Sun and Zhao *Journal of Inequalities and Applications* (2019) 2019:207
https://doi.org/10.1186/s13660-019-2161-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Oscillatory behavior of third-order neutral delay differential equations with distributed deviating arguments

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Abstract

The main contribution of this paper is to establish some new criteria, which ensure that every solution of third-order neutral delay differential equations with distributed deviating arguments is either oscillatory or tends to zero. The obtained theorems extend and improve several known results in the literature. Two examples are provided to illustrate the main results.

MSC: 34K11

Keywords: Oscillation; Asymptotic property; Distributed deviating arguments; Riccati transformation

1 Introduction

Over the last several years, an increasing attention has been given to the oscillation theory and asymptotic behavior of various classes of second-order and high-order differential equations and dynamic equations on time scales [1–17]. So far, much research activity concerns the oscillation problem of the third-order (TO) neutral differential and dynamic equations [18–26]. Recently, the research focus has been shifted to the study of the TO differential equations (DE) with distributed deviating arguments (DDA), and some results can be found in [27–35].

Li et al. [3] investigated

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + \int_a^b q(t,\xi)\left|x\left[g(t,\xi)\right]\right|^{\alpha-1}x\left[g(t,\xi)\right]\,d\sigma(\xi) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $0 \le p(t) \le p_0 < \infty$ and $\alpha \ge 1$ is a constant. Under the methods proposed by Li et al. [3, 20], Jiang and Li [30] studied the following equation with DDA:

$$\left(r(t)\left|z''(t)\right|^{\alpha-1}z''(t)\right)' + \int_a^b q(t,\xi)\left|x\left[g(t,\xi)\right]\right|^{\alpha-1}x\left[g(t,\xi)\right]\,d\sigma(\xi) = 0, \tag{1.1}$$

where $\alpha > 0$ is a constant, and obtained several theorems for (1.1) whenever

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)\,dt = \infty, \quad \text{or} \quad \int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)\,dt < \infty.$$

Furthermore, Elabbasy and Moaaz [31], and Wang et al. [32] examined a TODE with DDA under the assumption $0 \le p(t) \le P < 1$. However, the obtained oscillation theorems cannot be applied when $p(t) \ge 1$. Then Tunç [33] utilized a new technique, different from the existing methods, to give some criteria for a TODE with DDA, when $p(t) \ge 1$.

The main objective here is to establish several oscillation criteria for the TO neutral delay (ND) DE with DDA

$$E_2'(t) + \int_a^b q(t,\xi)\left|x\bigl(g(t,\xi)\bigr)\right|^{\alpha_3-1} x\bigl(g(t,\xi)\bigr)\,d\sigma(\xi) = 0, \tag{1.2}$$

where $t \ge t_0 > 0$,

$$\begin{aligned} E_2(t) &= r_2(t)\left|E_1'(t)\right|^{\alpha_2-1}E_1'(t),\\ E_1(t) &= r_1(t)\left|z'(t)\right|^{\alpha_1-1}z'(t),\\ z(t) &= x(t) + p(t)x\bigl(\tau(t)\bigr), \end{aligned} \tag{1.3}$$

and α_1, α_2, and α_3 are positive constants. We assume that:

(A1) $r_i(t) \in C([t_0,\infty),(0,\infty))$, $\int_{t_0}^{\infty} r_i^{-\frac{1}{\alpha_i}}(t)\,dt = \infty$, $i = 1,2$;
(A2) $p(t) \in C([t_0,\infty),[1,\infty))$ with $p(t) \not\equiv 1$, and $q(t,\xi) \in C([t_0,\infty) \times [a,b],[0,\infty))$ with $q(t,\xi) \not\equiv 0$ eventually;
(A3) $\tau(t) \in C^1([t_0,\infty),\mathbb{R})$, $\tau(t) \le t$, $\tau'(t) > 0$ and $\lim_{t\to\infty}\tau(t) = \infty$;
(A4) $g(t,\xi) \in C([t_0,\infty) \times [a,b],\mathbb{R})$ is a nondecreasing function for ξ, which satisfies $\liminf_{t\to\infty} g(t,\xi) = \infty$ for $\xi \in [a,b]$;
(A5) $\sigma(\xi) \in C([a,b],\mathbb{R})$ is nondecreasing and the integral of (1.2) is taken in the Riemann–Stieltjes sense.

This article is organized in the following manner. Section 2 presents three lemmas to prove our results. Section 3 establishes some new oscillation criteria for (1.2). Two examples finalize this article.

2 Some lemmas

For simplicity, we use some notations for sufficiently large t_1 with $t_1 \ge t_0$ as below:

$$\begin{aligned} &g_1(t) = g(t,a), \qquad g_2(t) = g(t,b), \qquad d_+(t) = \max\bigl\{0,d(t)\bigr\},\\ &\delta_1(t,t_1) = \int_{t_1}^{t}\left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}}ds, \qquad \delta_2(t,t_1) = \left(\frac{\delta_1(t,t_1)}{r_1(t)}\right)^{\frac{1}{\alpha_1}},\\ &\delta_3(t,t_1) = \int_{t_1}^{t}\delta_2(s,t_1)\,ds, \quad t \ge t_1. \end{aligned}$$

Furthermore, assume that

$$p_1(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right) > 0, \tag{2.1}$$

$$p_2(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{\delta_3(\tau^{-1}(\tau^{-1}(t)),t_1)}{p(\tau^{-1}(\tau^{-1}(t)))\delta_3(\tau^{-1}(t),t_1)}\right) > 0, \tag{2.2}$$

$$q_1(t) = \int_a^b q(t,\xi)p_1^{\alpha_3}\bigl(g(t,\xi)\bigr)\,d\sigma(\xi),$$
$$q_2(t) = \int_a^b q(t,\xi)p_2^{\alpha_3}\bigl(g(t,\xi)\bigr)\,d\sigma(\xi).$$

