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Some results on integral inequalities via Riemann–Liouville fractional integrals



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Abstract

In current continuation, we have incorporated the notion of $s - (\alpha, m)$ -convex functions and have established new integral inequalities. In order to generalize Hermite–Hadamard-type inequalities, some new integral inequalities of Hermite–Hadamard and Simpson type using $s - (\alpha, m)$ -convex function via Riemann–Liouville fractional integrals are obtained that reproduce the results presented by (Appl. Math. Lett. 11(5):91–95, 1998; Comput. Math. Appl. 47(2–3): 207–216, 2004; J. Inequal. Appl. 2013:158, 2013). Applications to special means are also provided.

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1 Introduction

Let \mathbb{R} be the set of real numbers, $I \subseteq \mathbb{R}$ be an interval, and $\eta : I \subset \mathbb{R} \to \mathbb{R}$ be a convex in the classical sense function which satisfies the inequality

 $\eta(\kappa s_1 + (1-\kappa)s_2) \le \kappa \eta(s_1) + (1-\kappa)\eta(s_2)$

whenever $s_1, s_2 \in I$ and $\kappa \in [0, 1]$. Numerous authors have presented inequalities for convex functions, however, because of its wide applicability and importance, one of the most notable is Hermite–Hadamard inequality, which is expressed as follows [4]:

Let $\eta : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval *I* of real numbers and $s_1, s_2 \in I$ with $s_1 < s_2$. Then

$$\eta\left(\frac{s_1+s_2}{2}\right) \leq \frac{1}{s_2-s_1} \int_{s_1}^{s_2} g(t) \, dt \leq \frac{\eta(s_1)+\eta(s_2)}{2}.$$

Both inequalities hold reversed if η is concave. In the field of mathematical inequalities, Hermite–Hadamard inequalities have been considered by numerous mathematicians because of their pertinence and handiness. Various researchers have extended the Hermite– Hadamard inequality to different structures utilizing the classical convex functions. First we recall some important definitions and results which we have used in this paper.

M. Muddassar [5] presented the class of $s - (\alpha, m)$ -convex functions as follows:

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Definition 1 A function $\eta : [0, \infty) \to [0, \infty)$ is said to be $s - (\alpha, m)$ -convex in the first sense, or f belongs to the class $K_{m,1}^{\alpha,s}$, if for every $s_1, s_2 \in [0, \infty]$ and $\kappa \in [0, 1]$, the following inequality holds:

$$\eta(\kappa s_1 + m(1-\kappa)s_2) \leq \kappa^{\alpha s}\eta(s_1) + m(1-\kappa^{\alpha s})\eta(s_2),$$

where $(\alpha, m) \in [0, 1]^2$, for some fixed $s \in (0, 1]$.

Definition 2 A function $\eta : [0, \infty) \to [0, \infty)$ is said to be $s - (\alpha, m)$ -convex function in the second sense, or f belongs to the class $K_{m,2}^{\alpha,s}$, if for every $s_1, s_2 \in [0, \infty]$ and $\kappa \in [0, 1]$, the following inequality holds:

$$\eta(\kappa s_1 + m(1-\kappa)s_2) \leq (\kappa^{\alpha})^s \eta(s_1) + m(1-\kappa^{\alpha})^s \eta(s_2),$$

where $(\alpha, m) \in [0, 1]^2$, for some fixed $s \in (0, 1]$.

Definition 3 Let $f \in L_1[a, b]$. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$, with $a \ge 0$, are defined by

$$J_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad a < x,$$

and

$$J_{b^-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt, \quad x < b,$$

respectively, where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

It is to be noted that $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties relating to this operator can be found in [6], and for useful details on Hermite–Hadamard and Simpson's type inequalities connected with fractional integral inequalities, the interested readers are directed to [7, 8].

In [1], Dragomir and Agarwal obtained inequalities for differentiable convex mappings which are connected with the right-hand side of Hermite–Hadamard (trapezoid) inequality and applied them to obtain some elementary inequalities for real numbers and in numerical integration.

