gamma(rj+\tau+1)}, \tag{10}$$

where $\tau \in \mathbb{C}_{-1}^{+}$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$.

It can be easily verified that

$$L_n^{(r,\tau)}(x^r) = \mathrm{Z}_n^{\tau}[x;r], \tag{11}$$

$$L_n^{\tau}(x) = \mathrm{Z}_n^{\tau}[x;1]. \tag{12}$$

Further, the polynomial $\mathrm{Z}_n^{(\mu,\tau)}[x;r]$ is defined [30] as:

$$\mathrm{Z}_n^{(\mu,\tau)}[x;r] = \sum_{j=0}^{n}(-1)^j\frac{\Gamma(rn+\tau+1)x^{rj}}{j!\Gamma(rj+\tau+1)\Gamma(\mu n-\eta j+1)}. \tag{13}$$

From Eqs. (10) and (13), we obtain

$$\mathrm{Z}_n^{\tau}[x;r] = \mathrm{Z}_n^{(1,\tau)}[x;r]. \tag{14}$$

If $\mu \in \mathbb{N}$, then Eq. (12) can be written in the following form:

$$\mathrm{Z}_n^{(\mu,\tau)}[x;r] = \frac{\Gamma(rn+\tau+1)}{\Gamma(\mu n+1)}\sum_{j=0}^{n}(-1)^j\frac{(-\mu n)_{\mu j}x^{rj}}{j!\Gamma(rj+\tau+1)(-1)^{(\mu-1)j}}. \tag{15}$$

The set of polynomials $L_n^{(\mu,\tau)}[\vartheta;x]$ is defined [30] as:

$$L_n^{(\mu,\tau)}[\vartheta;x] = \sum_{j=0}^{n}(-1)^j\frac{\Gamma(rn+\tau+1)x^{j}}{j!\Gamma(rj+\tau+1)\Gamma(\vartheta n-\vartheta j+1)}, \tag{16}$$

where $\mu,\vartheta \in \mathbb{C}^{+}, \tau \in \mathbb{C}_{-1}^{+}, n \in \mathbb{N}$.

2 Main integral equalities

Throughout this paper, we assume that $\delta_i,\xi_i,\nu,\kappa_i,\alpha \in \mathbb{C}; \min_{1\le i\le r}(\Re(\delta_i),\Re(\xi_i),\Re(\nu),\Re(\kappa_i)) > 0$, $(\rho_i,\varepsilon_i) > 0$ and $\rho_i < \Re(\xi_i)+\varepsilon_i$; $i = 1,2,\ldots,r$.

Theorem 1 *The following integral equality holds*:

$$\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\nu-1}(1-u)^{\alpha-1}E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\left(z_1u^{\xi_1},\ldots,z_ru^{\xi_r}\right)du = E_{\xi_i;\nu+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1,\ldots,z_r). \tag{17}$$

Proof Applying Eq. (7) to the left-hand side of Eq. (17), we obtain

$$= \sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r}}{\Gamma(\nu+\sum_{i=1}^{r}\xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}}$$

$$
\begin{aligned}
&\times \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\nu+\sum_{i=1}^r m_i\xi_i-1}(1-u)^{\alpha-1}\,du \\
&= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r}\mathrm{B}(\alpha,\nu+\sum_{i=1}^r \xi_i m_i)}{\Gamma(\alpha)\Gamma(\nu+\sum_{i=1}^r \xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r}}{\Gamma(\nu+\alpha+\sum_{i=1}^r \xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&= E_{\xi_i;\nu+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1,\ldots,z_r).
\end{aligned}
$$

This completes the proof of Theorem 1. □

Theorem 2 *The following integral equality holds*:

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1} E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\bigl(z_1(s-t)^{\xi_1},\ldots,z_r(s-t)^{\xi_r}\bigr)\,ds \\
&\quad= (x-t)^{\alpha+\nu-1} E_{\xi_i;\nu+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\bigl(z_1(x-t)^{\xi_1},\ldots,z_r(x-t)^{\xi_r}\bigr).
\end{aligned}
\tag{18}
$$

Proof Using Eq. (6) in the left-hand side of Eq. (18), we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1} \\
&\quad\times \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1(s-t)^{\xi_1})^{m_1}\cdots(z_r(s-t)^{\xi_r})^{m_r}}{\Gamma(\nu+\sum_{i=1}^r \xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}}\,ds.
\end{aligned}
$$

Changing the variable s to $u = \frac{s-t}{x-t}$, we obtain

$$
\begin{aligned}
&= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r}}{\Gamma(\nu+\sum_{i=1}^r \xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \frac{1}{\Gamma(\alpha)} \\
&\quad\times \int_0^1 \bigl(x-t-u(x-t)\bigr)^{\alpha-1}\bigl(t+u(x-t)-t\bigr)^{\nu+\sum_{i=1}^r \xi_i m_i-1}(x-t)\,du \\
&= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1(x-t)^{\xi_1})^{m_1}\cdots(z_r(x-t)^{\xi_r})^{m_r}}{\Gamma(\nu+\sum_{i=1}^r \xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&\quad\times \frac{(x-t)^{\alpha+\nu-1}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1}(u)^{\nu+\sum_{i=1}^r \xi_i m_i-1}\,du \\
&= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1(x-t)^{\xi_1})^{m_1}\cdots(z_r(x-t)^{\xi_r})^{m_r}}{(x-t)^{-\alpha-\nu+1}\Gamma(\nu+\alpha+\sum_{i=1}^r \xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}},
\end{aligned}
$$

from which, after little simplification, we easily arrive at the required Eq. (18). □

Theorem 3 *If* $\omega_i,\tau,\in\mathbb{C};\min_{1\le i\le r}(\Re(\omega_i),\Re(\tau))>0$, *then the following integral equality holds*:

$$
\begin{aligned}
&\int_0^x t^{\tau-1}(x-t)^{\nu-1} E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;1}\bigl(z_1(x-t)^{\xi_1},\ldots,z_r(x-t)^{\xi_r}\bigr) \\
&\quad\times E_{\xi_i;\tau;\varepsilon_i}^{\omega_i;\kappa_i;1}\bigl(z_1(t)^{\xi_1},\ldots,z_r(t)^{\xi_r}\bigr)\,dt = x^{\tau+\nu-1} E_{\xi_i;\nu+\tau;\varepsilon_i}^{(\omega_i+\delta_i);\kappa_i;1}\bigl(z_1x^{\xi_1},\ldots,z_rx^{\xi_r}\bigr).
\end{aligned}
\tag{19}
$$

