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# A predictor-corrector iterative method for solving linear least squares problems and perturbation error analysis

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## Abstract

The motivation of the present work concerns two objectives. Firstly, a predictor-corrector iterative method of convergence order  $p = 45$  requiring 10 matrix by matrix multiplications per iteration is proposed for computing the Moore–Penrose inverse of a nonzero matrix of rank  $= r$ . Convergence and a priori error analysis of the proposed method are given. Secondly, the numerical solution to the general linear least squares problems by an algorithm using the proposed method and the perturbation error analysis are provided. Furthermore, experiments are conducted on the ill-posed problem of one-dimensional image restoration and on some test problems from Harwell–Boeing collection. Obtained numerical results show the applicability, stability, and the estimated order of convergence of the proposed method.

**MSC:** Primary 65F20; secondary 65F30

**Keywords:** Linear least squares problems; Perturbation error analysis; Moore–Penrose inverse; Image restoration problem; Matrix algorithms

## 1 Introduction

The numerical solution of many problems in mathematical physics requires the solution of an algebraic linear system of equations. When the coefficient matrix of a consistent linear system is singular or rectangular matrix, then the need for Moore–Penrose inverse occurs for the computation of minimum norm least squares solution.

Let  $\mathbb{C}^{n_1 \times n_2}$  and  $\mathbb{C}_r^{n_1 \times n_2}$  denote the sets of all complex  $n_1 \times n_2$  matrices and all complex  $n_1 \times n_2$  matrices of rank  $= r$ , respectively. By  $A^*$ ,  $R(A)$ , and  $P_{R(A)}$  we present the conjugate transpose of  $A$ , range of  $A$ , and the orthogonal projector onto  $R(A)$  respectively, and as usual  $I$  denotes the unit matrix of appropriate order. The generalized inverse or Moore–Penrose inverse of a matrix  $A \in \mathbb{C}^{n_1 \times n_2}$  denoted by  $A^\dagger \in \mathbb{C}^{n_2 \times n_1}$  is the unique matrix satisfying the following four Penrose equations (see [1]):

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A. \quad (1.1)$$

Many important approaches for computing the Moore–Penrose generalized inversion have been developed. Among these methods, direct methods usually tend to require a predictable amount of resources in terms of time and storage, which normally puts them out

of interest especially for large scaled systems with sparse matrices. Special interest is given to propose and investigate iterative methods for computing the Moore–Penrose inverse  $A^\dagger$ . Let  $A \in \mathbb{C}_r^{n_1 \times n_2}$ , we survey on iterative methods by which  $V_m$  denotes the approximate Moore–Penrose inverse of  $A$  at the  $m$ th iteration, and  $Mms$  stands for the matrix by matrix multiplications per iteration. The authors of [2] proposed the sequence

$$V_m = \alpha \sum_{i=0}^m A^* (I - \alpha A A^*)^i, \quad m = 0, 1, \dots, \tag{1.2}$$

that converges to  $A^\dagger$  as  $m \rightarrow \infty$  under the assumption

$$0 < \alpha < \frac{2}{\lambda_1(A^*A)}, \tag{1.3}$$

where

$$\lambda_1(A^*A) \geq \lambda_2(A^*A) \geq \dots \geq \lambda_r(A^*A) > 0 \tag{1.4}$$

are the nonzero eigenvalues of  $A^*A$ . Later in [3] (see also [4]) the iterative method (1.2) was written as follows:

$$V_0 = \alpha A^*, \tag{1.5}$$

$$V_{m+1} = V_m + \alpha (I - V_m A) A^*, \quad m = 0, 1, \dots, \tag{1.6}$$

and it converges to  $A^\dagger$  linearly. The following iterative algorithm [5, 6]

$$V_0 = \alpha A^*, \tag{1.7}$$

$$V_{m+1} = V_m (2I - A V_m), \quad m = 0, 1, \dots,$$

yields  $A^\dagger$  as the limit of the sequence  $\{V_m\}$ ,  $m = 0, 1, \dots$ , when  $\alpha$  satisfies (1.3). This method is a variant of the well-known Schulz method of 2nd order. A  $p$ th order iterative method for computing  $A^\dagger$  is studied in [7] and [8] for  $p \geq 2$  and  $p$  being integer as follows:

$$\begin{aligned} V_0 &= \alpha A^*, \\ \bar{T}_m &= I - V_m A, \\ P_m &= \sum_{i=0}^{p-1} \bar{T}_m^i, \end{aligned} \tag{1.8}$$

$$V_{m+1} = P_m V_m = \left( \sum_{i=0}^{p-1} \bar{T}_m^i \right) V_m, \quad m = 0, 1, \dots$$