Theorem 1 Let $\eta : I \subset R \to R$ be a differentiable mapping on I where $s_1, s_2 \in I$ with $s_1 < s_2$. If $|\eta'|^q$ is convex on $[s_1, s_2]$, for some $q \ge 1$, then the following inequality holds:

$$\left|\frac{\eta(s_1) + \eta(s_2)}{2} - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \eta(u) \, du\right| \le \frac{s_2 - s_1}{8} \left[\left|\eta'(s_1)\right| + \left|\eta'(s_2)\right| \right]. \tag{1}$$

In [9], a variant of Hermite-Hadamard-type inequalities was obtained as follows.

Theorem 2 Let $\eta : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I and let $s_1, s_2 \in I$ with $s_1 < s_2$. If $|\eta'|$ is a convex function on $[s_1, s_2]$, then the following inequality holds:

$$\left|\frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \eta(u) \, du - \eta\left(\frac{s_1 + s_2}{2}\right)\right| \le \frac{s_2 - s_1}{8} \left[\left|\eta'(s_1)\right| + \left|\eta'(s_2)\right|\right]. \tag{2}$$

.

In [2], Yang obtained Hermite-Hadamard (trapezoid) inequalities for differentiable mapping for concave functions.

Theorem 3 Let $I \subset R$ be an open interval, $l, m, n, P, Q \in I$ with $l \leq P \leq n \leq Q \leq m (n \neq l, m)$ $l, m, n \in R$ and $\eta : [s_1, s_2] \to R$ be a differentiable function. If $|\eta'|^q$ is concave on $[s_1, s_2]$, and $1 \leq \theta \leq q$, then

$$\left| (P-l)\eta(l) + (m-Q)\eta(m) + (Q-P)\eta(n) - \int_{s_1}^{s_2} \eta(u) \, du \right| \le K(P,Q,n,\theta) J(P,Q,n,\theta),$$
(3)

where

$$K(P,Q,n,\theta) = \left(\frac{1}{2} \left[(P-l)^2 + (n-P)^2 + (Q-n)^2 + (m-Q)^2 \right] \right)^{\frac{(\theta-1)}{\theta}},$$

and

 $J(P, Q, n, \theta)$

$$\begin{split} &= \left(\frac{1}{2}\Big[(P-l)^2+(n-P)^2\Big] \left|\eta' \left(\frac{(P-l)^2+(n-P)^2(2n-3l+P)}{3[(P-l)^2+(n-P)^2]}+l\right)\right|^\theta\right)^{\frac{(\theta-1)}{\theta}} \\ &+ \left(\frac{1}{2}\Big[(Q-n)^2+(m-Q)^2\Big] \left|\eta' \left(m-\frac{(Q-n)^2(3m-2n-Q)+(m-Q)^3}{3[(Q-n)^2+(m-Q)^2]}\right)\right|^\theta\right)^{\frac{(\theta-1)}{\theta}}. \end{split}$$

Proposition 1 Under the assumptions of Theorem 3 with P = Q = n = (l + m)/2 and $\theta = 1$, we get the following inequality:

$$\left|\frac{\eta(s_1) + \eta(s_2)}{2} - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \eta(u) \, du\right| \le \frac{s_2 - s_1}{8} \left[\left|\eta'\left(\frac{5s_1 + s_2}{6}\right)\right| + \left|\eta'\left(\frac{s_1 + 5s_2}{6}\right)\right| \right]. \tag{4}$$

Proposition 2 Under the assumptions of Theorem 3 with $P = s_1$, $Q = s_2$, n = (l + m)/2 and $\theta = 1$, we get the following inequality:

$$\left| \eta\left(\frac{s_1 + s_2}{2}\right) - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \eta(u) \, du \right| \le \frac{s_2 - s_1}{8} \left[\left| \eta'\left(\frac{2s_1 + s_2}{3}\right) \right| + \left| \eta'\left(\frac{s_1 + 2s_2}{3}\right) \right| \right].$$
(5)

The aim of this paper is to build up Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integral using the $s - (\alpha, m)$ convexity, as well as concavity, for functions whose absolute values of the first derivative are convex. The results presented in this paper provide extension of those given in earlier works. The interested readers are referred to [3, 10-25].