Proof Applying Eq. (7) to the left-hand side of Eq. (19), we obtain

$$\begin{aligned}
&= \sum_{m_1,\dots,m_r=0}^{\infty} \sum_{l_1,\dots,l_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(\omega_1)_{\rho_1 l_1}\cdots(\omega_r)_{\rho_r l_r}}{\Gamma(\nu+\sum_{i=1}^r \xi_i m_i)\Gamma(\tau+\sum_{i=1}^r \xi_i l_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&\quad \times \frac{(z_1)^{m_1+l_1}\cdots(z_r)^{m_r+l_r}}{(\kappa_1)_{\varepsilon_1 l_1}\cdots(\kappa_r)_{\varepsilon_r l_r}} \int_0^x t^{\tau+\sum_{i=1}^r l_i\xi_i-1}(x-t)^{\nu+\sum_{i=1}^r m_i\xi_i-1}\,dt \\
&= \sum_{m_1,\dots,m_r=0}^{\infty} \sum_{l_1,\dots,l_r=0}^{\infty} \frac{x^{\tau+\nu-1}(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(\omega_1)_{\rho_1 l_1}\cdots(\omega_r)_{\rho_r l_r}}{\Gamma(\nu+\sum_{i=1}^r \xi_i m_i)\Gamma(\tau+\sum_{i=1}^r \xi_i l_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&\quad \times \frac{(z_1x^{\xi_1})^{m_1+l_1}\cdots(z_rx^{\xi_r})^{m_r+l_r}}{(\kappa_1)_{\varepsilon_1 l_1}\cdots(\kappa_r)_{\varepsilon_r l_r}} B\left(\tau+\sum_{i=1}^r l_i\xi_i, \nu+\sum_{i=1}^r m_i\xi_i\right) \\
&= x^{\tau+\nu-1} \sum_{m_1,\dots,m_r=0}^{\infty} \sum_{l_1,\dots,l_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(\omega_1)_{\rho_1 l_1}\cdots(\omega_r)_{\rho_r l_r}}{\Gamma(\tau+\nu+\sum_{i=1}^r (m_i+l_i)\xi_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&\quad \times \frac{(z_1x^{\xi_1})^{m_1+l_1}\cdots(z_rx^{\xi_r})^{m_r+l_r}}{(\kappa_1)_{\varepsilon_1 l_1}\cdots(\kappa_r)_{\varepsilon_r l_r}} \\
&= x^{\tau+\nu-1} \sum_{m_1,\dots,m_r=0}^{\infty} \sum_{l_1,\dots,l_r=0}^{\infty} \binom{m_1}{l_1}\cdots\binom{m_r}{l_r} \frac{(\delta_1)_{(m_1-l_1)\rho_1}\cdots(\delta_r)_{(m_r-l_r)\rho_r}}{\Gamma(\tau+\nu+\sum_{i=1}^r (m_i)\xi_i)} \\
&\quad \times \frac{(\omega_1)_{l_1\rho_1}\cdots(\omega_r)_{l_r\rho_r}(z_1x^{\xi_1})^{m_1}\cdots(z_rx^{\xi_r})^{m_r}}{(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}}.
\end{aligned} \tag{20}$$

Substituting $\rho_1 = \cdots = \rho_r = 1$ in Eq. (20) reduces it to

$$\begin{aligned}
&\int_0^x t^{\tau-1}(x-t)^{\nu-1} E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;1}\bigl(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\bigr) \\
&\quad \times E_{\xi_i;\tau;\varepsilon_i}^{\omega_i;\kappa_i;1}\bigl(z_1(t)^{\xi_1},\dots,z_r(t)^{\xi_r}\bigr)\,du \\
&\quad = x^{\tau+\nu-1} \sum_{m_1,\dots,m_r=0}^{\infty} \sum_{l_1,\dots,l_r=0}^{\infty} \binom{m_1}{l_1}\cdots\binom{m_r}{l_r} \frac{(\delta_1)_{(m_1-l_1)}\cdots(\delta_r)_{(m_r-l_r)}}{\Gamma(\tau+\nu+\sum_{i=1}^r (m_i)\xi_i)} \\
&\quad \times \frac{(\omega_1)_{l_1}\cdots(\omega_r)_{l_r}(z_1x^{\xi_1})^{m_1}\cdots(z_rx^{\xi_r})^{m_r}}{(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}}.
\end{aligned} \tag{21}$$

Now applying the formula $(\alpha+\beta)_m = \sum_{n=0}^{\infty}\binom{m}{n}(\alpha)_n(\beta)_{m-n}$, Eq. (21) is reduced to the following form:

$$\begin{aligned}
&= x^{\tau+\nu-1} \sum_{m_1,\dots,m_r=0}^{\infty} \sum_{l_1,\dots,l_r=0}^{\infty} \frac{(\omega_1+\delta_1)_{m_1}\cdots(\omega_r+\delta_r)_{m_r}(z_1x^{\xi_1})^{m_1}\cdots(z_rx^{\xi_r})^{m_r}}{\Gamma(\tau+\nu+\sum_{i=1}^r (m_i)\xi_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&= x^{\tau+\nu-1} E_{\xi_i;\nu+\tau;\varepsilon_i}^{(\omega_i+\delta_i);\kappa_i;1}\bigl(z_1x^{\xi_1},\dots,z_rx^{\xi_r}\bigr).
\end{aligned}$$

□

Theorem 4 *The following integral equality holds*:

$$\int_0^z t^{\nu-1} E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\bigl(z_1t^{\xi_1},\dots,z_rt^{\xi_r}\bigr)\,dt = z^{\nu} E_{\xi_i;\nu+1;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\bigl(z_1z^{\xi_1},\dots,z_rz^{\xi_r}\bigr). \tag{22}$$

Proof Applying Eq. (6) to the left-hand side of Eq. (22), we obtain

$$\begin{aligned}
&= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r}}{\Gamma(\nu+\sum_{i=1}^{r}\xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \int_0^z t^{\nu+\sum_{i=1}^{r} m_i\xi_i-1}\,dt \\
&= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r} z^{\nu+\sum_{i=1}^{r} m_i\xi_i}}{(\nu+\sum_{i=1}^{r} m_i\xi_i)\Gamma(\nu+\sum_{i=1}^{r}\xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&= z^{\nu} \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1 z^{\xi_1})^{m_1}\cdots(z_r z^{\xi_r})^{m_r}}{\Gamma(\nu+1+\sum_{i=1}^{r}\xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\
&= z^{\nu} E_{\xi_i;\nu+1;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\bigl(z_1 z^{\xi_1},\ldots,z_r z^{\xi_r}\bigr).
\end{aligned}$$

This completes the proof of Theorem 4. □

Remark 2.1 Upon setting $\kappa_1 = \cdots = \kappa_r = \varepsilon_1 = \cdots = \varepsilon_r = 1$ and $r = 1$, Eqs. (17), (18), (19), and (22) reduce to a result given by Shukla and Prajapati [31, Eqs. (2.4.1), (2.4.2), (2.4.3), (2.4.4)].