Method (1.8) is also called hyperpower iterative method and for  $n_1 \leq n_2$  is usually presented as

$$V_{m+1} = V_m \sum_{j=0}^{p-1} T_m^j, \quad m = 0, 1, \dots, \tag{1.9}$$

where  $T_m = I - AV_m$ , performing  $p$  *Mms*. In [7], when  $p = 3$ , method (1.9) is rewritten in the form

$$V_{m+1} = V_m(3I - 3AV_m + (AV_m)^2), \quad m = 0, 1, \dots, \tag{1.10}$$

and in Horner’s form for the inversion of a square nonsingular matrix is given in [9] as

$$V_{m+1} = V_m(3I - AV_m(3I - AV_m)), \quad m = 0, 1, \dots, \tag{1.11}$$

and has at least third order of convergence [7, 10]. There are different representations of the hyperpower method (1.9) which differ in the evaluation of the power sum  $\sum_{j=0}^{p-1} T_m^j$ . The computational effort in the  $p$ th order method decreases when the number of matrix by matrix multiplications and additions in the polynomial are reduced through factorizations if nested loops are used (see [11] and [12]). When  $p = 5$ , an effective form of the hyperpower iterative method for inverse of a square nonsingular matrix is presented in [11] as follows:

$$V_{m+1} = V_m(I + (I + T_m^2)(T_m + T_m^2)), \quad m = 0, 1, \dots, \tag{1.12}$$

and requires 4 *Mms*. To compute the inner inverse of matrix  $A \in C^{n_1 \times n_2}$ , the following iterative procedure is proposed in [13]:

$$V_{m+1} = V_m \left[ pI - \frac{p(p-1)}{2} AV_m + \dots + (-1)^{p-1} (AV_m)^{p-1} \right], \quad p \geq 2, \tag{1.13}$$

$m = 0, 1, 2, \dots$ , which requires  $p$  *Mms*. Equation (1.13) is an equivalent form of (1.9). In [14] the authors showed that the iterative procedure (1.13) can be used to compute the Moore–Penrose inverse of a matrix. The authors of [15] proved that the sequences (1.13) indeed are convergent to a fixed inner inverse of the matrix which is the Moore–Penrose inverse of the matrix, and they provided numerical results for the third order iterations. The following 7th order method is given by [16] for computing generalized inverse  $A_{T,S}^{(2)}$ :

$$V_{m+1} = V_m(I + (T_m + T_m^4)(I + T_m + T_m^2)), \quad m = 0, 1, \dots. \tag{1.14}$$

The method in (1.14) performs 5 *Mms*. Recently several systematic algorithms for factorizations of the hyperpower iterative family (1.9) of arbitrary orders have been given in [17] for computing outer inverses. Among these methods the 11th, 15th, and 19th order methods are

$$V_{m+1} = V_m(I + T_m(I + (T_m + T_m^2 + T_m^3)(I + T_m^3 + T_m^6))), \quad m = 0, 1, \dots \tag{1.15}$$

$$V_{m+1} = V_m(I + (T_m + T_m^2)(I + (T_m^2 + T_m^4)(I + T_m^4 + T_m^8))), \quad m = 0, 1, \dots \tag{1.16}$$

$$V_{m+1} = V_m(I + (T_m + T_m^2)(I + T_m^2 + T_m^4)(I + T_m^6 + T_m^{12})), \quad m = 0, 1, \dots, \tag{1.17}$$

given that each uses 7, 7, and 8 *Mms*, respectively.

In [18] for a given integer parameter  $k \geq 1$ , two classes of iterative methods for matrix inversion of a square nonsingular matrix were proposed, and these methods were used

to construct inverse preconditioners for solving linear systems. Class 1 methods converge with order of convergence  $p = 3 * 2^k + 1$  and Class 2 methods converge with order  $p = 5 * 2^k - 1$ , requiring  $2k + 4$  *Mms* and  $3k + 4$  *Mms*, respectively. Most recently, high order iterative methods with a recurrence formula for approximate matrix inversion have been proposed in [19] with orders of convergence  $p = 4k + 3$ , where  $k \geq 1$  is an integer requiring  $k + 4$  matrix multiplications per iteration. The methods of orders  $p = 7, 11, 15, 19$  are used to construct approximate Schur-block incomplete LU preconditioners (Schur-BILU) for regularized solution of discrete ill-posed problems.

