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Positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions

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Abstract

In this paper we use the fixed point index to study the existence of positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions. Here we use appropriate nonnegative matrices to depict the coupling behavior for our nonlinearities.

MSC: 34B18; 34B10; 34B15

Keywords: Hadamard fractional differential equations; Integral boundary conditions; Positive solutions; Fixed point index

1 Introduction

In this paper we consider the system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions

$$\begin{cases} D^{\beta}u(t)+f_1(t,u(t),v(t))=0, & 1<t<e,\\ D^{\beta}v(t)+f_2(t,u(t),v(t))=0, & 1<t<e,\\ u(1)=v(1)=u'(1)=v'(1)=0,\\ u(e)=\int_1^e h(s)v(s)\frac{ds}{s},\\ v(e)=\int_1^e g(s)u(s)\frac{ds}{s}, \end{cases} \tag{1.1}$$

where $\beta\in(2,3]$, D^{β} is the Hadamard fractional derivative of fractional order β, and f_i $(i=1,2)$, h, g satisfy the following conditions:

(H1) f_i $(i=1,2)$ are nonnegative continuous functions on $[1,e]\times\mathbb{R}^+\times\mathbb{R}^+$,

(H2) $h,g\geq 0$ $(\not\equiv 0)$ on $[1,e]$ with $\int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t}\cdot\int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t}\in(0,1)$.

Fractional-order differential equations is a rapidly developing area of research; we refer the reader to [1–48] and the references therein. In [1–9], the authors used iterative techniques to study existence and uniqueness of solutions for fractional boundary value problems. In [1] the authors studied positive solutions for the p-Laplacian fractional Riemann–

Stieltjes integral boundary value problem

$$\begin{cases} -D_t^{\beta}(\varphi_p(-D_t^{\alpha}z))(t) = f(t, z(t), D_t^{\gamma}z(t)), \quad t \in (0,1), \\ D_t^{\alpha}z(0) = D_t^{\alpha+1}z(0) = D_t^{\gamma}z(0) = 0, \\ D_t^{\alpha}z(1) = 0, \qquad D_t^{\gamma}z(1) = \int_0^1 D_t^{\gamma}z(s)\,dA(s), \end{cases} \tag{1.2}$$

where D_t^{α}, D_t^{β}, D_t^{γ} are the Riemann–Liouville fractional derivatives, and they not only obtained existence and uniqueness of positive solutions for (1.2), but also constructed an iteration sequence for the unique positive solution. In [10–32], the authors used fixed point methods to study the existence of (positive) solutions fractional order equations. In [10] Mawhin's continuation theorem was used to study the following fractional order boundary value problem at resonance:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), x'(t)), \quad t \in [0,T], \\ x(0) = \alpha I_{\eta}^{\gamma,\delta}x(\zeta), \qquad x(T) = \beta\, {}^{\rho}I^{p}x(\xi), \quad 0 < \zeta, \xi \le T, \end{cases} \tag{1.3}$$

where ${}^cD^q$ is the Caputo fractional derivative, $I_{\eta}^{\gamma,\delta}$ is a Erdélyi–Kober type integral, and ${}^{\rho}I^{p}$ denotes the generalized Riemann–Liouville type integral boundary conditions. For fractional differential systems, see [23–32]. In [23], using the Leray–Schauder alternative and the Banach contraction principle, the authors studied existence and uniqueness of solutions for the system of nonlinear Caputo type sequential fractional integro-differential equations

$$\begin{cases} ({}^cD^{\alpha} + \lambda\,{}^cD^{\alpha-1})u(t) = f(t, u(t), v(t), {}^cD^{p_1}v(t), I^{q_1}v(t)), \quad t \in (0,1), \\ ({}^cD^{\beta} + \mu\,{}^cD^{\beta-1})v(t) = g(t, u(t), {}^cD^{p_2}u(t), I^{q_2}u(t), v(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, \qquad u(1) = \int_0^1 u(s)\,dH_1(s) + \int_0^1 v(s)\,dH_2(s), \\ v(0) = v'(0) = v''(0) = 0, \qquad v(1) = \int_0^1 u(s)\,dK_1(s) + \int_0^1 v(s)\,dK_2(s). \end{cases} \tag{1.4}$$

Hadamard fractional differential equations are also popular in the literature; see [33–48] and the references therein. In [33], the authors used the Banach contraction principle, the Leray–Schauder's alternative, and Krasnoselskii's fixed-point theorem to study the existence and uniqueness of solutions for the coupled system of nonlinear sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions

$$\begin{cases} {}^CD^{p_1}\,{}^HD^{q_1}x(t) = f(t, x(t), y(t)), \quad t \in [a,b], \\ {}^HD^{q_2}\,{}^CD^{p_2}x(t) = g(t, x(t), y(t)), \quad t \in [a,b], \\ \alpha_1 x(a) + \alpha_2\,{}^CD^{p_2}y(a) = 0, \qquad \beta_1 x(b) + \beta_2\,{}^CD^{p_2}y(b) = 0, \\ \alpha_3 y(a) + \alpha_4\,{}^HD^{q_1}x(a) = 0, \qquad \beta_3 y(b) + \beta_4\,{}^HD^{q_1}x(b) = 0, \end{cases} \tag{1.5}$$

where ${}^CD^{p_i}$, ${}^HD^{q_i}$ are respectively the Caputo and Hadamard fractional derivatives. In [34] the authors established positive solutions for the coupled Hadamard fractional integral boundary value problems

$$\begin{cases} D^\alpha u(t) + \lambda f(t,u(t),v(t)) = 0, \quad t \in (1,e), \lambda > 0, \\ D^\beta v(t) + \lambda g(t,u(t),v(t)) = 0, \quad t \in (1,e), \lambda > 0, \\ u^{(j)}(1) = v^{(j)}(1) = 0, \quad 0 \le j \le n-2, \\ u(e) = \mu \int_1^e v(s)\frac{ds}{s}, \\ v(e) = \nu \int_1^e u(s)\frac{ds}{s}, \end{cases} \tag{1.6}$$

where $\alpha, \beta \in (n-1,n]$ and $n \ge 3$, D^α, D^β are the Hadamard fractional derivatives and their nonlinearities f, g satisfy the following conditions:

$(\mathrm{H})_{\mathrm{Yang1}}$ There exists $[\theta_1,\theta_2] \subset (1,e)$ such that $\liminf_{u\to+\infty} \min_{t\in[\theta_1,\theta_2]} \frac{f(t,u,v)}{u} = +\infty$ and $\liminf_{v\to+\infty} \min_{t\in[\theta_1,\theta_2]} \frac{g(t,u,v)}{v} = +\infty$;

or

$(\mathrm{H})_{\mathrm{Yang2}}$ There exists $[\theta_1,\theta_2] \subset (1,e)$ such that $\liminf_{v\to+\infty} \min_{t\in[\theta_1,\theta_2]} \frac{f(t,u,v)}{v} = +\infty$ and $\liminf_{u\to+\infty} \min_{t\in[\theta_1,\theta_2]} \frac{g(t,u,v)}{u} = +\infty$.

Motivated by the above, in this paper we study the existence of positive solutions for the system of nonlinear Hadamard fractional differential equations (1.1) involving coupled integral boundary conditions. We use appropriate nonnegative matrices to depict the coupling behavior for our nonlinearities, and note that they can grow both superlinearly and sublinearly. We remark here that our conditions for nonlinear terms are not as restrictive as those in $(\mathrm{H})_{\mathrm{Yang1}}$ and $(\mathrm{H})_{\mathrm{Yang2}}$; see (H3)–(H6) in Sect. 3.