2 Main results

In order to prove our main results, we need the following integral inequality

Lemma 1 Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with a < b, such that f' is integrable. Then the following inequality for Riemann-Liouville fractional integrals holds with $0 < \alpha < 1$:

$$\left(1-\frac{2}{2^{\alpha}}\lambda\right)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+(b)}{2^{\alpha}}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]$$

$$\leq \frac{b-a}{2^{\alpha+1}} [L_1 + L_2 + (L_3 + L_4)],$$

where
$$L_1 = \int_0^1 [(1-t)^{\alpha} - \lambda] f'(ta + (1-t)\frac{a+b}{2}) dt$$
, $L_2 = \int_0^1 [\lambda - (1-t)^{\alpha}] f'(tb + (1-t)\frac{a+b}{2}) dt$, $L_3 = \int_0^1 [2^{\alpha} - \lambda - (2-t)^{\alpha}] f'(t\frac{a+b}{2} + (1-t)a) dt$, $L_4 = \int_0^1 [\lambda - 2^{\alpha} + (2-t)^{\alpha}] f'(t\frac{a+b}{2} + (1-t)b) dt$.

Proof A simple proof of this inequality can be obtained by integrating by parts. The details are left to the interested readers. \Box

Using Lemma 1 the following results can be obtained.

Theorem 4 Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with a < b such that f' is integrable. If |f'| is $s - (\alpha, m)$ convex on [a, b], then the following inequality for Riemann–Liouville fractional integrals holds with $0 < \alpha \le 1$:

$$\begin{split} \left| \left(1 - \frac{2}{2^{\alpha}} \lambda \right) f\left(\frac{a+b}{2} \right) + \lambda \frac{f(a) + f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)] \right| \\ &\leq \left[\left\{ M_{1} | f'(a) | + 2m(M_{2} - M_{1}) | f'\left(\frac{a+b}{2} \right) | + M_{1} | f'(b) | \right\} \\ &+ \left\{ M_{3} | f'(a) | + m(M_{4} - M_{3}) | f'\left(\frac{a+b}{2} \right) | + M_{3} | f'(b) | \right\} \right], \end{split}$$
(6)
$$M_{1} = \int_{0}^{1} \left| (1-t)^{\alpha} - \lambda | t^{\alpha s} dt = \frac{(\alpha s)!}{(\alpha s + s + 1)!} - \frac{\lambda}{(s+1)}, \end{aligned}$$
$$M_{2} = \int_{0}^{1} \left| (1-t)^{\alpha} - \lambda | dt \\ &= \frac{1-2(1-\zeta)^{\alpha+1}}{\alpha+1} + (1-2\zeta)\lambda, \end{aligned}$$
$$M_{3} = \int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda | t^{\alpha s} dt \\ &= \frac{2^{\alpha s}}{s+1} - 2^{\alpha s} \left[\frac{1}{(s+1)} - \frac{\alpha s}{2(s+1)} \right] - \frac{\lambda}{(s+1)}, \end{aligned}$$
$$M_{4} = \int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda | dt \\ &= \frac{1+2^{\alpha+1}-2(2-\xi)^{\alpha+1}}{\alpha+1} + (2^{\alpha} - \lambda)(1-2\xi), \end{split}$$

where

$$\zeta = 1 - \lambda^{\frac{1}{\alpha}} \quad and \quad \xi = 2 - \left(2^{\alpha} - \lambda\right)^{\frac{1}{\alpha}}.$$