The original contribution of this study is the construction of a predictor-corrector iterative method (PCIM) of convergence order  $p = 45$  requiring 10 *Mms* for computing the Moore–Penrose inverse of nonzero matrix  $A \in \mathbb{C}_r^{n_1 \times n_2}$ . Also, perturbation error analysis of the proposed method for the minimum norm solution of least squares problems is provided. To verify the theoretical results, experimental analysis are conducted for the one-dimensional image restoration problem and for some well-conditioned and moderately ill-conditioned problems from Harwell–Boeing collection. Next, a benchmark example is used to numerically compute the order of convergence of the proposed method. Concluding remarks are given in the last section.

Some versions of the predictor-corrector variational iteration method for the solution of integral equations of fractional order were studied in [20] and [21].

## 2 A predictor-corrector iterative method for approximate Moore–Penrose inverse

Let  $I$  denote the  $n_1 \times n_1$  unit matrix and let  $A \in \mathbb{C}_r^{n_1 \times n_2}$  and the nonzero eigenvalues of  $A^*A$  be denoted by (1.4). We define the following matrix-valued functions consisting of matrix multiplications and additions:

$$\Omega(T_m) = T_m + T_m^2, \tag{2.1}$$

$$\Psi(T_m) = I + T_m^2, \tag{2.2}$$

$$\Gamma(T_m) = T_m^4, \tag{2.3}$$

where  $T_m = (I - AV_m)$ . We propose the following:

*Predictor-corrector iterative method (PCIM).*

$$\begin{aligned} In : V_0 &= \alpha A^* \\ G : T_m &= I - AV_m, \quad \Phi(T_m) = \Psi(T_m)\Omega(T_m) \\ P : V_{m+\frac{1}{2}} &= V_m(I + \Phi(T_m)), \quad T_{m+\frac{1}{2}} = I - AV_{m+\frac{1}{2}} \\ C : V_{m+1} &= V_{m+\frac{1}{2}}(I + \Phi(T_{m+\frac{1}{2}})(I + \Gamma(T_{m+\frac{1}{2}}))) \\ &\text{for } m = 0, 1, 2, \dots \end{aligned} \tag{2.4}$$

Here,  $In$  is the initial step,  $G$  is the generator,  $P$  is the predictor at the fractional step  $m + \frac{1}{2}$ , and  $C$  is the corrector for the approximate Moore–Penrose inverse  $V_{m+1}$  of  $A$  at the  $m + 1$ th iteration in (2.4). Let  $R_m = P_{R(A)} - AV_m$  be the residual at the  $m$ th iteration and consider

the following form of (2.4) which requires  $P_{R(A)}$ :

$$\begin{aligned}
 In : V_0 &= \alpha A^*, \\
 G : R_m &= P_{R(A)} - AV_m, \quad \Phi(R_m) = \Psi(R_m)\Omega(R_m), \\
 P : V_{m+\frac{1}{2}} &= V_m(P_{R(A)} + \Phi(R_m)), \quad R_{m+\frac{1}{2}} = P_{R(A)} - AV_{m+\frac{1}{2}}, \\
 C : V_{m+1} &= V_{m+\frac{1}{2}}(P_{R(A)} + \Phi(R_{m+\frac{1}{2}})(I + \Gamma(R_{m+\frac{1}{2}}))) \\
 &\text{for } m = 0, 1, 2, \dots
 \end{aligned} \tag{2.5}$$

**Lemma 2.1** *Let  $A \in \mathbb{C}_r^{n_1 \times n_2}$ , if in the initial step  $In$  of (2.5)  $\alpha$  satisfies (1.3), then for the sequence  $\{V_{m+1}\}$  obtained at  $m + 1$ th iteration by PCIM (2.5) the following hold true for every  $m = 0, 1, 2, \dots$ :*

$$(i) \quad V_{m+1}P_{R(A)} = V_{m+1}, \quad (ii) \quad P_{R(A^*)}V_{m+1} = V_{m+1}, \tag{2.6}$$

$$(iii) \quad (AV_{m+1})^* = AV_{m+1}, \quad (iv) \quad (V_{m+1}A)^* = V_{m+1}A. \tag{2.7}$$