2 Preliminaries

In this section, we first provide some material for Hadamard fractional calculus; for details, see the book [49].

Definition 2.1 The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s)\frac{ds}{s}, \quad n-1 < q < n,$$

where $n = [q] + 1$; $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 The Hadamard fractional integral of order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t (\log t - \log s)^{q-1} g(s)\frac{ds}{s}, \quad q > 0,$$

provided the integral exists.

In what follows, we calculate the Green's functions associated with (1.1) and study some properties of these Green's functions.

Lemma 2.3 (see [34, Lemma 2.3]) *Let $x, y \in C[1,e]$. Then the integral boundary value problem*

$$\begin{cases} D^{\beta}u(t) + x(t) = 0, \qquad D^{\beta}v(t) + y(t) = 0, \quad t \in (1,e), \\ u(1) = v(1) = u'(1) = v'(1) = 0, \\ u(e) = \int_1^e h(s)v(s)\frac{ds}{s}, \\ v(e) = \int_1^e g(s)u(s)\frac{ds}{s}, \end{cases} \tag{2.1}$$

can be transformed into the following Hammerstein type integral equations:

$$\begin{cases} u(t) = \int_1^e G_1(t,s)x(s)\frac{ds}{s} + \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}x(s)\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}y(s)\frac{ds}{s}, \\ v(t) = \int_1^e G_1(t,s)y(s)\frac{ds}{s} + \frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}y(s)\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}x(s)\frac{ds}{s}, \end{cases} \tag{2.2}$$

where

$$G_1(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} (\log t)^{\beta-1}(1-\log s)^{\beta-1} - (\log t - \log s)^{\beta-1}, & 1 \le s \le t \le e, \\ (\log t)^{\beta-1}(1-\log s)^{\beta-1}, & 1 \le t \le s \le e; \end{cases} \tag{2.3}$$

here, $d_{g,h}$, d_g, d_h *are three positive constants defined in the proof.*

Proof From Lemma 2.3 of [34] we have

$$u(t) = c_{11}(\log t)^{\beta-1} + c_{12}(\log t)^{\beta-2} + c_{13}(\log t)^{\beta-3} - \frac{1}{\Gamma(\beta)}\int_1^t(\log t - \log s)^{\beta-1}x(s)\frac{ds}{s},$$
$$v(t) = c_{21}(\log t)^{\beta-1} + c_{22}(\log t)^{\beta-2} + c_{23}(\log t)^{\beta-3} - \frac{1}{\Gamma(\beta)}\int_1^t(\log t - \log s)^{\beta-1}y(s)\frac{ds}{s},$$

where $c_{1i}, c_{2i} \in \mathbb{R}$, $i = 1,2,3$. Note that $u(1) = v(1) = u'(1) = v'(1) = 0$ implies $c_{12}, c_{13}, c_{22}, c_{23} = 0$. Then we have

$$u(t) = c_{11}(\log t)^{\beta-1} - \frac{1}{\Gamma(\beta)}\int_1^t(\log t - \log s)^{\beta-1}x(s)\frac{ds}{s},$$
$$v(t) = c_{21}(\log t)^{\beta-1} - \frac{1}{\Gamma(\beta)}\int_1^t(\log t - \log s)^{\beta-1}y(s)\frac{ds}{s}.$$

Using the conditions $u(e) = \int_1^e h(s)v(s)\frac{ds}{s}$, $v(e) = \int_1^e g(s)u(s)\frac{ds}{s}$, we obtain

$$\begin{aligned} &c_{11} - \frac{1}{\Gamma(\beta)}\int_1^e(1-\log s)^{\beta-1}x(s)\frac{ds}{s} \\ &\quad = c_{21}\int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t} - \frac{1}{\Gamma(\beta)}\int_1^e h(t)\int_1^t(\log t - \log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t}, \\ &c_{21} - \frac{1}{\Gamma(\beta)}\int_1^e(1-\log s)^{\beta-1}y(s)\frac{ds}{s} \\ &\quad = c_{11}\int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t} - \frac{1}{\Gamma(\beta)}\int_1^e g(t)\int_1^t(\log t - \log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t}. \end{aligned}$$

This implies that

$$\begin{bmatrix} 1 & -\int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t} \\ -\int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t} & 1 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}x(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^e h(t)\int_1^t (\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t} \\ \frac{1}{\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}y(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^e g(t)\int_1^t (\log t-\log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t} \end{bmatrix}.$$

Let $d_{g,h} = 1 - \int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t} \cdot \int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t}$, $d_h = \int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t}$, $d_g = \int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t}$. Then

$$\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \frac{1}{d_{g,h}}\begin{bmatrix} 1 & \int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t} \\ \int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t} & 1 \end{bmatrix} \\ \cdot \begin{bmatrix} \frac{1}{\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}x(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^e h(t)\int_1^t (\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t} \\ \frac{1}{\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}y(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^e g(t)\int_1^t (\log t-\log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t} \end{bmatrix}.$$

Consequently, we have

$$\begin{aligned} u(t) ={}& \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}x(s)\frac{ds}{s} \\ &- \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e h(t)\int_1^t (\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t} \\ &+ \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}y(s)\frac{ds}{s} \\ &- \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e g(t)\int_1^t (\log t-\log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t} \\ &- \frac{1}{\Gamma(\beta)}\int_1^t (\log t-\log s)^{\beta-1}x(s)\frac{ds}{s} \\ ={}& \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t}\int_1^e (1-\log s)^{\beta-1}y(s)\frac{ds}{s} \\ &- \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e h(t)\int_1^t (\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t} \\ &+ \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}x(s)\frac{ds}{s} \\ &- \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e g(t)\int_1^t (\log t-\log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t} \\ &- \frac{1}{\Gamma(\beta)}\int_1^t (\log t-\log s)^{\beta-1}x(s)\frac{ds}{s} + \frac{(\log t)^{\beta-1}}{\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}x(s)\frac{ds}{s} \\ &- \frac{(\log t)^{\beta-1}}{\Gamma(\beta)}\int_1^e (1-\log s)^{\beta-1}x(s)\frac{ds}{s} \\ ={}& \int_1^e G_1(t,s)x(s)\frac{ds}{s} + \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\biggl[\int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t}\int_1^e (1-\log s)^{\beta-1}x(s)\frac{ds}{s} \\ &- \int_1^e g(t)\int_1^t (\log t-\log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t}\biggr] \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\biggl[\int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t}\int_1^e(1-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
&-\int_1^e h(t)\int_1^t(\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t}\biggr]\\
={}&\int_1^e G_1(t,s)x(s)\frac{ds}{s}+\frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\biggl[\int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t}\int_1^e(1-\log s)^{\beta-1}x(s)\frac{ds}{s}\\
&-\int_1^e g(t)\int_s^e(\log t-\log s)^{\beta-1}x(s)\frac{dt}{t}\frac{ds}{s}\biggr]\\
&+\frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\biggl[\int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t}\int_1^e(1-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
&-\int_1^e h(t)\int_s^e(\log t-\log s)^{\beta-1}y(s)\frac{dt}{t}\frac{ds}{s}\biggr]\\
={}&\int_1^e G_1(t,s)x(s)\frac{ds}{s}+\frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}x(s)\frac{ds}{s}\\
&+\frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}y(s)\frac{ds}{s}.
\end{aligned}
$$