Remark 2.2 By setting the parameters in Eqs. (17), (18), (19), and (22), we obtain the known results established by Khan [32, Eqs. (2.4.4), (2.4.5), (2.4.6), (2.4.7)].

Lemma 2.1 ([33, Eq. 2.13]) *If $\mu_1,\mu_2,\alpha',\beta' \in \mathbb{C}^+$, $\tau_1,\tau_2 \in \mathbb{C}^+_{-1}$ and $n,m \in \mathbb{N}$, then the following formula holds*:

$$\begin{aligned}
&L_n^{(\mu_1,\tau_1)}\bigl(\beta',x\bigr)L_m^{\mu_1,\tau_1}\bigl(\alpha',x\bigr) \\
&\quad= \sum_{h=0}^{n+m}\sum_{r=0}^{h} \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(h-r+1)\Gamma(\alpha'(m-h+r)+1)} \\
&\qquad\times \frac{(-x)^h}{\Gamma(r+1)\Gamma(\beta'(n-r)+1)\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)}.
\end{aligned} \tag{23}$$

Theorem 5 *If $\alpha',\beta',\eta \in \mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2 \in \mathbb{C}^+_{-1}$ and $n,m \in \mathbb{N}$, then the following integral equality holds*:

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\nu-1}(1-u)^{\alpha-1} L_n^{(\mu_1,\tau_1)}\bigl(\beta',\eta(1-u)\bigr)L_m^{(\mu_2,\tau_2)}\bigl(\alpha',\eta(1-u)\bigr) \\
&\qquad\times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\bigl(z_1 u^{\xi_1},\ldots,z_r u^{\xi_r}\bigr)\,du \\
&\quad= \sum_{h=0}^{n+m}(\eta)^h \Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1,\ldots,z_r),
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} &= \sum_{r=0}^{h}\binom{h}{r}\frac{(-1)^h(\alpha)_h}{\Gamma(h+1)\Gamma(\alpha'(m-h+r)+1)\Gamma(\beta'(n-r)+1)} \\
&\quad\times \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)}.
\end{aligned} \tag{25}$$

Proof Using Eqs. (6) and (23) in the left-hand side of Eq. (24), we obtain

$$\begin{aligned} I &= \frac{1}{\Gamma(\alpha)} \int_0^1 u^{\nu-1}(1-u)^{\alpha-1} \sum_{h=0}^{n+m} \sum_{r=0}^{h} \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(h-r+1)\Gamma(\alpha'(m-h+r)+1)} \\ &\quad \times \frac{(-\eta(1-u))^h}{\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)\Gamma(r+1)\Gamma(\beta'(n-r)+1)} \\ &\quad \times \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1 u^{\xi_1})^{m_1}\cdots(z_r u^{\xi_r})^{m_r}}{\Gamma(\nu+\sum_{i=1}^{r}\xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}}\,du \\ &= \frac{1}{\Gamma(\alpha)} \sum_{h=0}^{n+m} \sum_{r=0}^{h} \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(h-r+1)\Gamma(\alpha'(m-h+r)+1)\Gamma(r+1)\Gamma(\beta'(n-r)+1)} \\ &\quad \times \frac{(-\eta)^h}{\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)} \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}}{\Gamma(\nu+\sum_{i=1}^{r}\xi_i m_i)} \\ &\quad \times \frac{(z_1)^{m_1}\cdots(z_r)^{m_r}}{(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \int_0^1 u^{\nu+\sum_{i=1}^{r}\xi_i m_i-1}(1-u)^{\alpha+h-1}\,du \\ &= \frac{1}{\Gamma(\alpha)} \sum_{h=0}^{n+m} \sum_{r=0}^{h} \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(h-r+1)\Gamma(\alpha(m-h+r)+1)\Gamma(r+1)\Gamma(\beta'(n-r)+1)} \\ &\quad \times \frac{(-\eta)^h\Gamma(\alpha+h)}{\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)} \\ &\quad \times \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\delta_1)_{\rho_1 m_1}\cdots(\delta_r)_{\rho_r m_r}(z_1)^{m_1}\cdots(z_r)^{m_r}}{\Gamma(\nu+\alpha+h+\sum_{i=1}^{r}\xi_i m_i)(\kappa_1)_{\varepsilon_1 m_1}\cdots(\kappa_r)_{\varepsilon_r m_r}} \\ &= \frac{1}{\Gamma(\alpha)} \sum_{h=0}^{n+m} \sum_{r=0}^{h} \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(h-r+1)\Gamma(\alpha'(m-h+r)+1)\Gamma(r+1)\Gamma(\beta'(n-r)+1)} \\ &\quad \times \frac{(-\eta)^h\Gamma(\alpha+h)}{\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)} E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1,\ldots,z_r). \end{aligned} \tag{26}$$

Using the fact that $\binom{x}{n} = \frac{(-1)^n}{n!}(-x)_n$, where $(-x)_n = (-1)^n(x-n+1)_n$, on the right-hand side of Eq. (26), yields

$$\begin{aligned} &= \sum_{h=0}^{n+m} \sum_{r=0}^{h} \binom{h}{r} \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(h+1)\Gamma(\alpha'(m-h+r)+1)\Gamma(\beta'(n-r)+1)} \\ &\quad \times \frac{(-\eta)^h(\alpha)_h}{\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)} E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1,\ldots,z_r), \end{aligned}$$

from which, after little rearrangement, we easily arrive at the required Eq. (24). □

Theorem 6 *If $\alpha',\beta',\eta \in \mathbb{C}^+$; $\mu_1,\mu_2,\tau_1,\tau_2 \in \mathbb{C}^+_{-1}$ and $n,m \in \mathbb{N}$, then the following integral equality holds*:

$$\frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1} L_n^{(\mu_1,\tau_1)}\big(\beta',\eta(x-s)\big) L_m^{(\mu_2,\tau_2)}\big(\alpha',\eta(x-s)\big)$$

$$\times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}\big)\,ds = (x-t)^{\alpha+\nu-1}\sum_{h=0}^{n+m}(\eta)^h$$

$$\times (x-t)^h \Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\big), \tag{27}$$

where $\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (25).