*Proof* The proof follows from PCIM (2.5) using mathematical induction. We give the proof of (i) and (iii) and the proof of (ii) and (iv) can be given analogously. By (III) of Theorem 1 [7],  $AA^\dagger = P_{R(A)}$  and  $A^\dagger A = P_{R(A^*)}$ , and on the basis of Moore–Penrose equations (1.1), the projection  $P_{R(A)}$  satisfies the following:

$$P_{R(A)}^2 = P_{R(A)}, \quad P_{R(A)}^* = P_{R(A)}, \quad P_{R(A)}A = A; \quad A^*P_{R(A)} = A^*. \tag{2.8}$$

(i) For  $m = 0$ , in the initial step  $In$  of PCIM (2.5) we have  $V_0 = \alpha A^*$  implying  $V_0P_{R(A)} = \alpha A^*P_{R(A)} = \alpha A^* = V_0$ . Assume that  $V_mP_{R(A)} = V_m$  holds true for  $m$ , then using (2.8) for the predictor step  $P$  of (2.5) we obtain

$$\begin{aligned}
 V_{m+\frac{1}{2}}P_{R(A)} &= V_m(P_{R(A)} + \Phi(R_m))P_{R(A)} \\
 &= V_m(P_{R(A)} + (P_{R(A)} - AV_m) + (P_{R(A)} - AV_m)^2 + (P_{R(A)} - AV_m)^3 \\
 &\quad + (P_{R(A)} - AV_m)^4)P_{R(A)} = V_{m+\frac{1}{2}},
 \end{aligned} \tag{2.9}$$

$$\begin{aligned}
 R_{m+\frac{1}{2}} &= P_{R(A)} - AV_{m+\frac{1}{2}} = P_{R(A)} - AV_m(P_{R(A)} + \Phi(R_m)) \\
 &= P_{R(A)} - AV_m(P_{R(A)} + R_m + R_m^2 + R_m^3 + R_m^4) = R_m^5.
 \end{aligned} \tag{2.10}$$

Also in the corrector step  $C$  of (2.5) using (2.10) we get

$$\Gamma(R_{m+\frac{1}{2}}) = R_{m+\frac{1}{2}}^4 = R_m^{20}, \tag{2.11}$$

$$\Phi(R_{m+\frac{1}{2}}) = \Psi(R_{m+\frac{1}{2}})\Omega(R_{m+\frac{1}{2}}) = (I + R_m^{10})(R_m^5 + R_m^{10}). \tag{2.12}$$

Using (2.8) and from the assumption  $V_mP_{R(A)} = V_m$ , equations (2.11) and (2.12) imply

$$\Gamma(R_{m+\frac{1}{2}})P_{R(A)} = (P_{R(A)} - AV_m)^{20}P_{R(A)} = \Gamma(R_{m+\frac{1}{2}}), \tag{2.13}$$

$$\Phi(R_{m+\frac{1}{2}})P_{R(A)} = (I + R_m^{10})(R_m^5 + R_m^{10})P_{R(A)} = \Phi(R_{m+\frac{1}{2}}). \tag{2.14}$$

Using (2.8)–(2.14) in the corrector step  $C$  of (2.5), we obtain

$$\begin{aligned} V_{m+1}P_{R(A)} &= V_{m+\frac{1}{2}}(P_{R(A)} + \Phi(R_{m+\frac{1}{2}})(I + \Gamma(R_{m+\frac{1}{2}})))P_{R(A)} \\ &= V_{m+\frac{1}{2}}(P_{R(A)} + \Phi(R_{m+\frac{1}{2}})P_{R(A)} + \Phi(R_{m+\frac{1}{2}})\Gamma(R_{m+\frac{1}{2}})P_{R(A)}) \\ &= V_{m+\frac{1}{2}}(P_{R(A)} + \Phi(R_{m+\frac{1}{2}})(I + \Gamma(R_{m+\frac{1}{2}}))) = V_{m+1}. \end{aligned} \tag{2.15}$$