Similarly, we also obtain that

$$
\begin{aligned}
v(t)={}&c_{21}(\log t)^{\beta-1}-\frac{1}{\Gamma(\beta)}\int_1^t(\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\beta)}\int_1^e(\log t)^{\beta-1}(1-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
&-\frac{1}{\Gamma(\beta)}\int_1^e(\log t)^{\beta-1}(1-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
={}&\int_1^e G_1(t,s)y(s)\frac{ds}{s}-\frac{1}{\Gamma(\beta)}\int_1^e(\log t)^{\beta-1}(1-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
&+\frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e(1-\log s)^{\beta-1}x(s)\frac{ds}{s}\\
&-\frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e h(t)\int_1^t(\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t}\\
&+\frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e(1-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
&-\frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e g(t)\int_1^t(\log t-\log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t}\\
={}&\int_1^e G_1(t,s)y(s)\frac{ds}{s}+\frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\biggl[\int_1^e h(t)(\log t)^{\beta-1}\frac{dt}{t}\int_1^e(1-\log s)^{\beta-1}y(s)\frac{ds}{s}\\
&-\int_1^e h(t)\int_1^t(\log t-\log s)^{\beta-1}y(s)\frac{ds}{s}\frac{dt}{t}\biggr]\\
&+\frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\biggl[\int_1^e g(t)(\log t)^{\beta-1}\frac{dt}{t}\int_1^e(1-\log s)^{\beta-1}x(s)\frac{ds}{s}\\
&-\int_1^e g(t)\int_1^t(\log t-\log s)^{\beta-1}x(s)\frac{ds}{s}\frac{dt}{t}\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_1^e G_1(t,s)y(s)\frac{ds}{s} + \frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}y(s)\frac{ds}{s} \\
&\quad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}x(s)\frac{ds}{s}.
\end{aligned}
$$

This completes the proof. □

From Lemma 2.3, we note (1.1) is equivalent to the Hammerstein type integral equations

$$
\begin{cases}
u(t) = \int_1^e G_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} \\
\qquad + \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s} \\
\qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s}, \\
v(t) = \int_1^e G_1(t,s)f_2(s,u(s),v(s))\frac{ds}{s} \\
\qquad + \frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s} \\
\qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s}.
\end{cases}
\tag{2.4}
$$

Lemma 2.4 *The function* $G_1(t,s)$ *satisfies the following inequalities*:

(I1) $\frac{1}{\Gamma(\beta)}(\log t)^{\beta-1}(1-\log t)\log s(1-\log s)^{\beta-1} \le G_1(t,s) \le \frac{\beta-1}{\Gamma(\beta)}\log s(1-\log s)^{\beta-1}$ *for* $t,s \in [1,e]$,

(I2) $\frac{1}{\Gamma(\beta)}(\log t)^{\beta-1}(1-\log t)\log s(1-\log s)^{\beta-1} \le G_1(t,s) \le \frac{\beta-1}{\Gamma(\beta)}(\log t)^{\beta-1}(1-\log t)$ *for* $t,s \in [1,e]$.

Proof We note a result from [14]. Let $\beta \in (n-1,n]$ with $n \in \mathbb{N}$, $n \ge 3$. Then the function

$$
G(z,l) = \frac{1}{\Gamma(\beta)}
\begin{cases}
z^{\beta-1}(1-l)^{\beta-1} - (z-l)^{\beta-1}, & 0 \le l \le z \le 1, \\
z^{\beta-1}(1-l)^{\beta-1}, & 0 \le z \le l \le 1,
\end{cases}
$$

has the following properties:

(R1) $G(z,l) = G(1-l,1-z)$ for $z,l \in [0,1]$;

(R2) $\Gamma(\beta)k(z)q(l) \le G(z,l) \le (\beta-1)q(l)$ for $z,l \in [0,1]$;

(R3) $\Gamma(\beta)k(z)q(l) \le G(z,l) \le (\beta-1)k(z)$ for $z,l \in [0,1]$, where $k(z) = \frac{z^{\beta-1}(1-z)}{\Gamma(\beta)}$, $q(l) = \frac{l(1-l)^{\beta-1}}{\Gamma(\beta)}$.

Now, we turn our attention to G_1. If $\log t$, $\log s$ are regarded as z, l, then from (R2), (R3) we have

$$
\begin{aligned}
&\Gamma(\beta)k(\log t)q(\log s) \le G(\log t,\log s) \le (\beta-1)q(\log s), \\
&\Gamma(\beta)k(\log t)q(\log s) \le G(\log t,\log s) \le (\beta-1)k(\log t), \quad \text{for } t,s \in [1,e].
\end{aligned}
$$

Thus (I1), (I2) hold. This completes the proof. □

Let $\mu(t) = \frac{1}{\Gamma(\beta)}\log t(1-\log t)^{\beta-1}$ for $t \in [1,e]$.

Lemma 2.5 *Let* $\kappa_1 = \frac{\beta^2\Gamma(\beta)}{\Gamma(2\beta+2)}$, $\kappa_2 = \frac{\beta-1}{\Gamma(\beta+2)}$. *Then, for any* $s \in [1,e]$, *the following inequalities hold*:

$$
\kappa_1\mu(s) \le \int_1^e G_1(t,s)\mu(t)\frac{dt}{t} \le \kappa_2\mu(s).
\tag{2.5}
$$

This is a direct result from Lemma 2.4(I1), so we omit its proof.

Let $E := C[1,e]$, $\|u\| := \max_{t\in[1,e]} |u(t)|$, $P := \{u \in E : u(t) \geq 0, \forall t \in [1,e]\}$. Then $(E, \|\cdot\|)$ is a real Banach space and P is a cone on E. From Lemma 2.3 and (2.4), we define operators $A_i : P \times P \to P$ as follows:

$$\begin{cases} A_1(u,v)(t) = \int_1^e G_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s}, \\ A_2(u,v)(t) = \int_1^e G_1(t,s)f_2(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s}, \end{cases} \tag{2.6}$$

and

$$A(u,v)(t) = \bigl(A_1(u,v), A_2(u,v)\bigr)(t) \quad \text{for } t \in [1,e]. \tag{2.7}$$

Note $A_i : P \times P \to P$, $A : P \times P \to P \times P$ are completely continuous operators and (u,v) solves (1.1) if and only if (u,v) is a fixed point of the operator A.

Lemma 2.6 *Let* $P_0 = \{z \in P : z(t) \geq \frac{(\log t)^{\beta-1}(1-\log t)}{\beta-1}\|z\|, \forall t \in [1,e]\}$. *Then* P_0 *is also a cone on* E, *and* $A_i(P \times P) \subset P_0$, $i = 1,2$.