Theorem 7 *If* $\alpha',\beta',\eta\in\mathbb{C}^+$; $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}_{-1}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$; $n,m\in\mathbb{N}$, *then the following integral equality holds*:

$$\int_0^x t^{\tau-1}(x-t)^{\nu-1}L_n^{(\mu_1,\tau_1)}(\beta',\eta t)L_m^{\mu_1,\tau_1}(\alpha',\eta t)$$

$$\times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;1}\big(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\big)E_{\xi_i;\tau;\varepsilon_i}^{\omega_i;\kappa_i;1}\big(z_1(t)^{\xi_1},\dots,z_r(t)^{\xi_r}\big)\,dt$$

$$= x^{\tau+\nu-1}\sum_{h=0}^{n+m}(\eta x)^h \Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} E_{\xi_i;\nu+\tau+h;\varepsilon_i}^{(\omega_i+\delta_i);\kappa_i;1}\big(z_1x^{\xi_1},\dots,z_rx^{\xi_r}\big), \tag{28}$$

where $\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (25).

Theorem 8 *If* $\alpha',\beta',\eta\in\mathbb{C}^+$; $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}_{-1}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$; $n,m\in\mathbb{N}$, *then the following integral equality holds*:

$$\int_0^z t^{\nu-1}L_n^{(\mu_1,\tau_1)}(\beta',\eta t)L_m^{\mu_1,\tau_1}(\alpha',\eta t)E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1t^{\xi_1},\dots,z_rt^{\xi_r}\big)\,dt$$

$$= z^{\nu-1}\sum_{h=0}^{n+m}(\eta z)^h \Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} E_{\xi_i;\nu+h+1;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1z^{\xi_1},\dots,z_rz^{\xi_r}\big), \tag{29}$$

where $\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (25).

Proof The proofs of Eqs. (27), (28), and (29) are the same as those of Eq. (25), which can be obtained from Eqs. (18), (19), and (22). □

Theorem 9 *If* $\alpha',\beta',\eta\in\mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}_{-1}^+$ *and* $n,m\in\mathbb{N}$, *then the following integral equality holds*:

$$\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\nu-1}(1-u)^{\alpha-1}L_n^{(\mu_1,\tau_1)}\big(\beta',\eta(1-u)\big)L_m^{(\mu_2,\tau_2)}\big(\alpha',\eta(1-u)\big)$$

$$\times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1u^{\xi_1},\dots,z_ru^{\xi_r}\big)\,du$$

$$= \sum_{h=0}^{n+m}(\eta)^h \wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1,\dots,z_r), \tag{30}$$

where

$$\wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} = \frac{\Gamma(\mu_1n+\tau_1+1)\Gamma(\mu_2m+\tau_2+1)}{\Gamma(\alpha'm+1)\Gamma(\beta'n+1)}$$

$$\times\sum_{r=0}^{h}\binom{h}{r}\frac{(-1)^{h-\alpha'(h-r)-\beta'r}(\alpha)_h(-\alpha'm)_{\alpha'(h-r)}(-\beta'n)_{\beta'r}}{\Gamma(h+1)\Gamma(\mu_1r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)}. \tag{31}$$

Proof Using $(-x)_n = (-1)^n(x-n+1)_n$, Eq. (26) is reduced to

$$
\begin{aligned}
&= \frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(\alpha' m+1)\Gamma(\beta' n+1)}\sum_{h=0}^{n+m}(\eta)^h\sum_{r=0}^{h}\binom{h}{r} \\
&\quad\times \frac{(-1)^{h-\alpha'(h-r)-\beta' r}(\alpha)_h(-\alpha' m)_{\alpha'(h-r)}(-\beta' n)_{\beta' r}}{\Gamma(h+1)\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)}E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+\alpha;\varepsilon_i}(z_1,\dots,z_r) \\
&= \sum_{h=0}^{n+m}(\eta)^h\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+\alpha;\varepsilon_i}(z_1,\dots,z_r),
\end{aligned}
$$

where $\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}$ is given by Eq. (31). □

Theorem 10 *If $\alpha',\beta',\eta\in\mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}^+_{-1}$ and $n,m\in\mathbb{N}$, then the following integral equality holds*:

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1}L_n^{(\mu_1,\tau_1)}\bigl(\beta',\eta(x-s)\bigr)L_m^{(\mu_2,\tau_2)}\bigl(\alpha',\eta(x-s)\bigr) \\
&\quad\times E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu;\varepsilon_i}\bigl(z_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}\bigr)\,ds = (x-t)^{\alpha+\nu-1}\sum_{h=0}^{n+m}(\eta)^h \\
&\quad\times (x-t)^h\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+\alpha;\varepsilon_i}\bigl(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\bigr),
\end{aligned}
\tag{32}
$$

where $\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}$ is given by Eq. (31).

Theorem 11 *If $\alpha',\beta',\eta\in\mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}^+_{-1}$, $\tau_1,\tau_2\in\mathbb{C}^+_{-1}$; $n,m\in\mathbb{N}$, then the following integral equality holds*:

$$
\begin{aligned}
&\int_0^x t^{\tau-1}(x-t)^{\nu-1}L_n^{(\mu_1,\tau_1)}\bigl(\beta',\eta t\bigr)L_m^{\mu_1,\tau_1}\bigl(\alpha',\eta t\bigr) \\
&\quad\times E^{\delta_i;\kappa_i;1}_{\xi_i;\nu;\varepsilon_i}\bigl(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\bigr)E^{\omega_i;\kappa_i;1}_{\xi_i;\tau;\varepsilon_i}\bigl(z_1(t)^{\xi_1},\dots,z_r(t)^{\xi_r}\bigr)\,dt \\
&\quad= x^{\tau+\nu-1}\sum_{h=0}^{n+m}(\eta x)^h\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}E^{(\omega_i+\delta_i);\kappa_i;1}_{\xi_i;\nu+\tau+h;\varepsilon_i}\bigl(z_1x^{\xi_1},\dots,z_rx^{\xi_r}\bigr),
\end{aligned}
\tag{33}
$$

where $\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}$ is given by Eq. (31).