(iii) For  $m = 0$ , in the initial step  $In$  of PCIM (2.5) we have  $V_0 = \alpha A^*$  implying  $(AV_0)^* = (A\alpha A^*)^* = \alpha AA^* = AV_0$ . Assume that  $(AV_m)^* = AV_m$  holds true, then from (2.8) in the predictor step  $P$  of (2.5) gives

$$R_m^* = (P_{R(A)} - AV_m)^* = R_m, \tag{2.16}$$

$$\Phi^*(R_m) = ((I + R_m^{10})(R_m^5 + R_m^{10}))^* = \Phi(R_m), \tag{2.17}$$

$$(AV_{m+\frac{1}{2}})^* = (P_{R(A)}^* + \Phi^*(R_m))(AV_m)^* = AV_{m+\frac{1}{2}}. \tag{2.18}$$

From the assumption and also using (2.8), (2.10)–(2.12), and (2.16) gives

$$R_{m+\frac{1}{2}}^* = R_{m+\frac{1}{2}}, \quad \Gamma^*(R_{m+\frac{1}{2}}) = \Gamma(R_{m+\frac{1}{2}}), \quad \Phi^*(R_{m+\frac{1}{2}}) = \Phi(R_{m+\frac{1}{2}}). \tag{2.19}$$

Using (2.8), (2.18), and (2.19) in the corrector step  $C$  of (2.5) yields

$$\begin{aligned} (AV_{m+1})^* &= (P_{R(A)} + \Phi(R_{m+\frac{1}{2}})(I + \Gamma(R_{m+\frac{1}{2}})))^*(AV_{m+\frac{1}{2}})^* \\ &= AV_{m+1}. \end{aligned} \tag{2.20}$$

□

*Remark 2.2* The validity of (2.6) and (2.7) for PCIM (2.5) implies that PCIM (2.4) is derived from PCIM (2.5). The computational significance of method (2.5) is limited by the need for knowledge of  $P_{R(A)}$ . Therefore, the PCIM given in (2.4) is more effective in computational aspects and is preferable for  $n_1 \leq n_2$ . If  $n_1 > n_2$ , then the dual version of PCIM (2.4) can be used.

**Theorem 2.3** *Let  $A \in \mathbb{C}_r^{n_1 \times n_2}$ , if in the initial step  $In$  of PCIM (2.4)  $\alpha$  satisfies (1.3), then the sequence  $\{V_{m+1}\}$  obtained by the proposed PCIM (2.4) converges to the Moore–Penrose inverse  $A^\dagger$  uniformly with  $p = 45$  order of convergence and with asymptotic convergence factor [18]  $ACF = \frac{10}{\ln 45} \approx 2.627$ . Furthermore, the following error estimate is valid:*

$$\|A^\dagger - V_{m+1}\|_2 \leq \frac{\alpha \|R_0\|_2^{45^{m+1}} \|A^*\|_2}{1 - \|R_0\|_2}, \tag{2.21}$$

where  $R_0 = P_{R(A)} - \alpha AA^*$ .

*Proof* To show the uniform convergence of  $\{V_{m+1}\}$  obtained by PCIM (2.4) to  $A^\dagger$ , we use the approach analogous to the proof of Theorem 3 of [7]. Let us define the residual in the dual version at initial approximation by

$$\overline{T}_0 = I - V_0A = I - \alpha A^*A. \tag{2.22}$$

Also, for any integer  $s > 0$ ,

$$T_0^s A = A \overline{T}_0^s \tag{2.23}$$

and for PCIM (2.4), for its dual version and from Lemma 2.1 using (2.10)–(2.12) for PCIM (2.5), we obtain the residual error at the  $(m + 1)$ th iteration as

$$T_{m+1} = I - AV_{m+1} = T_m^{45}, \quad m = 0, 1, \dots, \tag{2.24}$$

$$\overline{T}_{m+1} = I - V_{m+1}A = \overline{T}_m^{45}, \quad m = 0, 1, \dots, \tag{2.25}$$

$$R_{m+1} = P_{R(A)} - AV_{m+1} = R_m^{45}, \quad m = 0, 1, \dots, \tag{2.26}$$

respectively. It follows from (2.24)–(2.26) that, for each  $m$ ,

$$AV_{m+1} = I - T_0^{45^{m+1}}, \quad V_{m+1}A = I - \overline{T}_0^{45^{m+1}}, \tag{2.27}$$

$$V_{m+1}AV_{m+1} = V_{m+1} - \overline{T}_0^{45^{m+1}} V_{m+1}. \tag{2.28}$$

From Lemma 2.1, equations (2.6), (2.7) and on the basis of Corollary 4 to Theorem 2 and Theorem 1 of [7] the passage to the limit in (2.27) and (2.28) imply the uniform convergence

$$AV_{m+1} \rightarrow P_{R(A)}, \quad V_{m+1}A \rightarrow P_{R(A^*)}, \quad V_{m+1}AV_{m+1} \rightarrow A^\dagger. \tag{2.29}$$