Proof Using $s - (\alpha, m)$ convexity of |f'|, for all $t \in [0, 1]$, we obtain:

$$|L_1| \leq \int_0^1 \left((1-t)^{\alpha} - \lambda \right) \left| f'\left(ta + (1-t)\frac{a+b}{2} \right) \right| dt$$
$$= \int_0^1 \left((1-t)^{\alpha} - \lambda \right) \left| t^{\alpha s} f'(a) + m\left(1 - t^{\alpha s} \right) f'\left(\frac{a+b}{2} \right) \right| dt$$

$$\leq \int_{0}^{1} \left((1-t)^{\alpha} - \lambda \right) \left\{ t^{\alpha s} \left| f'(a) \right| + m(1-t^{\alpha s}) \left| f'\left(\frac{a+b}{2}\right) \right| \right\} dt \\ = M_{1} \left| f'(a) \right| + m(M_{2} - M_{1}) \left| f'\left(\frac{a+b}{2}\right) \right| dt \\ = \int_{0}^{1} \left((1-t)^{\alpha} - \lambda \right) \left| t^{\alpha s} f'(b) + m(1-t^{\alpha s}) f'\left(\frac{a+b}{2}\right) \right| dt \\ \leq \int_{0}^{1} \left((1-t)^{\alpha} - \lambda \right) \left\{ t^{\alpha s} \left| f'(b) \right| + m(1-t^{\alpha s}) \left| f'\left(\frac{a+b}{2}\right) \right| \right\} dt \\ = M_{1} \left| f'(b) \right| + m(M_{2} - M_{1}) \left| f'\left(\frac{a+b}{2}\right) \right| , \\ \left| L_{3} \right| = \int_{0}^{1} \left(2^{\alpha} - (2-t)^{\alpha} - \lambda \right) \left\{ \left| f''\left(t\frac{a+b}{2} + (1-t)a\right) \right| \right\} dt \\ = \int_{0}^{1} \left(2^{\alpha} - (2-t)^{\alpha} - \lambda \right) \left\{ \left| t^{\alpha s} f'\left(\frac{a+b}{2}\right) + m(1-t^{\alpha s}) f'(a) \right| \right\} dt \\ \leq \int_{0}^{1} \left(2^{\alpha} - (2-t)^{\alpha} - \lambda \right) \left\{ \left| t^{\alpha s} f'\left(\frac{a+b}{2}\right) \right| + m(1-t^{\alpha s}) \left| f'(a) \right| \right\} dt \\ = M_{3} \left| f'\left(\frac{a+b}{2}\right) \right| + m(M_{4} - M_{3}) \left| f'(a) \right| , \\ \left| L_{4} \right| \leq \int_{0}^{1} \left(2^{\alpha} - (2-t)^{\alpha} - \lambda \right) \left\{ \left| t^{\alpha s} f'\left(\frac{a+b}{2}\right) + m(1-t^{\alpha s}) f'(b) \right| \right\} dt \\ \leq \int_{0}^{1} \left(2^{\alpha} - (2-t)^{\alpha} - \lambda \right) \left\{ \left| t^{\alpha s} f'\left(\frac{a+b}{2}\right) + m(1-t^{\alpha s}) f'(b) \right| \right\} dt \\ = M_{3} \left| f'\left(\frac{a+b}{2}\right) \right| + m(M_{4} - M_{3}) \left| f'(b) \right| . \end{cases}$$

Corollary 1 Under the assumptions Theorem 4, with $\alpha = s = m = 1$,

$$\left| (1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) dx \right|$$

$$\leq \frac{b-a}{8} \left[2\lambda^{2} - 2\lambda + 1 \right] \left(\left| f'(a) \right| + \left| f'(b) \right| \right).$$
(7)

Remark 1 Taking $\lambda = 1$, inequality (7) reduces to inequality (1).

Remark 2 Taking $\lambda = 0$, inequality (7) reduces to inequality (2).