Theorem 12 *If $\alpha',\beta',\eta\in\mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}^+_{-1}$, $\tau_1,\tau_2\in\mathbb{C}^+_{-1}$; $n,m\in\mathbb{N}$, then the following integral equality holds*:

$$
\begin{aligned}
&\int_0^z t^{\nu-1}L_n^{(\mu_1,\tau_1)}\bigl(\beta',\eta t\bigr)L_m^{\mu_1,\tau_1}\bigl(\alpha',\eta t\bigr)E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu;\varepsilon_i}\bigl(z_1t^{\xi_1},\dots,z_rt^{\xi_r}\bigr)\,dt \\
&\quad= z^{\nu-1}\sum_{h=0}^{n+m}(\eta z)^h\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+1;\varepsilon_i}\bigl(z_1z^{\xi_1},\dots,z_rz^{\xi_r}\bigr),
\end{aligned}
\tag{34}
$$

where $\wp^{m,n,\alpha',\beta'}_{\mu_1,\tau_1,\mu_2,\tau_2}$ is given by Eq. (31).

Proof The proofs of Eqs. (32), (33), and (34) are the same as that of Eq. (30). □

Remark 2.3 Upon setting $\kappa_1 = \cdots = \kappa_r = \varepsilon_1 = \cdots \varepsilon_r = 1$, Eqs. (17), (18), (19), (22), (24), (27), (28), (29), (30), (32), (33), and (34) are reduced to Eqs. (2.1), (2.3), (2.5), (2.10), (2.20), (2.27), (2.28), (2.29), (2.31), (2.35), (2.36), and (2.37) established by Agarwal et al. [33].

3 Special cases

In this section, we emphasize special cases by selecting particular values of parameters.

(i) Putting $\alpha' = \beta' = 1$, the results in Eqs. (30), (32), (33), and (34) are reduced to the following form:

Corollary 1 *If* $\eta \in \mathbb{C}^+, \mu_1, \mu_2, \tau_1, \tau_2 \in \mathbb{C}^+_{-1}$ *and* $n, m \in \mathbb{N}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^1 u^{\nu-1}(1-u)^{\alpha-1} L_n^{(\mu_1,\tau_1)}\big(\eta(1-u)\big) L_m^{(\mu_2,\tau_2)}\big(\eta(1-u)\big) \\
&\quad \times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1 u^{\xi_1}, \ldots, z_r u^{\xi_r}\big)\, du \\
&\quad = \sum_{h=0}^{n+m} (\eta)^h \frac{\Gamma(\mu_1 n + \tau_1 + 1)\Gamma(\mu_2 m + \tau_2 + 1)}{\Gamma(m+1)\Gamma(n+1)} \sum_{r=0}^{h} \binom{h}{r} \\
&\quad \times \frac{(\alpha)_h (-m)_{(h-r)} (-n)_r}{\Gamma(h+1)\Gamma(\mu_1 r + \tau_1 + 1)\Gamma(\mu_2(h-r) + \tau_2 + 1)} E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1, \ldots, z_r).
\end{aligned} \tag{35}$$

Corollary 2 *If* $\eta \in \mathbb{C}^+, \mu_1, \mu_2, \tau_1, \tau_2 \in \mathbb{C}^+_{-1}$ *and* $n, m \in \mathbb{N}$, *then*

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1} L_n^{(\mu_1,\tau_1)}\big(\eta(x-s)\big) L_m^{(\mu_2,\tau_2)}\big(\eta(x-s)\big) \\
&\quad \times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1(s-t)^{\xi_1}, \ldots, z_r(s-t)^{\xi_r}\big)\, ds \\
&\quad = (x-t)^{\alpha+\nu-1} \sum_{h=0}^{n+m} (\eta)^h (x-t)^h \frac{\Gamma(\mu_1 n + \tau_1 + 1)\Gamma(\mu_2 m + \tau_2 + 1)}{\Gamma(m+1)\Gamma(n+1)} \\
&\quad \times \sum_{r=0}^{h} \binom{h}{r} \frac{(\alpha)_h (-m)_{(h-r)} (-n)_r}{\Gamma(h+1)\Gamma(\mu_1 r + \tau_1 + 1)\Gamma(\mu_2(h-r) + \tau_2 + 1)} \\
&\quad \times E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1(x-t)^{\xi_1}, \ldots, z_r(x-t)^{\xi_r}\big).
\end{aligned} \tag{36}$$

Corollary 3 *If* $\eta \in \mathbb{C}^+, \mu_1, \mu_2, \tau_1, \tau_2 \in \mathbb{C}^+_{-1},; n, m \in \mathbb{N}$, *then*

$$\begin{aligned}
&\int_0^x t^{\tau-1}(x-t)^{\nu-1} L_n^{(\mu_1,\tau_1)}(\eta t) L_m^{\mu_1,\tau_1}(\eta t) \\
&\quad \times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;1}\big(z_1(x-t)^{\xi_1}, \ldots, z_r(x-t)^{\xi_r}\big) E_{\xi_i;\tau;\varepsilon_i}^{\omega_i;\kappa_i;1}\big(z_1(t)^{\xi_1}, \ldots, z_r(t)^{\xi_r}\big)\, dt \\
&\quad = x^{\tau+\nu-1} \sum_{h=0}^{n+m} (\eta x)^h \frac{\Gamma(\mu_1 n + \tau_1 + 1)\Gamma(\mu_2 m + \tau_2 + 1)}{\Gamma(m+1)\Gamma(n+1)} \sum_{r=0}^{h} \binom{h}{r} \\
&\quad \times \frac{(\alpha)_h (-m)_{(h-r)} (-n)_r}{\Gamma(h+1)\Gamma(\mu_1 r + \tau_1 + 1)\Gamma(\mu_2(h-r) + \tau_2 + 1)} E_{\xi_i;\nu+\tau+h;\varepsilon_i}^{(\omega_i+\delta_i);\kappa_i;1}\big(z_1 x^{\xi_1}, \ldots, z_r x^{\xi_r}\big).
\end{aligned} \tag{37}$$