Therefore, the sequence  $\{V_{m+1}\}$  converges uniformly to  $A^\dagger$  with order  $p = 45$ . By denoting

$$\widehat{V}_{m+\frac{1}{2}} = V_{m+\frac{1}{2}} (I + T_m^5 + T_m^{10} + T_m^{15} + T_m^{20} + T_m^{25} + T_m^{30} + T_m^{35}), \tag{2.30}$$

the corrector step  $C$  of (2.4) can be rewritten as

$$V_{m+1} = V_{m+\frac{1}{2}} + \widehat{V}_{m+\frac{1}{2}} T_m^5. \tag{2.31}$$

We denote the error of PCIM (2.4) at the corrector step  $C$  by  $E_{m+1} = A^\dagger - V_{m+1}$ . Next, by using that  $A^\dagger AA^\dagger = A^\dagger$  and from Lemma 2.1 equations (2.6) and (2.8), we obtain

$$\begin{aligned} E_{m+1} - E_{m+1}R_{m+1} &= A^\dagger - V_{m+1} - (A^\dagger - V_{m+1})(P_{R(A)} - AV_{m+1}) \\ &= V_{m+1}(AA^\dagger - AV_{m+1}) = V_{m+1}R_{m+1}. \end{aligned} \tag{2.32}$$

Using (1.3) and on the basis of Theorem 2 of [6], we have  $\rho(R_0) < 1$ . Since  $R_0$  is Hermitian, this gives  $\rho(R_0) = \|R_0\|_2 < 1$  and using (2.26) and norm properties

$$\|R_{m+1}\|_2 \leq \|R_m\|_2^{45} \leq \|R_0\|_2^{45^{m+1}} < 1. \tag{2.33}$$

From (2.32) using second norm gives

$$\|E_{m+1}\|_2 \leq \frac{\|V_{m+1}R_{m+1}\|_2}{1 - \|R_{m+1}\|_2} \tag{2.34}$$

(see [7]). By using (2.26), (2.30), and (2.31), it follows that

$$\begin{aligned}
 V_{m+1}R_{m+1} &= V_{m+1}R_m^{40}R_m^5 \\
 &= \{V_{m+1}(P_{R(A)} + R_m^5 + R_m^{10} + R_m^{15} + R_m^{20} + R_m^{25} + R_m^{30} + R_m^{35} + R_m^{40}) \\
 &\quad - V_{m+1}(P_{R(A)} + R_m^5 + R_m^{10} + R_m^{15} + R_m^{20} + R_m^{25} + R_m^{30} + R_m^{35})\}R_m^5 \\
 &= \{V_{m+1}(P_{R(A)} + R_m^5 + R_m^{10} + R_m^{15} + R_m^{20} + R_m^{25} + R_m^{30} + R_m^{35} + R_m^{40}) \\
 &\quad - V_{m+\frac{1}{2}}(P_{R(A)} + R_m^5 + R_m^{10} + R_m^{15} + R_m^{20} + R_m^{25} + R_m^{30} + R_m^{35}) \\
 &\quad - \widehat{V}_{m+\frac{1}{2}}T_m^5(P_{R(A)} + R_m^5 + R_m^{10} + R_m^{15} + R_m^{20} + R_m^{25} + R_m^{30} + R_m^{35})\}R_m^5 \\
 &= (V_{m+1} - \widehat{V}_{m+\frac{1}{2}})(P_{R(A)} + R_m^5 + R_m^{10} + R_m^{15} + R_m^{20} \\
 &\quad + R_m^{25} + R_m^{30} + R_m^{35} + R_m^{40})R_m^5. \tag{2.35}
 \end{aligned}$$