Proof We only prove $A_1(P \times P) \subset P_0$. From Lemma 2.4(I1), for $t \in [1,e]$, we have

$$\begin{aligned} A_1(u,v)(t) &= \int_1^e G_1(t,s)f_1\bigl(s,u(s),v(s)\bigr)\frac{ds}{s} \\ &\quad + \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1\bigl(s,u(s),v(s)\bigr)\frac{ds}{s} \\ &\quad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2\bigl(s,u(s),v(s)\bigr)\frac{ds}{s} \\ &\leq \int_1^e \frac{\beta-1}{\Gamma(\beta)} \log s(1-\log s)^{\beta-1}f_1\bigl(s,u(s),v(s)\bigr)\frac{ds}{s} \\ &\quad + \frac{d_h}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1\bigl(s,u(s),v(s)\bigr)\frac{ds}{s} \\ &\quad + \frac{1}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2\bigl(s,u(s),v(s)\bigr)\frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} A_1(u,v)(t) &\geq \int_1^e \frac{(\log t)^{\beta-1}(1-\log t)}{\Gamma(\beta)} \log s(1-\log s)^{\beta-1}f_1\bigl(s,u(s),v(s)\bigr)\frac{ds}{s} \\ &\quad + \frac{d_h(\log t)^{\beta-1}(1-\log t)}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1\bigl(s,u(s),v(s)\bigr)\frac{ds}{s} \\ &\quad + \frac{(\log t)^{\beta-1}(1-\log t)}{d_{g,h}\Gamma(\beta)} \int_1^e \int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2\bigl(s,u(s),v(s)\bigr)\frac{ds}{s}. \end{aligned}$$

Note that $\beta - 1 > 1$, so we have

$$A_1(u,v)(t) \geq \frac{(\log t)^{\beta-1}(1-\log t)}{\beta - 1}\|A_1(u,v)\| \quad \text{for } u,v \in P, t \in [1,e].$$

This completes the proof. □

Lemma 2.7 (see [50]) *Let E be a real Banach space and P be a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If there exists $\omega_0 \in P \backslash \{0\}$ such that*

$$\omega - A\omega \neq \lambda\omega_0, \quad \forall \lambda \geq 0, \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 2.8 (see [50]) *Let E be a real Banach space and P be a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \forall \lambda \in [0,1], \omega \in \partial\Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

3 Main results

Let

$$\begin{aligned}
\kappa_3 &= \frac{\beta}{d_{g,h}\Gamma(2\beta+1)}\int_1^e g(t)(\log t)^{\beta-1}(1-\log t)\frac{dt}{t},\\
\kappa_4 &= \frac{\beta}{d_{g,h}\Gamma(2\beta+1)}\int_1^e h(t)(\log t)^{\beta-1}(1-\log t)\frac{dt}{t},\\
\kappa_5 &= \frac{\beta(\beta-1)}{d_{g,h}\Gamma(2\beta+1)}\int_1^e g(t)\frac{dt}{t}, \qquad \kappa_6 = \frac{\beta(\beta-1)}{d_{g,h}\Gamma(2\beta+1)}\int_1^e h(t)\frac{dt}{t}.
\end{aligned}$$

Now we list our assumptions for the nonlinearities f_i $(i = 1,2)$.

(H3) There are $a_{1i}, b_{1i} \geq 0$ $(i = 1,2)$ and $l_1, l_2 > 0$ such that

$$\begin{gathered}
a_{11}(\kappa_1 + \kappa_3 d_h) + a_{12}\kappa_4 < 1, \qquad b_{12}(\kappa_1 + d_g\kappa_4) + b_{11}\kappa_3 < 1,\\
\det\begin{pmatrix} b_{11}(\kappa_1 + \kappa_3 d_h) + b_{12}\kappa_4 & a_{11}(\kappa_1 + \kappa_3 d_h) + a_{12}\kappa_4 - 1 \\ b_{12}(\kappa_1 + d_g\kappa_4) + b_{11}\kappa_3 - 1 & a_{12}(\kappa_1 + d_g\kappa_4) + a_{11}\kappa_3 \end{pmatrix} > 0,\\
\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \geq \begin{pmatrix} a_{11}x + b_{11}y - l_1 \\ a_{12}x + b_{12}y - l_2 \end{pmatrix}, \quad \forall (t,x,y) \in [1,e] \times \mathbb{R}^+ \times \mathbb{R}^+.
\end{gathered}$$

(H4) There are $a_{2i}, b_{2i} \geq 0$ $(i = 1,2)$ and $r_1 > 0$ such that

$$(\kappa_2 + \kappa_5 d_h)a_{21} + \kappa_6 a_{22} < 1, \qquad (\kappa_2 + d_g\kappa_6)b_{22} + \kappa_5 b_{21} < 1,$$

$$\det\begin{pmatrix} 1 - (\kappa_2 + \kappa_5 d_h)a_{21} - \kappa_6 a_{22} & -(\kappa_2 + \kappa_5 d_h)b_{21} - \kappa_6 b_{22} \\ -(\kappa_2 + d_g\kappa_6)a_{22} - \kappa_5 a_{21} & 1 - (\kappa_2 + d_g\kappa_6)b_{22} - \kappa_5 b_{21} \end{pmatrix} > 0,$$

$$\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \leq \begin{pmatrix} a_{21}x + b_{21}y \\ a_{22}x + b_{22}y \end{pmatrix}, \quad \forall (t,x,y) \in [1,e] \times [0,r_1] \times [0,r_1].$$

(H5) There are $a_{3i}, b_{3i} \geq 0$ $(i = 1,2)$ and $r_2 > 0$ such that

$$a_{31}(\kappa_1 + \kappa_3 d_h) + a_{32}\kappa_4 < 1, \qquad b_{32}(\kappa_1 + d_g\kappa_4) + b_{31}\kappa_3 < 1,$$

$$\det\begin{pmatrix} b_{31}(\kappa_1 + \kappa_3 d_h) + b_{32}\kappa_4 & a_{31}(\kappa_1 + \kappa_3 d_h) + a_{32}\kappa_4 - 1 \\ b_{32}(\kappa_1 + d_g\kappa_4) + b_{31}\kappa_3 - 1 & a_{32}(\kappa_1 + d_g\kappa_4) + a_{31}\kappa_3 \end{pmatrix} > 0,$$

$$\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \geq \begin{pmatrix} a_{31}x + b_{31}y \\ a_{32}x + b_{32}y \end{pmatrix}, \quad \forall (t,x,y) \in [1,e] \times [0,r_2] \times [0,r_2].$$

(H6) There are $a_{4i}, b_{4i} \geq 0$ $(i = 1,2)$ and $l_3, l_4 > 0$ such that

$$(\kappa_2 + \kappa_5 d_h)a_{41} + \kappa_6 a_{42} < 1, \qquad (\kappa_2 + d_g\kappa_6)b_{42} + \kappa_5 b_{41} < 1,$$

$$\det\begin{pmatrix} 1 - (\kappa_2 + \kappa_5 d_h)a_{41} - \kappa_6 a_{42} & -(\kappa_2 + \kappa_5 d_h)b_{41} - \kappa_6 b_{42} \\ -(\kappa_2 + d_g\kappa_6)a_{42} - \kappa_5 a_{41} & 1 - (\kappa_2 + d_g\kappa_6)b_{42} - \kappa_5 b_{41} \end{pmatrix} > 0,$$

$$\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \leq \begin{pmatrix} a_{41}x + b_{41}y + l_3 \\ a_{42}x + b_{42}y + l_4 \end{pmatrix}, \quad \forall (t,x,y) \in [1,e] \times \mathbb{R}^+ \times \mathbb{R}^+.$$

Let $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

Theorem 3.1 *Suppose that* (H1)–(H4) *hold. Then* (1.1) *has a positive solution.*

Proof Let $S_1 = \{(u,v) \in P \times P : (u,v) = A(u,v) + \lambda(\varphi_1,\varphi_1), \forall \lambda \geq 0\}$, where φ_1 is a fixed element in P_0. We claim that S_1 is a bounded set in $P \times P$. Note if there exists $(u,v) \in S_1$ such that

$$u(t) = A_1(u,v)(t) + \lambda\varphi_1(t), \qquad v(t) = A_2(u,v)(t) + \lambda\varphi_1(t) \quad \text{for } t \in [1,e], \tag{3.1}$$

then this, together with Lemma 2.6, implies that

$$u, v \in P_0. \tag{3.2}$$

From (3.1) we have

$$u(t) \geq A_1(u,v)(t), \qquad v(t) \geq A_2(u,v)(t) \quad \text{for } t \in [1,e]. \tag{3.3}$$