Corollary 4 *If* $\eta \in \mathbb{C}^+, \mu_1, \mu_2, \tau_1, \tau_2 \in \mathbb{C}^+_{-1}, n, m \in \mathbb{N}$, *then*

$$\int_0^z t^{\nu-1} L_n^{(\mu_1,\tau_1)}(\eta t) L_m^{\mu_1,\tau_1}(\eta t) E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1 t^{\xi_1}, \ldots, z_r t^{\xi_r}\big)\, dt$$

$$= z^{\nu-1}\sum_{h=0}^{n+m}(\eta z)^h\frac{\Gamma(\mu_1 n+\tau_1+1)\Gamma(\mu_2 m+\tau_2+1)}{\Gamma(m+1)\Gamma(n+1)}\sum_{r=0}^{h}\binom{h}{r}$$
$$\times\frac{(\alpha)_h(-m)_{(h-r)}(-n)_r}{\Gamma(h+1)\Gamma(\mu_1 r+\tau_1+1)\Gamma(\mu_2(h-r)+\tau_2+1)}E_{\xi_i;\nu+h+1;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1 z^{\xi_1},\dots,z_r z^{\xi_r}\big). \quad (38)$$

(ii) Putting $\mu_1=\mu_2=\alpha'=\beta'=1$ and using Eq. (12), the results in Eqs. (30), (32), (33), and (34) reduced to the following form:

Corollary 5 *If $\eta\in\mathbb{C}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$ and $n,m\in\mathbb{N}$, then*

$$\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\nu-1}(1-u)^{\alpha-1}L_n^{(1,\tau_1)}\big(\eta(1-u);1\big)L_m^{(1,\tau_2)}\big(\eta(1-u);1\big)$$
$$\times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1 u^{\xi_1},\dots,z_r u^{\xi_r}\big)\,du$$
$$=\sum_{h=0}^{n+m}(\eta)^h\frac{\Gamma(n+\tau_1+1)\Gamma(m+\tau_2+1)}{\Gamma(m+1)\Gamma(n+1)}$$
$$\times\sum_{r=0}^{h}\binom{h}{r}\frac{(\alpha)_h(-m)_{(h-r)}(-n)_r}{\Gamma(h+1)\Gamma(r+\tau_1+1)\Gamma((h-r)+\tau_2+1)}E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}(z_1,\dots,z_r). \quad (39)$$

Corollary 6 *If $\eta\in\mathbb{C}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$ and $n,m\in\mathbb{N}$, then*

$$\frac{1}{\Gamma(\alpha)}\int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1}L_n^{(1,\tau_1)}\big(\eta(x-s);1\big)L_m^{(1,\tau_2)}\big(\eta(x-s);1\big)$$
$$\times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}\big)\,ds$$
$$=(x-t)^{\alpha+\nu-1}\sum_{h=0}^{n+m}(\eta)^h(x-t)^h\frac{\Gamma(n+\tau_1+1)\Gamma(m+\tau_2+1)}{\Gamma(m+1)\Gamma(n+1)}$$
$$\times\sum_{r=0}^{h}\binom{h}{r}\frac{(\alpha)_h(-m)_{(h-r)}(-n)_r}{\Gamma(h+1)\Gamma(r+\tau_1+1)\Gamma((h-r)+\tau_2+1)}$$
$$\times E_{\xi_i;\nu+h+\alpha;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\big). \quad (40)$$

Corollary 7 *If $\eta\in\mathbb{C}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$, $n,m\in\mathbb{N}$, then*

$$\int_0^x t^{\tau-1}(x-t)^{\nu-1}L_n^{(\mu_1,\tau_1)}(\eta t;1)L_m^{\mu_1,\tau_1}(\eta t;1)$$
$$\times E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;1}\big(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\big)E_{\xi_i;\tau;\varepsilon_i}^{\omega_i;\kappa_i;1}\big(z_1(t)^{\xi_1},\dots,z_r(t)^{\xi_r}\big)\,dt$$
$$=x^{\tau+\nu-1}\sum_{h=0}^{n+m}(\eta x)^h\frac{\Gamma(n+\tau_1+1)\Gamma(m+\tau_2+1)}{\Gamma(m+1)\Gamma(n+1)}\sum_{r=0}^{h}\binom{h}{r}$$
$$\times\frac{(\alpha)_h(-m)_{(h-r)}(-n)_r}{\Gamma(h+1)\Gamma(r+\tau_1+1)\Gamma((h-r)+\tau_2+1)}E_{\xi_i;\nu+\tau+h;\varepsilon_i}^{(\omega_i+\delta_i);\kappa_i;1}\big(z_1 x^{\xi_1},\dots,z_r x^{\xi_r}\big). \quad (41)$$

Corollary 8 *If $\eta\in\mathbb{C}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$, $n,m\in\mathbb{N}$, then*

$$\int_0^z t^{\nu-1}L_n^{(\mu_1,\tau_1)}(\eta t;1)L_m^{\mu_1,\tau_1}(\eta t;1)E_{\xi_i;\nu;\varepsilon_i}^{\delta_i;\kappa_i;\rho_i}\big(z_1 t^{\xi_1},\dots,z_r t^{\xi_r}\big)\,dt$$

$$= z^{\nu-1} \sum_{h=0}^{n+m} (\eta z)^h \frac{\Gamma(n+\tau_1+1)\Gamma(m+\tau_2+1)}{\Gamma(m+1)\Gamma(n+1)} \sum_{r=0}^{h} \binom{h}{r} \\ \times \frac{(\alpha)_h(-m)_{(h-r)}(-n)_r}{\Gamma(h+1)\Gamma(r+\tau_1+1)\Gamma((h-r)+\tau_2+1)} E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+1;\varepsilon_i}\big(z_1 z^{\xi_1},\dots,z_r z^{\xi_r}\big). \tag{42}$$

(iii) Putting $\mu_1 = \mu_1 = 0, \alpha' = \beta' = 1$, the results in Eqs. (30), (32), (33), and (34) are reduced to the following form:

Corollary 9 *If $\eta \in \mathbb{C}^+, \tau_1, \tau_2 \in \mathbb{C}^+_{-1}$ and $n, m \in \mathbb{N}$, then*

$$\frac{1}{\Gamma(\alpha)} \int_0^1 u^{\nu-1}(1-u)^{\alpha-1}\big(1-\eta(1-u)\big)^m E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu;\varepsilon_i}\big(z_1 u^{\xi_1},\dots,z_r u^{\xi_r}\big)\,du \\ = \sum_{h=0}^{m} (\eta)^h(-m)_h(\alpha)_h E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+\alpha;\varepsilon_i}(z_1,\dots,z_r). \tag{43}$$