Lemma 2.1 *Assume that* (A1)–(A5) *hold. Furthermore, let $x(t)$ be an eventually positive solution of* (1.2). *Then $z(t)$ satisfies either*

$$\text{(I)}\quad z(t) > 0,\qquad z'(t) > 0,\qquad E_1'(t) > 0,\qquad E_2'(t) \le 0,$$

or

$$\text{(II)}\quad z(t) > 0,\qquad z'(t) < 0,\qquad E_1'(t) > 0,\qquad E_2'(t) \le 0.$$

Proof From the condition of Lemma 2.1, there exists a $t_1 \ge t_0$ such that

$$x(t) > 0,\qquad x\bigl(\tau(t)\bigr) > 0\quad\text{and}\quad x\bigl(g(t,\xi)\bigr) > 0,\quad \xi \in [a,b], \tag{2.3}$$

for $t \ge t_1$. Then (1.2) implies that

$$E_2'(t) = -\int_a^b q(t,\xi)\bigl|x\bigl(g(t,\xi)\bigr)\bigr|^{\alpha_3-1}x\bigl(g(t,\xi)\bigr)\,d\sigma(\xi) \le 0, \tag{2.4}$$

which means that $E_2(t)$ is nonincreasing and of constant sign, and $E_1'(t)$ is also of constant sign. We claim that $E_1'(t) > 0$. To prove this, assume on the contrary that $E_1'(t) < 0$ and there exists M_1, s.t. for $t \ge t_2 \ge t_1$,

$$E_2(t) \le -M_1 < 0.$$

Then

$$\bigl|E_1'(t)\bigr| \ge \left(\frac{M_1}{r_2(t)}\right)^{\frac{1}{\alpha_2}} > 0. \tag{2.5}$$

We integrate (2.5) to get

$$E_1(t) \le E_1(t_2) - M_1^{\frac{1}{\alpha_2}} \int_{t_2}^t \left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha_2}} ds.$$

Letting $t \to \infty$ and from (A1), we obtain $\lim_{t\to\infty} E_1(t) = -\infty$. Then there exist constants M_2 and $t_3 \ge t_2$ such that

$$E_1(t) \le -M_2 < 0,\qquad t \ge t_3,$$

which yields that

$$\bigl|z'(t)\bigr| \ge \left(\frac{M_2}{r_1(t)}\right)^{\frac{1}{\alpha_1}} > 0.$$

Integrating the latter inequality from t_3 to t, we have

$$z(t) \leq z(t_3) - M_2^{\frac{1}{\alpha_1}} \int_{t_3}^{t} \left(\frac{1}{r_1(s)}\right)^{\frac{1}{\alpha_1}} ds.$$

We use (A1) again to have $\lim_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Hence, $E_1'(t) > 0$ for $t \geq t_1$, and $z(t)$ has properties (I) and (II). □

Lemma 2.2 *Assume that* (A1)–(A5) *and* (2.1) *hold. Furthermore, suppose that* $x(t)$ *is an eventually positive solution of* (1.2) *and* $z(t)$ *has property* (II) *of Lemma* 2.1. *If*

$$\int_{t_0}^{\infty} \left[\frac{1}{r_1(u)} \int_{u}^{\infty} \left(\frac{1}{r_2(v)} \int_{v}^{\infty} q_1(s)\, ds\right)^{\frac{1}{\alpha_2}} dv\right]^{\frac{1}{\alpha_1}} du = \infty, \tag{2.6}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof One can see that (1.3) yields (see [33])

$$z\bigl(\tau^{-1}(t)\bigr) = x\bigl(\tau^{-1}(t)\bigr) + p\bigl(\tau^{-1}(t)\bigr)x(t),$$

which can be rewritten as

$$\begin{aligned} x(t) &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} && (2.7)\\ &\geq p_1(t)z\bigl(\tau^{-1}(t)\bigr). && (2.8) \end{aligned}$$

Combining (2.4) and (2.8), we get

$$\begin{aligned} E_2'(t) &\leq -\int_a^b q(t,\xi)p_1^{\alpha_3}\bigl(g(t,\xi)\bigr)z^{\alpha_3}\bigl(\tau^{-1}\bigl(g(t,\xi)\bigr)\bigr)\, d\sigma(\xi)\\ &\leq -q_1(t)z^{\alpha_3}\bigl(\tau^{-1}\bigl(g_2(t)\bigr)\bigr), && (2.9) \end{aligned}$$

based on the fact that $z'(t) < 0$ for $t \geq t_1$. Since $z(t)$ has property (II) of Lemma 2.1, one gets $\lim_{t\to\infty} z(t) = l \geq 0$. We claim that $l = 0$. Indeed, if we assume on the contrary that $l > 0$, then there exists $t_2 \geq t_1$ s.t. $\tau^{-1}(g_2(t)) \geq t_2$ and $z(\tau^{-1}(g_2(t))) \geq l$, $t \geq t_2$. Inequality (2.9) then yields

$$E_2'(t) \leq -l^{\alpha_3} q_1(t). \tag{2.10}$$

We integrate (2.10) to get

$$r_2(t)\bigl(E_1'(t)\bigr)^{\alpha_2} \geq l^{\alpha_3} \int_t^{\infty} q_1(s)\, ds,$$

which indicates that

$$E_1'(t) \geq \left(\frac{l^{\alpha_3}}{r_2(t)} \int_t^{\infty} q_1(s)\, ds\right)^{\frac{1}{\alpha_2}}. \tag{2.11}$$