From the definitions of A_i ($i = 1, 2$), multiplying by $\mu(t)$ and integrating from 1 to e, Lemmas 2.4 and 2.5 enable us to obtain

$$\begin{aligned}
&\begin{pmatrix} \int_1^e u(t)\mu(t)\frac{dt}{t} \\ \int_1^e v(t)\mu(t)\frac{dt}{t} \end{pmatrix} \\
&\quad\geq \begin{pmatrix} \int_1^e \mu(t)(\int_1^e G_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s})\frac{dt}{t} \\ \int_1^e \mu(t)(\int_1^e G_1(t,s)f_2(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s})\frac{dt}{t} \end{pmatrix} \\
&\quad\geq \begin{pmatrix} (\kappa_1+\kappa_3 d_h)\int_1^e \mu(t)f_1(t,u(t),v(t))\frac{dt}{t} + \kappa_4\int_1^e \mu(t)f_2(t,u(t),v(t))\frac{dt}{t} \\ (\kappa_1+d_g\kappa_4)\int_1^e \mu(t)f_2(t,u(t),v(t))\frac{dt}{t} + \kappa_3\int_1^e \mu(t)f_1(t,u(t),v(t))\frac{dt}{t} \end{pmatrix}.
\end{aligned} \tag{3.4}$$

Combining this with (H3), we have

$$\begin{aligned}
&\begin{pmatrix} \int_1^e u(t)\mu(t)\frac{dt}{t} \\ \int_1^e v(t)\mu(t)\frac{dt}{t} \end{pmatrix} \\
&\quad\geq \begin{pmatrix} (\kappa_1+\kappa_3 d_h)\int_1^e \mu(t)(a_{11}u(t)+b_{11}v(t)-l_1)\frac{dt}{t} \\ \qquad + \kappa_4\int_1^e \mu(t)(a_{12}u(t)+b_{12}v(t)-l_2)\frac{dt}{t} \\ (\kappa_1+d_g\kappa_4)\int_1^e \mu(t)(a_{12}u(t)+b_{12}v(t)-l_2)\frac{dt}{t} \\ \qquad + \kappa_3\int_1^e \mu(t)(a_{11}u(t)+b_{11}v(t)-l_1)\frac{dt}{t} \end{pmatrix},
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\begin{pmatrix} b_{11}(\kappa_1+\kappa_3 d_h)+b_{12}\kappa_4 & a_{11}(\kappa_1+\kappa_3 d_h)+a_{12}\kappa_4-1 \\ b_{12}(\kappa_1+d_g\kappa_4)+b_{11}\kappa_3-1 & a_{12}(\kappa_1+d_g\kappa_4)+a_{11}\kappa_3 \end{pmatrix}\begin{pmatrix} \int_1^e v(t)\mu(t)\frac{dt}{t} \\ \int_1^e u(t)\mu(t)\frac{dt}{t} \end{pmatrix} \\
&\quad\leq \begin{pmatrix} ((\kappa_1+\kappa_3 d_h)l_1+\kappa_4 l_2)\int_1^e \mu(t)\frac{dt}{t} \\ ((\kappa_1+d_g\kappa_4)l_2+\kappa_3 l_1)\int_1^e \mu(t)\frac{dt}{t} \end{pmatrix} = \begin{pmatrix} \frac{(\kappa_1+\kappa_3 d_h)l_1+\kappa_4 l_2}{\Gamma(\beta+2)} \\ \frac{(\kappa_1+d_g\kappa_4)l_2+\kappa_3 l_1}{\Gamma(\beta+2)} \end{pmatrix}.
\end{aligned}$$

Solving this matrix inequality, we have

$$\begin{pmatrix} \int_1^e v(t)\mu(t)\frac{dt}{t} \\ \int_1^e u(t)\mu(t)\frac{dt}{t} \end{pmatrix} \leq \frac{\begin{pmatrix} a_{12}(\kappa_1+d_g\kappa_4)+a_{11}\kappa_3 & 1-a_{11}(\kappa_1+\kappa_3 d_h)+a_{12}\kappa_4 \\ 1-b_{12}(\kappa_1+d_g\kappa_4)+b_{11}\kappa_3 & b_{11}(\kappa_1+\kappa_3 d_h)+b_{12}\kappa_4 \end{pmatrix}}{\det\begin{pmatrix} b_{11}(\kappa_1+\kappa_3 d_h)+b_{12}\kappa_4 & a_{11}(\kappa_1+\kappa_3 d_h)+a_{12}\kappa_4-1 \\ b_{12}(\kappa_1+d_g\kappa_4)+b_{11}\kappa_3-1 & a_{12}(\kappa_1+d_g\kappa_4)+a_{11}\kappa_3 \end{pmatrix}}\begin{pmatrix} \frac{(\kappa_1+\kappa_3 d_h)l_1+\kappa_4 l_2}{\Gamma(\beta+2)} \\ \frac{(\kappa_1+d_g\kappa_4)l_2+\kappa_3 l_1}{\Gamma(\beta+2)} \end{pmatrix}.$$

Hence, there exist $M_1 > 0$, $M_2 > 0$ such that

$$\begin{pmatrix} \int_1^e v(t)\mu(t)\frac{dt}{t} \\ \int_1^e u(t)\mu(t)\frac{dt}{t} \end{pmatrix} \leq \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

Note (3.2), and we find

$$\begin{pmatrix} \|v\| \\ \|u\| \end{pmatrix} \leq \begin{pmatrix} \frac{M_1(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)} \\ \frac{M_2(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)} \end{pmatrix}.$$

This proves that S_1 is bounded in $P \times P$. As a result, if we choose $R_1 > \{r_1, \frac{M_1(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)}, \frac{M_2(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)}\}$ (r_1 is defined by (H4)), then we have

$$(u,v) \neq A(u,v) + \lambda(\varphi_1, \varphi_1), \quad \text{for } (u,v) \in \partial B_{R_1} \cap (P \times P), \forall \lambda \geq 0.$$

From Lemma 2.7 we have

$$i\bigl(A, B_{R_1} \cap (P \times P), P \times P\bigr) = 0. \tag{3.6}$$

Next we claim that

$$(u,v) \neq \lambda A(u,v), \quad \text{for } (u,v) \in \partial B_{r_1} \cap (P \times P), \forall \lambda \in [0,1], \tag{3.7}$$

where r_1 is defined by (H4). Suppose (3.7) is not true. Then there exist $(u,v) \in \partial B_{r_1} \cap (P \times P)$ and $\lambda \in [0,1]$ such that $(u,v) = \lambda A(u,v)$, which implies that

$$u(t) \leq A_1(u,v)(t), \qquad v(t) \leq A_2(u,v)(t) \quad \text{for } t \in [1,e]. \tag{3.8}$$

Multiplying by $\mu(t)$ and integrating from 1 to e, Lemmas 2.4 and 2.5 enable us to obtain

$$\begin{aligned}
&\begin{pmatrix} \int_1^e u(t)\mu(t)\frac{dt}{t} \\ \int_1^e v(t)\mu(t)\frac{dt}{t} \end{pmatrix} \\
&\quad \leq \begin{pmatrix} \int_1^e \mu(t)(\int_1^e G_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{d_h(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s})\frac{dt}{t} \\ \int_1^e \mu(t)(\int_1^e G_1(t,s)f_2(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{d_g(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t}f_2(s,u(s),v(s))\frac{ds}{s} \\ \qquad + \frac{(\log t)^{\beta-1}}{d_{g,h}\Gamma(\beta)}\int_1^e\int_1^e g(t)G_1(t,s)\frac{dt}{t}f_1(s,u(s),v(s))\frac{ds}{s})\frac{dt}{t} \end{pmatrix} \\
&\quad \leq \begin{pmatrix} (\kappa_2 + \kappa_5 d_h)\int_1^e \mu(t)f_1(t,u(t),v(t))\frac{dt}{t} + \kappa_6\int_1^e \mu(t)f_2(t,u(t),v(t))\frac{dt}{t} \\ (\kappa_2 + d_g\kappa_6)\int_1^e \mu(t)f_2(t,u(t),v(t))\frac{dt}{t} + \kappa_5\int_1^e \mu(t)f_1(t,u(t),v(t))\frac{dt}{t} \end{pmatrix}.
\end{aligned} \tag{3.9}$$