Corollary 10 *If $\eta \in \mathbb{C}^+, \tau_1, \tau_2 \in \mathbb{C}^+_{-1}$ and $n, m \in \mathbb{N}$, then*

$$\frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1}\big(1-\eta(x-s)\big)^m \\ \times E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu;\varepsilon_i}\big(z_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}\big)\,ds = (x-t)^{\alpha+\nu-1} \sum_{h=0}^{m} \big(\eta(x-t)\big)^h \\ \times (-m)_h(\alpha)_h E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+\alpha;\varepsilon_i}\big(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\big). \tag{44}$$

Corollary 11 *If $\eta \in \mathbb{C}^+, \tau_1, \tau_2 \in \mathbb{C}^+_{-1}$ and $n, m \in \mathbb{N}$, then*

$$\int_0^x t^{\tau-1}(x-t)^{\nu-1}(1-\eta t)^m E^{\delta_i;\kappa_i;1}_{\xi_i;\nu;\varepsilon_i}\big(z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\big) \\ \times E^{\omega_i;\kappa_i;1}_{\xi_i;\tau;\varepsilon_i}\big(z_1(t)^{\xi_1},\dots,z_r(t)^{\xi_r}\big)\,dt \\ = x^{\tau+\nu-1} \sum_{h=0}^{m} (\eta x)^h(-m)_h(\alpha)_h E^{(\omega_i+\delta_i);\kappa_i;1}_{\xi_i;\nu+\tau+h;\varepsilon_i}\big(z_1 x^{\xi_1},\dots,z_r x^{\xi_r}\big). \tag{45}$$

Corollary 12 *If $\eta \in \mathbb{C}^+, \tau_1, \tau_2 \in \mathbb{C}^+_{-1}$ and $n, m \in \mathbb{N}$, then*

$$\int_0^z t^{\nu-1}(1-\eta t)^m E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu;\varepsilon_i}\big(z_1 t^{\xi_1},\dots,z_r t^{\xi_r}\big)\,dt \\ = z^{\nu-1} \sum_{h=0}^{m} (\eta z)^h(-m)_h(\alpha)_h E^{\delta_i;\kappa_i;\rho_i}_{\xi_i;\nu+h+1;\varepsilon_i}\big(z_1 z^{\xi_1},\dots,z_r z^{\xi_r}\big). \tag{46}$$

Remark 3.1 If we put $\kappa_1 = \cdots = \kappa_r = \varepsilon_1 = \cdots = \varepsilon_r = 1$, then Eqs. (35)–(46) are reduced to Eqs. (3.1), (3.2), (3.3), (3.4), (3.6), (3.7), (3.8), (3.9), (3.11), (3.16), (3.17), (3.18) established in Agarwal et al. [33].

(iv) Setting $\rho_i = \kappa_i = \varepsilon_i = 1, \xi_1 = \cdots = \xi_r = 1$, the multivariate Mittag-Leffler function reduces to the confluent hypergeometric series, and we have the following results, which are obtained from the main results in (17), (18), (19), (22), (24), (27), (28), (29), (30), (32), (33), and (34).

Corollary 13 *The following integral equality holds*:

$$\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\nu-1}(1-u)^{\alpha-1}\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu;z_1u^{\xi_1},\dots,z_ru^{\xi_r}\bigr]\,du$$
$$=\varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu+\alpha;z_1,\dots,z_r]. \tag{47}$$

Corollary 14 *Then following integral equality holds*:

$$\frac{1}{\Gamma(\alpha)}\int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1}$$
$$\times\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu;z_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}\bigr]\,ds$$
$$=(x-t)^{\alpha+\nu-1}\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu+\alpha;z_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}\bigr]. \tag{48}$$

Corollary 15 *Then following integral equality holds*:

$$\int_0^x t^{\tau-1}(x-t)^{\nu-1}\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu;z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\bigr]$$
$$\times\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu;z_1t^{\xi_1},\dots,z_rt^{\xi_r}\bigr]\,dt$$
$$=x^{\tau+\nu-1}\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu+\tau;z_1t^{\xi_1},\dots,z_rt^{\xi_r}\bigr]. \tag{49}$$

Corollary 16 *Then following integral equality holds*:

$$\int_0^z t^{\nu-1}\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu;z_1t^{\xi_1},\dots,z_rt^{\xi_r}\bigr]\,dt$$
$$=z^{\nu-1}\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu+1;z_1z^{\xi_1},\dots,z_rz^{\xi_r}\bigr]. \tag{50}$$

Corollary 17 *If* $\alpha',\beta',\eta\in\mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}_{-1}^+$ *and* $n,m\in\mathbb{N}$, *then the following integral equality holds*:

$$\frac{1}{\Gamma(\alpha)}\int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1}L_n^{(\mu_1,\tau_1)}\bigl(\beta',\eta(x-s)\bigr)L_m^{(\mu_2,\tau_2)}\bigl(\alpha',\eta(x-s)\bigr)$$
$$\times\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu;_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}\bigr]\,ds$$
$$=(x-t)^{\alpha+\nu-1}\sum_{h=0}^{n+m}\bigl(\eta(x-t)\bigr)^h\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$$
$$\times\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu+h+\alpha;z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}\bigr], \tag{51}$$

where $\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (25).

Corollary 18 *If* $\alpha',\beta',\eta\in\mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}_{-1}^+$ *and* $n,m\in\mathbb{N}$, *then the following integral equality holds*:

$$\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\nu-1}(1-u)^{\alpha-1}L_n^{(\mu_1,\tau_1)}\bigl(\beta',\eta(1-u)\bigr)L_m^{(\mu_2,\tau_2)}\bigl(\alpha',\eta(1-u)\bigr)$$
$$\times\varphi_2^{(r)}\bigl[\delta_1,\dots,\delta_r;\nu;z_1u^{\xi_1},\dots,z_ru^{\xi_r}\bigr]\,du$$

$$= \sum_{h=0}^{n+m} (\eta)^h \Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu+\alpha;z_1,\dots,z_r], \tag{52}$$

where $\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (25).