Integrating (2.11) from t to ∞, we have

$$-E_1(t) \geq l^{\frac{\alpha_3}{\alpha_2}} \int_t^\infty \left(\frac{1}{r_2(v)} \int_v^\infty q_1(s)\,ds \right)^{\frac{1}{\alpha_2}} dv,$$

and then

$$\left|z'(t)\right| \geq l^{\frac{\alpha_3}{\alpha_1\alpha_2}} \left[\frac{1}{r_1(t)} \int_t^\infty \left(\frac{1}{r_2(v)} \int_v^\infty q_1(s)\,ds \right)^{\frac{1}{\alpha_2}} dv \right]^{\frac{1}{\alpha_1}},$$

since $z'(t) < 0$ for $t \geq t_2$. Integrating the latter inequality from t_2 to ∞, we get

$$z(t_2) > l^{\frac{\alpha_3}{\alpha_1\alpha_2}} \int_{t_2}^\infty \left[\frac{1}{r_1(u)} \int_u^\infty \left(\frac{1}{r_2(v)} \int_v^\infty q_1(s)\,ds \right)^{\frac{1}{\alpha_2}} dv \right]^{\frac{1}{\alpha_1}} du,$$

which contradicts (2.6). Thus, we obtain $l = 0$ and $\lim_{t\to\infty} x(t) = 0$, since $0 < x(t) \leq z(t)$. □

Lemma 2.3 *Assume that* (A1)–(A5) *and* (2.2) *hold. Furthermore, suppose that* $x(t)$ *is an eventually positive solution of* (1.2) *and* $z(t)$ *has property* (I) *of Lemma* 2.1. *Then for* $t \geq t_1 \geq t_0$,

$$\left[r_2(t)\left(\left(r_1(t)\left(z'(t)\right)^{\alpha_1}\right)'\right)^{\alpha_2}\right]' + q_2(t) z^{\alpha_3}\left(\tau^{-1}\left(g_1(t)\right)\right) \leq 0. \tag{2.12}$$

Proof Property (I) of $z(t)$ yields

$$E_2(t) = r_2(t)\left(\left(r_1(t)\left(z'(t)\right)^{\alpha_1}\right)'\right)^{\alpha_2} > 0.$$

Applying the monotonicity of $E_2^{\frac{1}{\alpha_2}}(t)$ gives

$$\begin{aligned} r_1(t)\left(z'(t)\right)^{\alpha_1} &= r_1(t_1)\left(z'(t_1)\right)^{\alpha_1} + \int_{t_1}^t \frac{r_2^{\frac{1}{\alpha_2}}(s)(r_1(s)(z'(s))^{\alpha_1})'}{r_2^{\frac{1}{\alpha_2}}(s)}\,ds \\ &\geq \delta_1(t,t_1) r_2^{\frac{1}{\alpha_2}}(t)\left(r_1(t)\left(z'(t)\right)^{\alpha_1}\right)'. \end{aligned} \tag{2.13}$$

It can be seen from (2.13) that

$$\left(\frac{r_1(t)(z'(t))^{\alpha_1}}{\delta_1(t,t_1)} \right)' \leq 0,$$

which, together with $r_1(t)(z'(t))^{\alpha_1} > 0$, yields that $z'(t)/\delta_2(t,t_1)$ is nonincreasing for $t \geq t_1$. Therefore,

$$z(t) = z(t_1) + \int_{t_1}^t \frac{z'(s)}{\delta_2(s,t_1)} \delta_2(s,t_1)\,ds \geq \frac{\delta_3(t,t_1)}{\delta_2(t,t_1)} z'(t), \tag{2.14}$$

which leads to

$$\left(\frac{z(t)}{\delta_3(t,t_1)} \right)' \leq 0. \tag{2.15}$$

Then for $t \geq t_2 \geq t_1$,

$$z\big(\tau^{-1}\big(\tau^{-1}(t)\big)\big) \leq \frac{\delta_3(\tau^{-1}(\tau^{-1}(t)), t_1)z(\tau^{-1}(t))}{\delta_3(\tau^{-1}(t), t_1)}, \tag{2.16}$$

due to $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$. Substituting (2.16) into (2.7), one has

$$x(t) \geq p_2(t)z\big(\tau^{-1}(t)\big),$$

which leads to

$$x\big(g(t,\xi)\big) \geq p_2\big(g(t,\xi)\big)z\big(\tau^{-1}\big(g_1(t)\big)\big). \tag{2.17}$$

Combining (1.2) and (2.17), we obtain (2.12). □

3 Main results

We respectively consider two cases $g_1(t) \leq \tau(t)$ and $g_1(t) \geq \tau(t)$ for $t \geq t_0$. Now, we begin with the first case.

Theorem 3.1 *Assume that* (A1)–(A5), (2.1), (2.2), *and* (2.6) *hold, and* $g_1(t) \leq \tau(t)$. *If there exists* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ *s.t.*

$$\begin{aligned} &\limsup_{t\to\infty} \int_{t_*}^{t} \biggl[\rho(s)q_2(s)\biggl(\frac{\delta_3(\tau^{-1}(g_1(s)), t_1)}{\delta_3(s,t_1)}\biggr)^{\alpha_3} \\ &\quad - \frac{\lambda(\rho'_+(s))^{\alpha_1\alpha_2+1}}{(\rho(s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}}\biggr]\,ds = \infty, \end{aligned} \tag{3.1}$$

for t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$, *where*

$$\lambda = \biggl(\frac{\alpha_1\alpha_2}{\alpha_3}\biggr)^{\alpha_1\alpha_2}\biggl(\frac{1}{\alpha_1\alpha_2+1}\biggr)^{\alpha_1\alpha_2+1},$$

$$\gamma(t) = \begin{cases} m_1(\delta_3(t,t_1))^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}, & m_1 \text{ is any positive constant}, \quad \text{if } \alpha_1\alpha_2 > \alpha_3, \\ m_2, \quad m_2 \text{ is any positive constant}, & \text{if } \alpha_1\alpha_2 \leq \alpha_3, \end{cases}$$

then every solution of (1.2) *is either oscillatory or tends to zero.*

Proof Suppose that (1.2) has a nonoscillatory solution $x(t)$. We may assume that (2.3) holds for $t \geq t_1 \geq t_0$. So we have that $z(t)$ is positive and satisfies the two properties for $t \geq t_1$.