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 5 Let f be defined as in Theorem 4 and suppose $|f'|^q$ is a convex on [a, b], with $q \ge 1$. Then the following inequality holds:

$$\left| \left(1 - \frac{2}{2^{\alpha}} \lambda \right) f\left(\frac{a+b}{2} \right) + \lambda \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^{+}}^{\alpha}f(b) + J_{b^{-}}^{\alpha}f(a) \right] \right| \\
\leq \frac{(b-a)}{2^{\alpha}} \\
\times \left[(M_{2})^{1-1/q} \left\{ \left(M_{1} \left| f'(b) \right|^{q} + m(M_{2} - M_{1}) \left| f'\left(\frac{a+b}{2} \right) \right|^{q} \right)^{1/q} \\
+ \left(M_{1} \left| f'(a) \right|^{q} + m(M_{2} - M_{1}) \left| f'\left(\frac{a+b}{2} \right) \right|^{q} \right)^{1/q} \right\} \\
+ (M_{4})^{1-1/q} \left\{ \left(M_{3} \left| f'(b) \right|^{q} + m(M_{4} - M_{3}) \left| f'\left(\frac{a+b}{2} \right) \right|^{q} \right)^{1/q} \\
+ \left(M_{3} \left| f'(b) \right|^{q} + m(M_{4} - M_{3}) \left| f'\left(\frac{a+b}{2} \right) \right|^{q} \right)^{1/q} \right\} \right].$$
(8)

Proof Using the well-known power-mean integral inequality for q > 1 and convexity of $|f'|^q$, we have

$$\begin{split} |L_{1}| &\leq \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| dt\right)^{1-1/q} \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^{q} dt\right)^{1/q} \\ &\leq \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| dt\right)^{1-1/q} \\ &\times \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| dt\right)^{q} + m(M_{2}-M_{1}) \left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)^{1/q} \\ &\leq (M_{2})^{1-1/q} \left(M_{1}|f'(a)|^{q} + m(M_{2}-M_{1}) \left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)^{1/q}, \\ |L_{2}| &\leq \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| dt\right)^{1-1/q} \\ &\times \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| dt\right)^{1-1/q} \\ &\leq \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| dt\right)^{1-1/q} \\ &\times \left(\int_{0}^{1} \left|\left((1-t)^{\alpha}-\lambda\right)\right| dt\right)^{1-1/q} \\ &\leq (M_{2})^{1-1/q} \left(M_{1}|f'(b)|^{q} + m(M_{2}-M_{1}) \left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)^{1/q}, \\ |L_{3}| &\leq \left(\int_{0}^{1} \left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right| dt\right)^{1-1/q} \\ &\times \left(\int_{0}^{1} \left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right| dt\right)^{1-1/q} \\ &\leq \left(\int_{0}^{1} \left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right| dt\right)^{1-1/q} \\ &\leq \left(\int_{0}^{1} \left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right| dt\right)^{1-1/q} \end{split}$$

$$\times \left(\int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda \right| \left\{ t^{\alpha s} \left| f'\left(\frac{a+b}{2}\right) \right|^{q} + m(1-t^{\alpha s}) \left| f'(a) \right|^{q} \right\} dt \right)^{1/q}$$

$$\leq (M_{4})^{1-1/q} \left(M_{3} \left| f'\left(\frac{a+b}{2}\right) \right|^{q} + m(M_{4} - M_{3}) \left| f'(a) \right|^{q} \right)^{1/q},$$

$$|L_{4}| \leq \left(\int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda \right| dt \right)^{1-1/q}$$

$$\times \left(\int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda \right| dt \right)^{1-1/q}$$

$$\leq \left(\int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda \right| dt \right)^{1-1/q}$$

$$\times \left(\int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda \right| dt \right)^{1-1/q}$$

$$\times \left(\int_{0}^{1} \left| 2^{\alpha} - (2-t)^{\alpha} - \lambda \right| dt \right)^{1-1/q}$$

$$\leq (M_{4})^{1-1/q} \left(M_{3} \left| f'\left(\frac{a+b}{2}\right) \right|^{q} + m(M_{4} - M_{3}) \left| f'(b) \right|^{q} \right)^{1/q}.$$

This completes the proof.