Substituting (H4) into this matrix inequality, we obtain

$$\begin{aligned}
&\begin{pmatrix} \int_1^e u(t)\mu(t)\frac{dt}{t} \\ \int_1^e v(t)\mu(t)\frac{dt}{t} \end{pmatrix} \\
&\quad \leq \begin{pmatrix} (\kappa_2 + \kappa_5 d_h)\int_1^e \mu(t)(a_{21}u(t) + b_{21}v(t))\frac{dt}{t} + \kappa_6\int_1^e \mu(t)(a_{22}u(t) + b_{22}v(t))\frac{dt}{t} \\ (\kappa_2 + d_g\kappa_6)\int_1^e \mu(t)(a_{22}u(t) + b_{22}v(t))\frac{dt}{t} + \kappa_5\int_1^e \mu(t)(a_{21}u(t) + b_{21}v(t))\frac{dt}{t} \end{pmatrix}.
\end{aligned} \tag{3.10}$$

Consequently, we get

$$\begin{pmatrix} 1 - (\kappa_2 + \kappa_5 d_h)a_{21} - \kappa_6 a_{22} & -(\kappa_2 + \kappa_5 d_h)b_{21} - \kappa_6 b_{22} \\ -(\kappa_2 + d_g\kappa_6)a_{22} - \kappa_5 a_{21} & 1 - (\kappa_2 + d_g\kappa_6)b_{22} - \kappa_5 b_{21} \end{pmatrix} \begin{pmatrix} \int_1^e u(t)\mu(t)\frac{dt}{t} \\ \int_1^e v(t)\mu(t)\frac{dt}{t} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, (H4) implies that

$$\begin{pmatrix} \int_1^e u(t)\mu(t)\frac{dt}{t} \\ \int_1^e v(t)\mu(t)\frac{dt}{t} \end{pmatrix} \leq \frac{\begin{pmatrix} 1-(\kappa_2+d_g\kappa_6)b_{22}-\kappa_5 b_{21} & (\kappa_2+\kappa_5 d_h)b_{21}+\kappa_6 b_{22} \\ (\kappa_2+d_g\kappa_6)a_{22}+\kappa_5 a_{21} & 1-(\kappa_2+\kappa_5 d_h)a_{21}-\kappa_6 a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}}{\det\begin{pmatrix} 1-(\kappa_2+\kappa_5 d_h)a_{21}-\kappa_6 a_{22} & -(\kappa_2+\kappa_5 d_h)b_{21}-\kappa_6 b_{22} \\ -(\kappa_2+d_g\kappa_6)a_{22}-\kappa_5 a_{21} & 1-(\kappa_2+d_g\kappa_6)b_{22}-\kappa_5 b_{21} \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence,

$$\int_1^e u(t)\mu(t)\frac{dt}{t} = 0, \qquad \int_1^e v(t)\mu(t)\frac{dt}{t} = 0.$$

Note that $\mu(t) \not\equiv 0$ for $t \in [1,e]$, so $u(t) = v(t) \equiv 0$, $t \in [1,e]$, which implies that $\|u\| = \|v\| = 0$, contradicting $(u,v) \in \partial B_{r_1} \cap (P \times P)$. As a result, (3.7) holds. From Lemma 2.8 we have

$$i\big(A, B_{r_1} \cap (P \times P), P \times P\big) = 1. \tag{3.11}$$

From (3.6) and (3.11) we have

$$\begin{aligned} &i\big(A, (B_{R_1} \backslash \overline{B}_{r_1}) \cap (P \times P), P \times P\big) \\ &\quad = i\big(A, B_{R_1} \cap (P \times P), P \times P\big) - i\big(A, B_{r_1} \cap (P \times P), P \times P\big) = 0 - 1 = -1. \end{aligned}$$

Therefore the operator A has at least one fixed point on $(B_{R_1} \backslash \overline{B}_{r_1}) \cap (P \times P)$. Equivalently, (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.2 *Suppose that* (H1)–(H2), (H5)–(H6) *hold. Then* (1.1) *has a positive solution.*

Proof We use similar methods as in Theorem 3.1 to prove this theorem. We first claim that

$$(u,v) \neq A(u,v) + \lambda(\varphi_2,\varphi_2), \quad \text{for } (u,v) \in \partial B_{r_2} \cap (P \times P), \forall \lambda \geq 0, \tag{3.12}$$

where $\varphi_2 \in P$ is a given element. Suppose the claim is not true. Then there exist $(u,v) \in \partial B_{r_2} \cap (P \times P)$ and $\lambda \geq 0$ such that $(u,v) = A(u,v) + \lambda(\varphi_2,\varphi_2)$, which implies that

$$u(t) \geq A_1(u,v)(t), \qquad v(t) \geq A_2(u,v)(t) \quad \text{for } t \in [1,e].$$

Similar to (3.4), (3.5), from (H5) we obtain

$$\begin{aligned} &\begin{pmatrix} \int_1^e u(t)\mu(t)\frac{dt}{t} \\ \int_1^e v(t)\mu(t)\frac{dt}{t} \end{pmatrix} \\ &\quad \geq \begin{pmatrix} (\kappa_1 + \kappa_3 d_h)\int_1^e \mu(t)(a_{31}u(t) + b_{31}v(t))\frac{dt}{t} + \kappa_4 \int_1^e \mu(t)(a_{32}u(t) + b_{32}v(t))\frac{dt}{t} \\ (\kappa_1 + d_g\kappa_4)\int_1^e \mu(t)(a_{32}u(t) + b_{32}v(t))\frac{dt}{t} + \kappa_3 \int_1^e \mu(t)(a_{31}u(t) + b_{31}v(t))\frac{dt}{t} \end{pmatrix}, \end{aligned}$$

and

$$\begin{pmatrix} b_{31}(\kappa_1 + \kappa_3 d_h) + b_{32}\kappa_4 & a_{31}(\kappa_1 + \kappa_3 d_h) + a_{32}\kappa_4 - 1 \\ b_{32}(\kappa_1 + d_g\kappa_4) + b_{31}\kappa_3 - 1 & a_{32}(\kappa_1 + d_g\kappa_4) + a_{31}\kappa_3 \end{pmatrix}\begin{pmatrix} \int_1^e v(t)\mu(t)\frac{dt}{t} \\ \int_1^e u(t)\mu(t)\frac{dt}{t} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $u(t)=v(t)\equiv 0$ for $t\in[1,e]$, and $\|u\|=\|v\|=0$, which contradicts $(u,v)\in\partial B_{r_2}\cap(P\times P)$. Consequently, (3.12) holds, and from Lemma 2.7 we have

$$i\bigl(A,B_{r_2}\cap(P\times P),P\times P\bigr)=0. \tag{3.13}$$

Let $S_2=\{(u,v)\in P\times P:(u,v)=\lambda A(u,v),\forall\lambda\in[0,1]\}$. Now we prove that S_2 is bounded in $P\times P$. Note if there exists $(u,v)\in S_2$, then