We first consider property (I). Define $\omega(t)$ by

$$\omega(t) = \rho(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)}, \quad t \geq t_1. \tag{3.2}$$

Then $\omega(t) > 0$ and

$$\begin{aligned} \omega'(t) = \rho'(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)} + \rho(t)\biggl[\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(t)} \\ - \frac{\alpha_3 r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}z'(t)}{z^{\alpha_3+1}(t)}\biggr] \end{aligned} \tag{3.3}$$

$$\begin{aligned} &= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(t)} \\ &\quad - \alpha_3\rho(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}z'(t)}{z^{\alpha_3+1}(t)}. \end{aligned} \tag{3.4}$$

Based on (2.12), we have

$$\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(t)} \le -q_2(t)\left(\frac{z(\tau^{-1}(g_1(t)))}{z(t)}\right)^{\alpha_3}. \tag{3.5}$$

Since $g_1(t) \le \tau(t)$, we get $\tau^{-1}(g_1(t)) \le t$. Applying (2.15), we obtain

$$\frac{z(\tau^{-1}(g_1(t)))}{z(t)} \ge \frac{\delta_3(\tau^{-1}(g_1(t)), t_1)}{\delta_3(t, t_1)}. \tag{3.6}$$

From (2.13), we have

$$z'(t) \ge \delta_2(t, t_1)\big(r_2(t)\big(\big(r_1(t)\big(z'(t)\big)^{\alpha_1}\big)'\big)^{\alpha_2}\big)^{\frac{1}{\alpha_1\alpha_2}}. \tag{3.7}$$

We combine (3.4)–(3.7) to conclude that

$$\begin{aligned} \omega'(t) &\le \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \rho(t)q_2(t)\left(\frac{\delta_3(\tau^{-1}(g_1(t)), t_1)}{\delta_3(t, t_1)}\right)^{\alpha_3} \\ &\quad - \frac{\alpha_3\delta_2(t, t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)} z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t)\omega^{\frac{1}{\alpha_1\alpha_2}+1}(t). \end{aligned} \tag{3.8}$$

Next, we will compute $z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t)$ and consider the following two cases:

Case (i). Assume that $\alpha_1\alpha_2 > \alpha_3$. Since $z(t)/\delta_3(t, t_1)$ is nonincreasing, due to (2.15), there exist constants $h_1 > 0$ and $t_2 \ge t_1$ such that

$$\frac{z(t)}{\delta_3(t, t_1)} \le \frac{z(t_2)}{\delta_3(t_2, t_1)} = h_1, \quad t \ge t_2,$$

and

$$z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t) \ge m_1\big(\delta_3(t, t_1)\big)^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}, \tag{3.9}$$

where $m_1 = h_1^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}$.

Case (ii). Assume that $\alpha_1\alpha_2 \le \alpha_3$. Since $z'(t) > 0$, there exists $h_2 > 0$ such that

$$z(t) \ge z(t_1) = h_2, \quad t \ge t_1,$$

and

$$z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t) \ge m_2, \quad t \ge t_1, \tag{3.10}$$

where $m_2 = h_2^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}$. We combine (3.8) with (3.9) and (3.10) to have

$$\begin{aligned}\omega'(t) &\leq \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \rho(t)q_2(t)\left(\frac{\delta_3(\tau^{-1}(g_1(t)),t_1)}{\delta_3(t,t_1)}\right)^{\alpha_3}\\ &\quad - \frac{\alpha_3\gamma(t)\delta_2(t,t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)}\omega^{\frac{1}{\alpha_1\alpha_2}+1}(t).\end{aligned} \tag{3.11}$$

By using the inequality (see [18])

$$Bu - Au^{\frac{1}{\alpha}+1} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha}, \quad A > 0, \tag{3.12}$$

where

$$B = \frac{\rho'_+(t)}{\rho(t)}, \qquad A = \frac{\alpha_3\gamma(t)\delta_2(t,t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)}, \quad \alpha = \alpha_1\alpha_2, u = \omega(t),$$

we obtain

$$\begin{aligned}&\frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha_3\gamma(t)\delta_2(t,t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)}\omega^{\frac{1}{\alpha_1\alpha_2}+1}(t)\\ &\quad\leq \left(\frac{\alpha_1\alpha_2}{\alpha_3\rho(t)\gamma(t)\delta_2(t,t_1)}\right)^{\alpha_1\alpha_2}\left(\frac{\rho'_+(t)}{\alpha_1\alpha_2+1}\right)^{\alpha_1\alpha_2+1}\\ &\quad= \frac{\lambda(\rho'_+(t))^{\alpha_1\alpha_2+1}}{(\rho(t)\gamma(t)\delta_2(t,t_1))^{\alpha_1\alpha_2}}.\end{aligned} \tag{3.13}$$

We combine (3.11) and (3.13) to conclude that

$$\omega'(t) \leq -\rho(t)q_2(t)\left(\frac{\delta_3(\tau^{-1}(g_1(t)),t_1)}{\delta_3(t,t_1)}\right)^{\alpha_3} + \frac{\lambda(\rho'_+(t))^{\alpha_1\alpha_2+1}}{(\rho(t)\gamma(t)\delta_2(t,t_1))^{\alpha_1\alpha_2}}.$$

We integrate the latter inequality to make

$$\begin{aligned}&\int_{t_2}^t\left[\rho(s)q_2(s)\left(\frac{\delta_3(\tau^{-1}(g_1(s)),t_1)}{\delta_3(s,t_1)}\right)^{\alpha_3} - \frac{\lambda(\rho'_+(s))^{\alpha_1\alpha_2+1}}{(\rho(s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}}\right]ds\\ &\quad\leq \omega(t_2) - \omega(t) < \omega(t_2),\end{aligned}$$

which contradicts (3.1).