Corollary 2 Under the assumptions Theorem 5, with $\alpha = s = m = 1$, $\lambda = 0$ in inequality (8), the following inequality holds:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right|$$

$$\leq \frac{b-a}{8} \left[\left(\frac{1}{3} \left| f'(a) \right| + \frac{2}{3} \left| f'\left(\frac{a+b}{2}\right) \right|^{q} \right)^{\frac{1}{q}} + \left(\frac{1}{3} \left| f'(b) \right| + \frac{2}{3} \left| f'\left(\frac{a+b}{2}\right) \right|^{q} \right)^{\frac{1}{q}} \right].$$

Corollary 3 Under the assumptions Theorem 5, with $\alpha = s = m = 1$, $\lambda = 1$ in inequality (8), the following inequality holds:

$$\frac{|f(a)+f(b)|}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \bigg|$$

$$\leq \frac{b-a}{8} \bigg[\bigg(\frac{2}{3} |f'(a)| + \frac{1}{3} \bigg| f'\bigg(\frac{a+b}{2} \bigg) \bigg|^{q} \bigg)^{\frac{1}{q}} + \bigg(\frac{2}{3} |f'(b)| + \frac{1}{3} \bigg| f'\bigg(\frac{a+b}{2} \bigg) \bigg|^{q} \bigg)^{\frac{1}{q}} \bigg].$$

Corollary 4 Under the assumptions Theorem 5, with $\alpha = s = m = 1$, $\lambda = \frac{1}{3}$ in inequality (8), the following inequality holds:

$$\begin{aligned} \left| \frac{1}{3} \left\{ 2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right\} &- \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \\ &\leq \frac{5(b-a)}{72} \times \left[\left(\frac{16}{45} \left| f'(a) \right|^{q} + \frac{29}{45} \left| f'\left(\frac{a+b}{2}\right) \right|^{q} \right)^{\frac{1}{q}} \right] \\ &+ \left(\frac{16}{45} \left| f'(b) \right|^{q} + \frac{29}{45} \left| f'\left(\frac{a+b}{2}\right) \right|^{q} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Theorem 6 Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with a < b such that f' is integrable. If $|f'|^q$ is concave on [a,b], then the following inequality for Riemann–Liouville

fractional integrals holds with $0 < \alpha \le 1$ *:*

$$\begin{split} \left(1 - \frac{2}{2^{\alpha}}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^{+}}^{\alpha}f(b) + J_{b^{-}}^{\alpha}f(a)\right] \\ &\leq \frac{(b-a)}{2^{\alpha+1}} \times \left[M_{2}\left\{\left|f'\left(\frac{M_{5}a+(M_{2}-M_{5})(\frac{a+b}{2})}{M_{2}}\right)\right| + \left|f'\left(\frac{M_{5}b+(M_{2}-M_{5})(\frac{a+b}{2})}{M_{2}}\right)\right|\right\} \\ &+ M_{4}\left|f'\left(\frac{M_{6}(\frac{a+b}{2})+(M_{4}-M_{6})a}{M_{4}}\right)\right| + \left|f'\left(\frac{M_{6}(\frac{a+b}{2})+(M_{4}-M_{6})b}{M_{4}}\right)\right|\right], \qquad (9) \\ M_{5} = \int_{0}^{1}\left|(1-t)^{\alpha}-\lambda\right|t\,dt \\ &= \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1} + \frac{\lambda}{2}(1-2\zeta^{2}), \\ M_{6} = \int_{0}^{1}\left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right|t\,dt \\ &= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1} + \left(2^{\alpha}-\lambda\right)\left(\frac{1}{2}-\xi^{2}\right). \end{split}$$

Proof Using the concavity of $|f'|^q$ and the power-mean inequality, we obtain

$$\begin{split} \left|f'(\lambda a + (1-\lambda)b)\right|^q &> \lambda \left|f'(a)\right|^q + (1-\lambda) \left|f'(b)\right|^q \\ &\geq \left(\lambda \left|f'(a)\right| + (1-\lambda)f'(b)\right|\right)^q. \end{split}$$