$$u(t)\leq A_1(u,v)(t),\qquad v(t)\leq A_2(u,v)(t)\quad\text{for } t\in[1,e],$$

and similar to (3.9), (3.10), and by (H6) we have

$$\begin{aligned}
&\begin{pmatrix}\int_1^e u(t)\mu(t)\frac{dt}{t}\\ \int_1^e v(t)\mu(t)\frac{dt}{t}\end{pmatrix}\\
&\quad\leq\begin{pmatrix}(\kappa_2+\kappa_5 d_h)\int_1^e\mu(t)(a_{41}u(t)+b_{41}v(t)+l_3)\frac{dt}{t}\\ +\kappa_6\int_1^e\mu(t)(a_{42}u(t)+b_{42}v(t)+l_4)\frac{dt}{t}\\ (\kappa_2+d_g\kappa_6)\int_1^e\mu(t)(a_{42}u(t)+b_{42}v(t)+l_4)\frac{dt}{t}\\ +\kappa_5\int_1^e\mu(t)(a_{41}u(t)+b_{41}v(t)+l_3))\frac{dt}{t}\end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\begin{pmatrix}1-(\kappa_2+\kappa_5 d_h)a_{41}-\kappa_6 a_{42} & -(\kappa_2+\kappa_5 d_h)b_{41}-\kappa_6 b_{42}\\ -(\kappa_2+d_g\kappa_6)a_{42}-\kappa_5 a_{41} & 1-(\kappa_2+d_g\kappa_6)b_{42}-\kappa_5 b_{41}\end{pmatrix}\begin{pmatrix}\int_1^e u(t)\mu(t)\frac{dt}{t}\\ \int_1^e v(t)\mu(t)\frac{dt}{t}\end{pmatrix}\\
&\quad\leq\begin{pmatrix}\frac{(\kappa_2+\kappa_5 d_h)l_3+\kappa_6 l_4}{\Gamma(\beta+2)}\\ \frac{(\kappa_2+d_g\kappa_6)l_4+\kappa_5 l_3}{\Gamma(\beta+2)}\end{pmatrix}.
\end{aligned}$$

Solving this matrix inequality, we have

$$\begin{pmatrix}\int_1^e u(t)\mu(t)\frac{dt}{t}\\ \int_1^e v(t)\mu(t)\frac{dt}{t}\end{pmatrix}\leq\frac{\begin{pmatrix}1-(\kappa_2+d_g\kappa_6)b_{42}-\kappa_5 b_{41} & (\kappa_2+\kappa_5 d_h)b_{41}+\kappa_6 b_{42}\\ (\kappa_2+d_g\kappa_6)a_{42}+\kappa_5 a_{41} & 1-(\kappa_2+\kappa_5 d_h)a_{41}-\kappa_6 a_{42}\end{pmatrix}}{\det\begin{pmatrix}1-(\kappa_2+\kappa_5 d_h)a_{41}-\kappa_6 a_{42} & -(\kappa_2+\kappa_5 d_h)b_{41}-\kappa_6 b_{42}\\ -(\kappa_2+d_g\kappa_6)a_{42}-\kappa_5 a_{41} & 1-(\kappa_2+d_g\kappa_6)b_{42}-\kappa_5 b_{41}\end{pmatrix}}\begin{pmatrix}\frac{(\kappa_2+\kappa_5 d_h)l_3+\kappa_6 l_4}{\Gamma(\beta+2)}\\ \frac{(\kappa_2+d_g\kappa_6)l_4+\kappa_5 l_3}{\Gamma(\beta+2)}\end{pmatrix}.$$

Hence, there exist $M_3>0$, $M_4>0$ such that

$$\begin{pmatrix}\int_1^e u(t)\mu(t)\frac{dt}{t}\\ \int_1^e v(t)\mu(t)\frac{dt}{t}\end{pmatrix}\leq\begin{pmatrix}M_3\\ M_4\end{pmatrix}.$$

Note that $(u,v)\in S_2$, and from Lemma 2.6, we find $u,v\in P_0$. Thus, we obtain

$$\begin{pmatrix}\|u\|\\ \|v\|\end{pmatrix}\leq\begin{pmatrix}\frac{M_3(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)}\\ \frac{M_4(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)}\end{pmatrix}.$$

This proves that S_2 is bounded in $P\times P$. As a result, if we take $R_2>\{r_2,\frac{M_3(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)},\frac{M_4(\beta-1)\Gamma(2\beta+2)}{\beta^2\Gamma(\beta)}\}$ (r_2 is defined by (H5)), we conclude that

$$(u,v)\neq\lambda A(u,v),\quad\text{for }(u,v)\in\partial B_{R_2}\cap(P\times P),\forall\lambda\in[0,1]. \tag{3.14}$$

From Lemma 2.8 we have

$$i\big(A, B_{R_2} \cap (P \times P), P \times P\big) = 1. \tag{3.15}$$

From (3.13) and (3.15) we have

$$\begin{aligned} & i\big(A, (B_{R_2} \backslash \overline{B}_{r_2}) \cap (P \times P), P \times P\big) \\ &\quad = i\big(A, B_{R_2} \cap (P \times P), P \times P\big) - i\big(A, B_{r_2} \cap (P \times P), P \times P\big) = 1 - 0 = 1. \end{aligned}$$

Therefore the operator A has at least one fixed point on $(B_{R_2} \backslash \overline{B}_{r_2}) \cap (P \times P)$. Equivalently, (1.1) has at least one positive solution. This completes the proof. □

Example 3.3 Let $\beta = 2.5$, $h(t) = g(t) = \log t$ for $t \in [1, e]$. Then $d_h = d_g = \int_1^e (\log t)^\beta \frac{dt}{t} = \frac{2}{7}$, $d_{g,h} = 1 - \int_1^e h(t)(\log t)^{\beta-1} \frac{dt}{t} \cdot \int_1^e g(t)(\log t)^{\beta-1} \frac{dt}{t} = 1 - \frac{4}{49} = \frac{45}{49}$. This implies that (H2) holds. Next, we calculate κ_i $(i = 1, 2, 3, 4, 5, 6)$ as follows:

$$\begin{aligned} \kappa_1 &= \frac{\beta^2 \Gamma(\beta)}{\Gamma(2\beta + 2)} = \frac{2.5^2 \Gamma(2.5)}{\Gamma(7)} \approx 0.01154, \\ \kappa_2 &= \frac{\beta - 1}{\Gamma(\beta + 2)} = \frac{1.5}{\Gamma(4.5)} \approx 0.129, \\ \kappa_3 = \kappa_4 &= \frac{\beta}{d_{g,h} \Gamma(2\beta + 1)} \int_1^e (\log t)(\log t)^{\beta-1}(1 - \log t) \frac{dt}{t} \\ &= \frac{2.5}{\frac{45}{49}\Gamma(6)} \int_1^e (\log t)^{2.5}(1 - \log t) \frac{dt}{t} \approx 0.00144, \\ \kappa_5 = \kappa_6 &= \frac{\beta(\beta - 1)}{d_{g,h}\Gamma(2\beta + 1)} \int_1^e (\log t) \frac{dt}{t} = \frac{2.5 \times 1.5}{\frac{45}{49}\Gamma(6)} \int_1^e (\log t) \frac{dt}{t} \approx 0.017. \end{aligned}$$