Secondly, we investigate property (II) and deduce $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.2 *Assume that* (A1)–(A5), (2.1), (2.2), *and* (2.6) *hold. Furthermore, suppose that* $g_1(t) \leq \tau(t)$ *and* $\alpha_1\alpha_2 = \alpha_3$. *If there exists* $\rho(t)$ *s.t.*

$$\limsup_{t\to\infty}\int_{t_*}^t\left[\rho(s)q_2(s)\left(\frac{\delta_3(\tau^{-1}(g_1(s)),t_1)}{\delta_3(s,t_1)}\right)^{\alpha_3} - \frac{\rho'_+(s)}{(\delta_3(s,t_1))^{\alpha_3}}\right]ds = \infty, \tag{3.14}$$

for t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$, *then we get the same conclusion as in Theorem* 3.1.

Proof Suppose that (2.3) holds for $t \geq t_1 \geq t_0$. Then $z(t)$ satisfies the two properties.

We first consider property (I). From (2.13) and (2.14), we have

$$\begin{aligned} r_1(t)z^{\alpha_1}(t) &\geq \left(\frac{\delta_3(t,t_1)}{\delta_2(t,t_1)}\right)^{\alpha_1} r_1(t)\bigl(z'(t)\bigr)^{\alpha_1} \\ &\geq \left(\frac{\delta_3(t,t_1)}{\delta_2(t,t_1)}\right)^{\alpha_1} \delta_1(t,t_1) r_2^{\frac{1}{\alpha_2}}(t)\bigl(r_1(t)\bigl(z'(t)\bigr)^{\alpha_1}\bigr)', \end{aligned}$$

and

$$z^{\alpha_3}(t) \geq \bigl(\delta_3(t,t_1)\bigr)^{\alpha_3} r_2(t)\bigl(\bigl(r_1(t)\bigl(z'(t)\bigr)^{\alpha_1}\bigr)'\bigr)^{\alpha_2}. \tag{3.15}$$

Define $\omega(t)$ by (3.2). As in the proof of Theorem 3.1, we get (3.3). Since $E_2(t) > 0$, (3.3) indicates

$$\omega'(t) \leq \rho'_+(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)} + \rho(t)\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(t)}.$$

Combining the latter inequality with (3.5), (3.6), and (3.15), we see that

$$\omega'(t) \leq \frac{\rho'_+(t)}{(\delta_3(t,t_1))^{\alpha_3}} - \rho(t)q_2(t)\left(\frac{\delta_3(\tau^{-1}(g_1(t)),t_1)}{\delta_3(t,t_1)}\right)^{\alpha_3}. \tag{3.16}$$

An integration of (3.16) from t_2 ($t_2 \geq t_1$) to t leads to

$$\int_{t_2}^{t}\left[\rho(s)q_2(s)\left(\frac{\delta_3(\tau^{-1}(g_1(s)),t_1)}{\delta_3(s,t_1)}\right)^{\alpha_3} - \frac{\rho'_+(s)}{(\delta_3(s,t_1))^{\alpha_3}}\right]ds \leq \omega(t_2) - \omega(t) < \omega(t_2),$$

for all sufficiently large t, which contradicts (3.14).

Secondly, if property (II) holds, then $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.3 *Assume that* (A1)–(A5), (2.1), (2.2), *and* (2.6) *hold. Furthermore, suppose that* $g_1(t) \leq \tau(t)$ *and* $\alpha_1\alpha_2 = \alpha_3 \geq 1$. *If there exists* $\rho(t)$ *s.t.*

$$\begin{aligned} \limsup_{t\to\infty} \int_{t_*}^{t} \biggl[&\rho(s)q_2(s)\left(\frac{\delta_3(\tau^{-1}(g_1(s)),t_1)}{\delta_3(s,t_1)}\right)^{\alpha_3} \\ &- \frac{(\rho'_+(s))^2}{4\alpha_3\rho(s)\gamma(s)\delta_2(s,t_1)(\delta_3(s,t_1))^{\alpha_3-1}}\biggr]ds = \infty, \end{aligned} \tag{3.17}$$

for t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$, *where* $\gamma(t)$ *is given in Theorem* 3.1, *then we get the same conclusion as in Theorem* 3.1.

Proof Suppose that (2.3) holds for $t \geq t_1 \geq t_0$. Then $z(t)$ satisfies the two properties.

We first consider property (I). Proceeding as in the proof of Theorem 3.1, we get (3.11), and

$$\begin{aligned} \omega'(t) \leq{}& \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \rho(t)q_2(t)\left(\frac{\delta_3(\tau^{-1}(g_1(t)),t_1)}{\delta_3(t,t_1)}\right)^{\alpha_3} \\ &- \frac{\alpha_3\gamma(t)\delta_2(t,t_1)\omega^2(t)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)}\omega^{\frac{1}{\alpha_1\alpha_2}-1}(t). \end{aligned} \tag{3.18}$$

From (3.2) and (3.15), one has

$$\omega^{\frac{1}{\alpha_1\alpha_2}-1}(t) = \rho^{\frac{1}{\alpha_1\alpha_2}-1}(t)\left(\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)}\right)^{\frac{1}{\alpha_1\alpha_2}-1}$$
$$\geq \rho^{\frac{1}{\alpha_1\alpha_2}-1}(t)\big(\delta_3(t,t_1)\big)^{\alpha_3-1}. \tag{3.19}$$