Hence

$$\left|f'(\lambda a + (1-\lambda)b)\right| \ge \lambda \left|f'(a)\right| + (1-\lambda)\left|f'(b)\right|,$$

so $|f^\prime|$ is also concave. By the Jensen integral inequality, we have

$$\begin{split} & \left(1 - \frac{2}{2^{\alpha}}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a) + f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a^{+}}^{\alpha}f(b) + J_{b^{-}}^{\alpha}f(a)] \\ & \leq \left(\int_{0}^{1} \left|\left((1-t)^{\alpha} - \lambda\right)\right| dt\right) \left| f'\left(\frac{\int_{0}^{1} \left|\left((1-t)^{\alpha} - \lambda\right)\right| (ta + (1-t)\frac{a+b}{2}) dt}{\int_{0}^{1} \left|\left((1-t)^{\alpha} - \lambda\right)\right| dt}\right) \right| \\ & + \left(\int_{0}^{1} \left|\left((1-t)^{\alpha} - \lambda\right)\right| dt\right) \left| f'\left(\frac{\int_{0}^{1} \left|\left((1-t)^{\alpha} - \lambda\right)\right| (tb + (1-t)\frac{a+b}{2}) dt}{\int_{0}^{1} \left|\left((1-t)^{\alpha} - \lambda\right)\right| dt}\right) \right| \\ & + \left(\int_{0}^{1} \left|\left(2^{\alpha} - (2-t)^{\alpha} - \lambda\right)\right| dt\right) \left| f'\left(\frac{\int_{0}^{1} \left|\left(2^{\alpha} - (2-t)^{\alpha} - \lambda\right)\right| (t\frac{a+b}{2} + (1-t)a dt)}{\int_{0}^{1} \left|\left(2^{\alpha} - (2-t)^{\alpha} - \lambda\right)\right| dt}\right) \right| \\ & + \left(\int_{0}^{1} \left|\left(2^{\alpha} - (2-t)^{\alpha} - \lambda\right)\right| dt\right) \left| f'\left(\frac{\int_{0}^{1} \left|\left(2^{\alpha} - (2-t)^{\alpha} - \lambda\right)\right| (t\frac{a+b}{2} + (1-t)b dt)}{\int_{0}^{1} \left|\left(2^{\alpha} - (2-t)^{\alpha} - \lambda\right)\right| dt}\right) \right| , \\ & \left(1 - \frac{2}{2^{\alpha}}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a) + f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^{+}}^{\alpha}f(b) + J_{b^{-}}^{\alpha}f(a)\right] \\ & = M_{2} \left| f'\left(\frac{M_{5}a + (M_{2} - M_{5})(\frac{a+b}{2})}{M_{2}}\right) \right| + M_{2} \left| f'\left(\frac{M_{5}b + (M_{2} - M_{5})(\frac{a+b}{2})}{M_{2}}\right) \right| \end{split}$$

$$+ M_4 \left| f' \left(\frac{M_6(\frac{a+b}{2}) + (M_4 - M_6)a}{M_4} \right) \right| + M_4 \left| f' \left(\frac{M_6(\frac{a+b}{2}) + (M_4 - M_6)b}{M_4} \right) \right|.$$

Remark 3 Under the assumptions Theorem 6, with $\alpha = s = 1$, $\lambda = 0$ in inequality (9), we obtain inequality (4).

Remark 4 Under the assumptions Theorem 6, with $\alpha = s = 1$, $\lambda = 1$ in inequality (9), we obtain inequality (5).

Corollary 5 Under the assumptions Theorem 6, with $\alpha = s = 1$, $\lambda = \frac{1}{3}$ in inequality (9), the following inequality holds:

$$\left| \left[\frac{1}{6} f(a) + \frac{2}{3} f\left(\frac{a+b}{2}\right) + \frac{1}{6} f(b) \right] - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \\
\leq \frac{5(b-a)}{72} \left[\left| f'\left(\frac{29a+61b}{90}\right) \right| + \left| f'\left(\frac{61a+29b}{90}\right) \right| \right].$$
(10)

Remark 5 Inequality (10) is a generalization of the inequality obtained in [3, Theorem 8].