Case 1. Let $a_{11} = 10$, $a_{12} = 600$, $b_{11} = 630$, $b_{12} = 7$, $a_{21} = 3$, $a_{22} = 4$, $b_{21} = 3$, $b_{22} = 2$. Then we have

$$\begin{aligned} & a_{11}(\kappa_1 + \kappa_3 d_h) + a_{12}\kappa_4 = 10 \times 0.012 + 600 \times 0.00144 < 1, \\ & b_{12}(\kappa_1 + d_g \kappa_4) + b_{11}\kappa_3 = 7 \times 0.012 + 630 \times 0.00144 < 1, \\ & (\kappa_2 + \kappa_5 d_h)a_{21} + \kappa_6 a_{22} = 0.134 \times 3 + 0.017 \times 4 < 1, \\ & (\kappa_2 + d_g \kappa_6)b_{22} + \kappa_5 b_{21} = 0.134 \times 2 + 0.017 \times 3 < 1, \\ & \begin{vmatrix} b_{11}(\kappa_1 + \kappa_3 d_h) + b_{12}\kappa_4 & a_{11}(\kappa_1 + \kappa_3 d_h) + a_{12}\kappa_4 - 1 \\ b_{12}(\kappa_1 + d_g \kappa_4) + b_{11}\kappa_3 - 1 & a_{12}(\kappa_1 + d_g \kappa_4) + a_{11}\kappa_3 \end{vmatrix} = \begin{vmatrix} 7.57 & -0.016 \\ -0.009 & 7.21 \end{vmatrix} > 0, \end{aligned}$$

and

$$\begin{vmatrix} 1 - (\kappa_2 + \kappa_5 d_h)a_{21} - \kappa_6 a_{22} & -(\kappa_2 + \kappa_5 d_h)b_{21} - \kappa_6 b_{22} \\ -(\kappa_2 + d_g \kappa_6)a_{22} - \kappa_5 a_{21} & 1 - (\kappa_2 + d_g \kappa_6)b_{22} - \kappa_5 b_{21} \end{vmatrix} = \begin{vmatrix} 0.53 & -0.436 \\ -0.587 & 0.681 \end{vmatrix} > 0.$$

Let $f_1(t,x,y) = (10x + 630y)^{\gamma_1}, f_2(t,x,y) = (600x + 7y)^{\gamma_2}$ for $t \in [1,e]$, $x,y \in \mathbb{R}^+$, $\gamma_1,\gamma_2 > 1$. Then we have

$$\liminf_{a_{11}x+b_{11}y\to+\infty} \frac{f_1(t,x,y)}{a_{11}x+b_{11}y} = \liminf_{10x+630y\to+\infty} \frac{(10x+630y)^{\gamma_1}}{10x+630y} = +\infty,$$

uniformly on $t \in [1,e]$,

$$\liminf_{a_{12}x+b_{12}y\to+\infty} \frac{f_2(t,x,y)}{a_{12}x+b_{12}y} = \liminf_{600x+7y\to+\infty} \frac{(600x+7y)^{\gamma_2}}{600x+7y} = +\infty, \quad \text{uniformly on } t \in [1,e],$$

$$\limsup_{a_{21}x+b_{21}y\to0^+} \frac{f_1(t,x,y)}{a_{21}x+b_{21}y} = \limsup_{3x+3y\to0^+} \frac{(10x+630y)^{\gamma_1}}{3x+3y} = 0, \quad \text{uniformly on } t \in [1,e],$$

and

$$\limsup_{a_{22}x+b_{22}y\to0^+} \frac{f_2(t,x,y)}{a_{22}x+b_{22}y} = \limsup_{4x+2y\to0^+} \frac{(600x+7y)^{\gamma_2}}{4x+2y} = 0, \quad \text{uniformly on } t \in [1,e].$$

As a result, (H3)–(H4) hold.

Case 2. Let $a_{31} = 8$, $a_{32} = 620$, $b_{31} = 630$, $b_{32} = 7$, $a_{41} = 3$, $a_{42} = 4$, $b_{41} = 3$, $b_{42} = 2$. Then we have

$$\begin{aligned}
&a_{31}(\kappa_1 + \kappa_3 d_h) + a_{32}\kappa_4 = 8 \times 0.012 + 620 \times 0.00144 < 1,\\
&b_{32}(\kappa_1 + d_g\kappa_4) + b_{31}\kappa_3 = 7 \times 0.012 + 630 \times 0.00144 < 1,\\
&(\kappa_2 + \kappa_5 d_h)a_{41} + \kappa_6 a_{42} = 0.134 \times 3 + 0.017 \times 4 < 1,\\
&(\kappa_2 + d_g\kappa_6)b_{42} + \kappa_5 b_{41} = 0.134 \times 2 + 0.017 \times 3 < 1,\\
&\begin{vmatrix} b_{31}(\kappa_1 + \kappa_3 d_h) + b_{32}\kappa_4 & a_{31}(\kappa_1 + \kappa_3 d_h) + a_{32}\kappa_4 - 1 \\ b_{32}(\kappa_1 + d_g\kappa_4) + b_{31}\kappa_3 - 1 & a_{32}(\kappa_1 + d_g\kappa_4) + a_{31}\kappa_3 \end{vmatrix} = \begin{vmatrix} 7.57 & -0.0112 \\ -0.009 & 7.45 \end{vmatrix} > 0,
\end{aligned}$$

and

$$\begin{vmatrix} 1 - (\kappa_2 + \kappa_5 d_h)a_{41} - \kappa_6 a_{42} & -(\kappa_2 + \kappa_5 d_h)b_{41} - \kappa_6 b_{42} \\ -(\kappa_2 + d_g\kappa_6)a_{42} - \kappa_5 a_{41} & 1 - (\kappa_2 + d_g\kappa_6)b_{42} - \kappa_5 b_{41} \end{vmatrix} = \begin{vmatrix} 0.53 & -0.436 \\ -0.587 & 0.681 \end{vmatrix} > 0.$$

Let $f_1(t,x,y) = (8x + 630y)^{\gamma_3}, f_2(t,x,y) = (620x + 7y)^{\gamma_4}$ for $t \in [1,e]$, $x,y \in \mathbb{R}^+$, $\gamma_3,\gamma_4 \in (0,1)$. Then we have

$$\begin{aligned}
&\liminf_{a_{31}x+b_{31}y\to0^+} \frac{f_1(t,x,y)}{a_{31}x+b_{31}y} = \liminf_{8x+630y\to0^+} \frac{(8x+630y)^{\gamma_3}}{8x+630y} = +\infty, \quad \text{uniformly on } t \in [1,e],\\
&\liminf_{a_{32}x+b_{32}y\to0^+} \frac{f_2(t,x,y)}{a_{32}x+b_{32}y} = \liminf_{620x+7y\to0^+} \frac{(620x+7y)^{\gamma_4}}{620x+7y} = +\infty, \quad \text{uniformly on } t \in [1,e],\\
&\limsup_{a_{41}x+b_{41}y\to+\infty} \frac{f_1(t,x,y)}{a_{41}x+b_{41}y} = \limsup_{3x+3y\to+\infty} \frac{(8x+630y)^{\gamma_3}}{3x+3y} = 0, \quad \text{uniformly on } t \in [1,e],
\end{aligned}$$

and

$$\limsup_{a_{42}x+b_{42}y\to+\infty} \frac{f_2(t,x,y)}{a_{42}x+b_{42}y} = \limsup_{4x+2y\to+\infty} \frac{(620x+7y)^{\gamma_4}}{4x+2y} = 0, \quad \text{uniformly on } t \in [1,e].$$

As a result, (H5)–(H6) hold.

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