We substitute (3.19) into (3.18) to see that

$$\omega'(t) \leq \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \rho(t)q_2(t)\left(\frac{\delta_3(\tau^{-1}(g_1(t)),t_1)}{\delta_3(t,t_1)}\right)^{\alpha_3} - \frac{\alpha_3\gamma(t)\delta_2(t,t_1)(\delta_3(t,t_1))^{\alpha_3-1}}{\rho(t)}\omega^2(t),$$

from which one gets

$$\omega'(t) \leq -\rho(t)q_2(t)\left(\frac{\delta_3(\tau^{-1}(g_1(t)),t_1)}{\delta_3(t,t_1)}\right)^{\alpha_3} + \frac{(\rho'_+(t))^2}{4\alpha_3\rho(t)\gamma(t)\delta_2(t,t_1)(\delta_3(t,t_1))^{\alpha_3-1}},$$

by completing the square with respect to $\omega(t)$. We integrate the latter inequality from t_2 ($t_2 \geq t_1$) to t to obtain

$$\int_{t_2}^{t}\left[\rho(s)q_2(s)\left(\frac{\delta_3(\tau^{-1}(g_1(s)),t_1)}{\delta_3(s,t_1)}\right)^{\alpha_3} - \frac{(\rho'_+(s))^2}{4\alpha_3\rho(s)\gamma(s)\delta_2(s,t_1)(\delta_3(s,t_1))^{\alpha_3-1}}\right]ds \leq \omega(t_2),$$

for all sufficiently large t, which contradicts (3.17).

Secondly, if property (II) holds, then $\lim_{t\to\infty} x(t) = 0$. □

Next, we consider $g_1(t) \geq \tau(t)$ for $t \geq t_0$.

Theorem 3.4 *Assume that conditions* (A1)–(A5), (2.1), (2.2), *and* (2.6) *hold, and* $g_1(t) \geq \tau(t)$. *If there exists* $\rho(t)$ *s.t.*

$$\limsup_{t\to\infty}\int_{t_*}^{t}\left[\rho(s)q_2(s) - \frac{\lambda(\rho'_+(s))^{\alpha_1\alpha_2+1}}{(\rho(s)\gamma(\tau(s))\delta_2(\tau(s),t_1))^{\alpha_1\alpha_2}}\right]ds = \infty, \tag{3.20}$$

for t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$, *then we get the same conclusion as in Theorem* 3.1.

Proof Suppose that (2.3) holds for $t \geq t_1 \geq t_0$. Then $z(t)$ satisfies the two properties.

We first consider property (I). Define $\nu(t)$ by

$$\nu(t) = \rho(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(\tau(t))}, \quad t \geq t_1. \tag{3.21}$$

Then $\nu(t) > 0$ and

$$\nu'(t) = \rho'(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(\tau(t))} + \rho(t)\left[\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(\tau(t))} - \frac{\alpha_3 r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}(z(\tau(t)))'}{z^{\alpha_3+1}(\tau(t))}\right] \tag{3.22}$$

$$\begin{aligned} &= \frac{\rho'(t)}{\rho(t)}\nu(t) + \rho(t)\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(\tau(t))} \\ &\quad - \alpha_3\rho(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}(z(\tau(t)))'}{z^{\alpha_3+1}(\tau(t))}. \end{aligned} \tag{3.23}$$

Since $\tau^{-1}(g_1(t)) \geq t \geq \tau(t)$ and $z'(t) > 0$, we have

$$\frac{z(\tau^{-1}(g_1(t)))}{z(\tau(t))} \geq 1, \quad t \geq t_1,$$

which indicates that

$$\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(\tau(t))} \leq -q_2(t), \tag{3.24}$$

due to (2.12). Based on (3.7), $E_2'(t) \leq 0$ and $\tau(t) \leq t$, so one has $\tau(t) \geq t_1$ and

$$\bigl(z\bigl(\tau(t)\bigr)\bigr)' \geq \delta_2\bigl(\tau(t), t_1\bigr)\bigl(r_2(t)\bigl(\bigl(r_1(t)\bigl(z'(t)\bigr)^{\alpha_1}\bigr)'\bigr)^{\alpha_2}\bigr)^{\frac{1}{\alpha_1\alpha_2}}, \tag{3.25}$$

for $t \geq t_2 > t_1$. Combining (3.9), (3.10), (3.23)–(3.25), we conclude that

$$\begin{aligned} \nu'(t) &\leq \frac{\rho_+'(t)}{\rho(t)}\nu(t) - \rho(t)q_2(t) \\ &\quad - \frac{\alpha_3\gamma(\tau(t))\delta_2(\tau(t), t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)}\nu^{\frac{1}{\alpha_1\alpha_2}+1}(t). \end{aligned} \tag{3.26}$$

Using (3.12) and (3.26) with

$$B = \frac{\rho_+'(t)}{\rho(t)}, \qquad A = \frac{\alpha_3\gamma(\tau(t))\delta_2(\tau(t), t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)},$$

one gets

$$\nu'(t) \leq -\rho(t)q_2(t) + \frac{\lambda(\rho_+'(t))^{\alpha_1\alpha_2+1}}{(\rho(t)\gamma(\tau(t))\delta_2(\tau(t), t_1))^{\alpha_1\alpha_2}}.$$

Integrating the latter inequality from t_2 to t, we have

$$\int_{t_2}^{t}\left[\rho(s)q_2(s) - \frac{\lambda(\rho_+'(s))^{\alpha_1\alpha_2+1}}{(\rho(s)\gamma(\tau(s))\delta_2(\tau(s), t_1))^{\alpha_1\alpha_2}}\right]ds \leq \omega(t_2),$$

for all sufficiently large t, which contradicts (3.20).

Secondly, if property (II) holds, then $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.5 *Assume that* (A1)–(A5), (2.1), (2.2), *and* (2.6) *hold. Furthermore, suppose that $g_1(t) \geq \tau(t)$ and $\alpha_1\alpha_2 = \alpha_3$. If there exists $\rho(t)$ s.t.*

$$\limsup_{t\to\infty}\int_{t_*}^{t}\left[\rho(s)q_2(s) - \frac{\rho_+'(s)}{(\delta_3(\tau(s), t_1))^{\alpha_3}}\right]ds = \infty, \tag{3.27}$$

for t_1 and t_* with $t_* \geq t_1 \geq t_0$, then we get the same conclusion as in Theorem 3.1.