Corollary 19 *If* $\alpha',\beta',\eta \in \mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2 \in \mathbb{C}_{-1}^+$, $\tau_1,\tau_2 \in \mathbb{C}_{-1}^+$; $n,m \in \mathbb{N}$, *then the following integral equality holds*:

$$\begin{aligned} &\int_0^x t^{\tau-1}(x-t)^{\nu-1} L_n^{(\mu_1,\tau_1)}(\beta',\eta t) L_m^{\mu_1,\tau_1}(\alpha',\eta t) \\ &\quad\times \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu;z_1(x-t)^{\xi_1},\dots,z_r(x-t)^{\xi_r}] \\ &\quad\times \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu;z_1(t)^{\xi_1},\dots,z_r(t)^{\xi_r}]\,dt \\ &= x^{\tau+\nu-1} \sum_{h=0}^{n+m} (\eta x)^h \Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu+\tau+h;z_1 x^{\xi_1},\dots,z_r x^{\xi_r}], \end{aligned} \tag{53}$$

where $\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (25).

Corollary 20 *If* $\alpha',\beta',\eta \in \mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2 \in \mathbb{C}_{-1}^+$, $\tau_1,\tau_2 \in \mathbb{C}_{-1}^+$; $n,m \in \mathbb{N}$, *then the following integral equality holds*:

$$\begin{aligned} &\int_0^z t^{\nu-1} L_n^{(\mu_1,\tau_1)}(\beta',\eta t) L_m^{\mu_1,\tau_1}(\alpha',\eta t) \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu;z_1 t^{\xi_1},\dots,z_r t^{\xi_r}]\,dt \\ &= z^{\nu-1} \sum_{h=0}^{n+m} (\eta z)^h \Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu+h+1;z_1 t^{\xi_1},\dots,z_r t^{\xi_r}], \end{aligned} \tag{54}$$

where $\Im_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (25).

Corollary 21 *If* $\alpha',\beta',\eta \in \mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2 \in \mathbb{C}_{-1}^+$ *and* $n,m \in \mathbb{N}$, *then the following integral equality holds*:

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^1 u^{\nu-1}(1-u)^{\alpha-1} L_n^{(\mu_1,\tau_1)}(\beta',\eta(1-u)) L_m^{(\mu_2,\tau_2)}(\alpha',\eta(1-u)) \\ &\quad\times \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu;z_1 u^{\xi_1},\dots,z_r u^{\xi_r}]\,du \\ &= \sum_{h=0}^{n+m} (\eta)^h \wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu+h+\alpha;z_1,\dots,z_r], \end{aligned} \tag{55}$$

where $\wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (31).

Corollary 22 *If* $\alpha',\beta',\eta \in \mathbb{C}^+$, $\mu_1,\mu_2,\tau_1,\tau_2 \in \mathbb{C}_{-1}^+$ *and* $n,m \in \mathbb{N}$, *then the following integral equality holds*:

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1} L_n^{(\mu_1,\tau_1)}(\beta',\eta(x-s)) L_m^{(\mu_2,\tau_2)}(\alpha',\eta(x-s)) \\ &\quad\times \varphi_2^{(r)}[\delta_1,\dots,\delta_r;\nu;z_1(s-t)^{\xi_1},\dots,z_r(s-t)^{\xi_r}]\,ds \end{aligned}$$

$$= (x-t)^{\alpha+\nu-1}\sum_{h=0}^{n+m}\bigl(\eta(x-t)\bigr)^h \wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'} \times \varphi_2^{(r)}\bigl[\delta_1,\ldots,\delta_r;\nu+h+\alpha;z_1(x-t)^{\xi_1},\ldots,z_r(x-t)^{\xi_r}\bigr], \tag{56}$$

where $\wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (31).

Corollary 23 *If* $\alpha',\beta',\eta\in\mathbb{C}^+,\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}_{-1}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$; $n,m\in\mathbb{N}$, *then the following integral equality holds*:

$$\int_0^x t^{\tau-1}(x-t)^{\nu-1}L_n^{(\mu_1,\tau_1)}(\beta',\eta t)L_m^{\mu_1,\tau_1}(\alpha',\eta t) \times \varphi_2^{(r)}\bigl[\delta_1,\ldots,\delta_r;\nu;z_1(x-t)^{\xi_1},\ldots,z_r(x-t)^{\xi_r}\bigr] \times \varphi_2^{(r)}\bigl[\delta_1,\ldots,\delta_r;\nu;z_1(t)^{\xi_1},\ldots,z_r(t)^{\xi_r}\bigr]\,dt = x^{\tau+\nu-1}\sum_{h=0}^{n+m}(\eta x)^h \wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}\varphi_2^{(r)}\bigl[\delta_1,\ldots,\delta_r;\nu+\tau+h;z_1x^{\xi_1},\ldots,z_rx^{\xi_r}\bigr], \tag{57}$$

where $\wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (31).

Corollary 24 *If* $\alpha',\beta',\eta\in\mathbb{C}^+,\mu_1,\mu_2,\tau_1,\tau_2\in\mathbb{C}_{-1}^+$, $\tau_1,\tau_2\in\mathbb{C}_{-1}^+$; $n,m\in\mathbb{N}$, *then the following integral equality holds*:

$$\int_0^z t^{\nu-1}L_n^{(\mu_1,\tau_1)}(\beta',\eta t)L_m^{\mu_1,\tau_1}(\alpha',\eta t)\varphi_2^{(r)}\bigl[\delta_1,\ldots,\delta_r;\nu;z_1t^{\xi_1},\ldots,z_rt^{\xi_r}\bigr]\,dt = z^{\nu-1}\sum_{h=0}^{n+m}(\eta z)^h \wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}\varphi_2^{(r)}\bigl[\delta_1,\ldots,\delta_r;\nu+h+1;z_1z^{\xi_1},\ldots,z_rz^{\xi_r}\bigr], \tag{58}$$

where $\wp_{\mu_1,\tau_1,\mu_2,\tau_2}^{m,n,\alpha',\beta'}$ *is given by Eq.* (31).

(v) Setting $\rho_i=\kappa_i=\varepsilon_i=\delta_i=\xi_i=\nu=r=1$, the multivariate Mittag-Leffler function is reduced to the exponential function $E_{1;1;1}^{1;1;1}(z)=E_{1,1}(z)=\exp(z)$; and we can find a similar line of results associated with exponential function from Eqs. (47)–(58).

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