3 Applications to special means

We consider some means for arbitrary positive real numbers a and b (see, for instance, [4]):

• The arithmetic mean

$$A = A(a,b) = \frac{a+b}{2}, \quad a,b \in \mathbb{R} \text{ with } a,b > 0;$$

• The geometric mean

$$G = G(a, b) = \sqrt{ab}, \quad a, b \in \mathbb{R} \text{ with } a, b > 0;$$

• The harmonic mean

$$H = H(a, b) = \frac{2ab}{a+b}, \quad a, b \in \mathbb{R} \setminus \{0\};$$

• The identric mean

$$I = I(a, b) = \begin{cases} a & \text{if } a = b, \\ \frac{1}{e} (\frac{b^b}{a^a})^{\frac{1}{b-a}} & \text{if } a \neq b, a, b > 0; \end{cases}$$

• The logarithmic mean

$$L = L(a, b) = \begin{cases} a & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b; \end{cases}$$

· Generalized logarithmic mean

$$L_n(a,b) = \begin{cases} a & \text{if } a = b, \\ \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} & \text{if } a \neq b, n \in \mathbb{Z} \setminus \{-1,0\}, a, b > 0. \end{cases}$$

Proposition 3 Let $n \in \mathbb{Z} \setminus \{-1, 0\}$ and a, b > 0. Then we have the following inequality:

$$\left|(1-\lambda)A^n(a,b)+\lambda A(a^n,b^n)-L_n^n(a,b)\right|\leq \frac{n(b-a)}{4}(2\lambda^2-2\lambda+1)A(\left|a^{n-1}\right|,\left|b^{n-1}\right|).$$

For $\lambda = 1$, we have

$$\left|\lambda A(a^n,b^n)-L_n^n(a,b)\right| \leq \frac{n(b-a)}{4} \left(2\lambda^2-2\lambda+1\right) A\left(\left|a^{n-1}\right|,\left|b^{n-1}\right|\right).$$

For $\lambda = 0$, we have

$$\left|\lambda A^n(a^n,b^n)-L_n^n(a,b)\right|\leq \frac{n(b-a)}{4}(2\lambda^2-2\lambda+1)A(\left|a^{n-1}\right|,\left|b^{n-1}\right|).$$

Proof The assertion follows from Corollary 1 for $f(x) = x^n$ and with *n* as specified above. \Box

Proposition 4 For all $0 < a \le b$, we have the following inequality:

$$\left|\ln\left[A(1+a,1+b)\right]I(1+b,1+a)\right| \leq \frac{2(b-a)}{3^{1+\frac{1}{q}}} \left(\frac{1}{1+a} + \frac{2}{A^q(1+b,1+a)}\right)^{\frac{1}{q}}.$$

Proof The first assertion follows from Corollary 2 for $f(x) = -\ln(1+x)x^n$.

$$\left|\ln\left[G(1+a,1+b)\right]I(1+b,1+a)\right| \leq \frac{2(b-a)}{3^{1+\frac{1}{q}}} \left(\frac{1}{1+a} + \frac{2}{A^q(1+b,1+a)}\right)^{\frac{1}{q}}.$$

The second assertion follows from Corollary 3 for $f(x) = -\ln(1 + x)$. Let $n \in \mathbb{Z} \setminus \{-1, 0\}$; a, b > 0 then we have the following inequality

$$\left|2A^{n}(a,b)+A\left(a^{n},b^{n}\right)-3L(a,b)\right| \leq \frac{5n(b-a)}{12}\frac{\left[29A^{q(n-1)}(a,b)+16b^{q(n-1)}\right]}{45^{\frac{1}{q}}}.$$

The third assertion follows from Corollary 4 for $f(x) = x^n$ and as *n* as specified above. \Box

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Competing interests

Authors have declared no conflict of interest.

Authors' contributions

Dr. SQ and Dr. JN analyzed the problem and suggested mathematical modeling. Dr. SIB and Dr. MB generalized the results by proposing various lemmas. This manuscript has been written by SEF, Prof. Dr. FA and Prof. Dr. XL. Both the mentioned professors also made some necessary corrections regarding mathematical formulations and response to the reviewer. All authors had read and approved the final manuscript.

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