Proof Suppose that (2.3) holds for $t \geq t_1 \geq t_0$. Then $z(t)$ satisfies the two properties.

We first consider property (I). Proceeding as in the proof of Theorem 3.4, we get (3.22), which implies

$$\nu'(t) \leq \rho'_+(t)\frac{r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(\tau(t))} + \rho(t)\frac{[r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}]'}{z^{\alpha_3}(\tau(t))}. \tag{3.28}$$

Applying (3.15), the monotonicity of $E_2(t)$ and the fact that $\tau(t) \leq t$, one has $\tau(t) \geq t_1$ and

$$z^{\alpha_3}(\tau(t)) \geq (\delta_3(\tau(t), t_1))^{\alpha_3} r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}, \tag{3.29}$$

for $t \geq t_2 > t_1$. Combining (3.24), (3.28), and (3.29), one gets

$$\nu'(t) \leq \frac{\rho'_+(t)}{(\delta_3(\tau(t), t_1))^{\alpha_3}} - \rho(t)q_2(t). \tag{3.30}$$

Upon integrating (3.30) from t_2 to t, we obtain a contradiction to (3.27).

Secondly, if property (II) holds, then $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.6 *Assume that* (A1)–(A5), (2.1), (2.2), *and* (2.6) *hold. Furthermore, suppose that* $g_1(t) \geq \tau(t)$ *and* $\alpha_1\alpha_2 = \alpha_3 \geq 1$. *If there exists* $\rho(t)$ *s.t.*

$$\limsup_{t\to\infty} \int_{t_*}^{t} \left[\rho(s)q_2(s) - \frac{(\rho'_+(s))^2}{4\alpha_3\rho(s)\gamma(\tau(s))\delta_2(\tau(s), t_1)(\delta_3(\tau(s), t_1))^{\alpha_3 - 1}}\right] ds = \infty,$$

for t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$, *then we get the same conclusion as in Theorem* 3.1.

We omit the proof of Theorem 3.6 here, since it is similar to that of Theorem 3.3.

4 Examples

The following examples are given to show the applications of Theorems 3.1 and 3.5.

Example 4.1 For $t > k_1 \geq 1$, consider a TONDDE with DDA

$$E'_2(t) + \int_{k_1}^{k_1+1} 10(t+\xi)\left|x\left(t - k_1 - \frac{1}{\xi}\right)\right|^{\frac{4}{3}} x\left(t - k_1 - \frac{1}{\xi}\right) d\xi = 0, \tag{4.1}$$

where

$$\begin{aligned} E_2(t) &= \left|E'_1(t)\right|^4 E'_1(t), \\ E_1(t) &= (t - k_1)\left|z'(t)\right|^{-\frac{2}{3}} z'(t), \\ z(t) &= x(t) + \frac{4t+5}{t+1}x(t - k_1). \end{aligned}$$

Let $\alpha_1 = 1/3$, $\alpha_2 = 5$, $\alpha_3 = 7/3$, $a = k_1$, $b = k_1 + 1$, $r_1(t) = t - k_1$, $r_2(t) = 1$, $\tau(t) = t - k_1$, $g(t,\xi) = t - k_1 - 1/\xi$, $\sigma(\xi) = \xi$, $p(t) = (4t+5)/(t+1)$, $q(t,\xi) = 10(t+\xi)$. Choose $t_0 = t_1 = k_1$. Then we obtain $\alpha_1\alpha_2 < \alpha_3$, $4 \leq p(t) < 5$,

$$g_1(t) = g(t, k_1) = t - k_1 - \frac{1}{k_1},$$

$$\delta_1(t,t_1) = \delta_1(t,k_1) = t-k_1,$$
$$\delta_2(t,t_1) = \left(\frac{\delta_1(t,k_1)}{t-k_1}\right)^3 = 1,$$
$$\delta_3(t,t_1) = \delta_3(t,k_1) = t-k_1,$$
$$\delta_3\bigl(\tau^{-1}(t),t_1\bigr) = \delta_3(t+k_1,k_1) = t,$$
$$\delta_3\bigl(\tau^{-1}\bigl(\tau^{-1}(t)\bigr),t_1\bigr) = \delta_3(t+2k_1,k_1) = t+k_1,$$
$$\delta_3\bigl(\tau^{-1}\bigl(g_1(t)\bigr),t_1\bigr) = \delta_3\left(t-\frac{1}{k_1},k_1\right) = t-k_1-\frac{1}{k_1}.$$

Furthermore, we deduce that

$$p_1(t) > \frac{1}{5}\left(1-\frac{1}{4}\right) = \frac{3}{20} > 0,$$
$$p_2(t) > \frac{1}{5}\left(1-\frac{1}{4}\cdot\frac{t+k_1}{t}\right) > \frac{1}{10} > 0,$$
$$q_1(t) > \int_{k_1}^{k_1+1}\frac{3}{20}\cdot 10(t+\xi)\,d\xi = \frac{3}{2}\left(t+k_1+\frac{1}{2}\right),$$
$$q_2(t) > \int_{k_1}^{k_1+1}\frac{1}{10}\cdot 10(t+\xi)\,d\xi = t+k_1+\frac{1}{2}.$$

It is easy to verify that

$$\begin{aligned}
&\int_{t_0}^{\infty}\left[\frac{1}{r_1(u)}\int_u^{\infty}\left(\frac{1}{r_2(v)}\int_v^{\infty}q_1(s)\,ds\right)^{\frac{1}{\alpha_2}}dv\right]^{\frac{1}{\alpha_1}}du\\
&\quad> \int_{k_1}^{\infty}\left[\frac{1}{u-k_1}\int_u^{\infty}\left(\int_v^{\infty}\frac{3}{2}\left(s+k_1+\frac{1}{2}\right)ds\right)^{\frac{1}{5}}dv\right]^3du\\
&\quad= \infty.
\end{aligned}$$

Therefore, conditions (A1)–(A5), (2.1), (2.2), and (2.6) hold, and $g_1(t) \le \tau(t)$. We choose $\rho(t) = t$ and $t_* = k_1+2$. Applying Theorem 3.1, it remains to check (3.1), where

$$\lambda = \left(\frac{5}{7}\right)^{\frac{5}{3}}\left(\frac{3}{8}\right)^{\frac{8}{3}}.$$

Then we get

$$\begin{aligned}
&\int_{t_*}^{t}\left[\rho(s)q_2(s)\left(\frac{\delta_3(\tau^{-1}(g_1(s)),t_1)}{\delta_3(s,t_1)}\right)^{\alpha_3} - \frac{\lambda(\rho'_+(s))^{\alpha_1\alpha_2+1}}{(\rho(s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}}\right]ds\\
&\quad> \int_{k_1+2}^{t}\left[s\left(s+k_1+\frac{1}{2}\right)\left(\frac{s-k_1-\frac{1}{k_1}}{s-k_1}\right)^{\frac{7}{3}} - \frac{(\frac{5}{7})^{\frac{5}{3}}(\frac{3}{8})^{\frac{8}{3}}}{(m_2 s)^{\frac{5}{3}}}\right]ds \to \infty,
\end{aligned}$$

as $t\to\infty$, since $\int_{k_1+2}^{t}s^{-\frac{5}{3}}\,ds < \infty$. Hence, we get the same conclusion as in Theorem 3.1.

Example 4.2 For $t > k_1 \geq 1$, consider a TONDDE with DDA

$$E_2'(t) + \int_{k_1}^{k_1+l} \frac{40\xi}{t} \left| x\left(\frac{t+\xi}{2}\right)\right|^2 x\left(\frac{t+\xi}{2}\right) d\xi = 0, \tag{4.2}$$

where l is a positive integer,

$$\begin{aligned}
&E_2(t) = \left|E_1'(t)\right|^{-\frac{2}{3}} E_1'(t),\\
&E_1(t) = (t-k_1)\left|z'(t)\right|^8 z'(t),\\
&z(t) = x(t) + \frac{5t+4k_1}{t+k_1} x\left(\frac{t}{2}\right).
\end{aligned}$$

Let $\alpha_1 = 9$, $\alpha_2 = 1/3$, $\alpha_3 = 3$, $a = k_1$, $b = k_1 + l$, $r_1(t) = t - k_1$, $r_2(t) = 1$, $\sigma(\xi) = \xi$,

$$\tau(t) = \frac{t}{2}, \qquad g(t,\xi) = \frac{t+\xi}{2}, \qquad p(t) = \frac{5t+4k_1}{t+k_1}, \qquad q(t,\xi) = \frac{40\xi}{t}.$$

Choose $t_0 = t_1 = k_1$. Then we have $\alpha_1\alpha_2 = \alpha_3$, $4 < p(t) < 5$,

$$\begin{aligned}
&g_1(t) = g(t,k_1) = \frac{t+k_1}{2},\\
&\delta_3\left(\tau^{-1}(t), t_1\right) = \delta_3(2t, k_1) = 2t - k_1,\\
&\delta_3\left(\tau^{-1}\left(\tau^{-1}(t)\right), t_1\right) = \delta_3(4t, k_1) = 4t - k_1,\\
&\delta_3\left(\tau(t), t_1\right) = \delta_3\left(\frac{t}{2}, k_1\right) = \frac{t}{2} - k_1 > \frac{t}{4},
\end{aligned}$$

where $t \geq t_2 > 4k_1$, $\delta_1(t,t_1)$, $\delta_2(t,t_1)$, and $\delta_3(t,t_1)$ are the same as in Example 4.1. Furthermore, we deduce that

$$\begin{aligned}
&p_1(t) > \frac{1}{5}\left(1 - \frac{1}{4}\right) = \frac{3}{20} > 0,\\
&p_2(t) > \frac{1}{5}\left(1 - \frac{1}{4} \cdot \frac{4t-k_1}{2t-k_1}\right) > \frac{1}{20} > 0,\\
&q_1(t) > \int_{k_1}^{k_1+1} \frac{3}{20} \cdot \frac{40\xi}{t} \, d\xi = \frac{6k_1 l + 3l^2}{t},\\
&q_2(t) > \int_{k_1}^{k_1+1} \frac{1}{20} \cdot \frac{40\xi}{t} \, d\xi = \frac{2k_1 l + l^2}{t}.
\end{aligned}$$

Clearly, (2.6) holds. Choosing $\rho(t) = t^2$ and $t_* = t_2$, one has

$$\begin{aligned}
&\int_{t_*}^{t} \left[\rho(s) q_2(s) - \frac{\rho_+'(s)}{(\delta_3(\tau(s), t_1))^{\alpha_3}}\right] ds\\
&\quad > \int_{t_2}^{t} \left[s^2 \cdot \frac{2k_1 l + l^2}{s} - \frac{2s}{(\frac{s}{4})^3}\right] ds \to \infty,
\end{aligned}$$

as $t \to \infty$, which means that (3.27) holds, and all conditions of Theorem 3.5 are satisfied. Hence, we get the same conclusion as in Theorem 3.1.

Funding
This research was supported by the National Natural Science Foundation of China (Grant No. 61803176, 61703180) and A Project of Shandong Province Higher Educational Science and Technology Program (Grant No. J18KA230).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 February 2019 Accepted: 17 July 2019 Published online: 25 July 2019

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