From equation (2.35) and using  $\|P_{R(A)}\|_2 = 1$  and second norm, we get

$$\begin{aligned}
 \|V_{m+1}R_{m+1}\|_2 &\leq \|R_m\|_2^5(1 + \|R_m\|_2^5 + \|R_m\|_2^{10} + \|R_m\|_2^{15} + \|R_m\|_2^{20} + \|R_m\|_2^{25} \\
 &\quad + \|R_m\|_2^{30} + \|R_m\|_2^{35} + \|R_m\|_2^{40})\|V_{m+1} - \widehat{V}_{m+\frac{1}{2}}\|_2. \tag{2.36}
 \end{aligned}$$

From (2.30) and (2.31) and using that  $V_{m+\frac{1}{2}}P_{R(A)}^s = V_{m+\frac{1}{2}}$  is valid for every positive integer  $s$  and also using second norm, it follows that

$$\begin{aligned}
 \|V_{m+1} - \widehat{V}_{m+\frac{1}{2}}\|_2 &= \|V_{m+\frac{1}{2}} + \widehat{V}_{m+\frac{1}{2}}(T_m^5 - I)\|_2 \\
 &= \|V_{m+\frac{1}{2}}T_m^{40}\|_2 \leq \|V_{m+\frac{1}{2}}R_m\|_2\|R_m\|_2^{39}. \tag{2.37}
 \end{aligned}$$

Also, in view of (2.33),

$$\begin{aligned}
 1 - \|R_{m+1}\|_2 &\geq 1 - \|R_m\|_2^{45} = (1 - \|R_m\|_2^5)(1 + \|R_m\|_2^5 + \|R_m\|_2^{10} + \|R_m\|_2^{15} \\
 &\quad + \|R_m\|_2^{20} + \|R_m\|_2^{25} + \|R_m\|_2^{30} + \|R_m\|_2^{35} + \|R_m\|_2^{40}). \tag{2.38}
 \end{aligned}$$

From the predictor step  $P$  of (2.4) and using  $\|P_{R(A)}\|_2 = 1$  and second norm, it follows that

$$\|V_{m+\frac{1}{2}}R_m\|_2 \leq \|V_mR_m\|_2(1 + \|R_m\|_2 + \|R_m\|_2^2 + \|R_m\|_2^3 + \|R_m\|_2^4). \tag{2.39}$$

Also

$$1 - \|R_m\|_2^5 = (1 - \|R_m\|_2)(1 + \|R_m\|_2 + \|R_m\|_2^2 + \|R_m\|_2^3 + \|R_m\|_2^4). \tag{2.40}$$

Using (2.34), (2.36)–(2.40) and by recursion, we obtain the following inequalities yielding (2.21):

$$\begin{aligned}
 \|A^\dagger - V_{m+1}\|_2 &\leq \frac{\|V_{m+1}R_{m+1}\|_2}{1 - \|R_{m+1}\|_2} \leq \frac{\|R_m\|_2^{44}\|V_{m+\frac{1}{2}}R_m\|_2}{1 - \|R_m\|_2^5} \leq \frac{\|R_m\|_2^{44}\|V_mR_m\|_2}{1 - \|R_m\|_2} \\
 &\leq \frac{\|R_0\|_2^{45^{m+1}}\|V_0\|_2}{1 - \|R_0\|_2} = \frac{\alpha\|R_0\|_2^{45^{m+1}}\|A^*\|_2}{1 - \|R_0\|_2}. \tag{2.41}
 \end{aligned}$$

The proposed PCIM (2.4) requires 5 *Mms* for the evaluation of the predictor step *P* and 5 *Mms* for the corrector step *C*. Therefore, total 10 *Mms* are needed giving the asymptotic convergence factor *ACF* [18] as  $\frac{10}{\ln 45} \approx 2.627$ . □

### 3 Perturbation error analysis

We consider the minimum norm solution of the linear system of equations

$$Ax = b, \quad A \in \mathbb{R}^{n_1 \times n_2}, b \in \mathbb{R}^{n_1}, \tag{3.1}$$

where  $b \in R(A)$  and  $x \in \mathbb{R}^{n_2}$  denotes the unknown solutions and  $\text{rank}(A) = r \leq \min(n_1, n_2)$ . The general least squares solution of (3.1) or minimum norm least squares solution is the solution of the problem

$$\min_{x \in S} \|x\|_2, \quad S = \{x \in \mathbb{R}^{n_2} \mid \|Ax - b\|_2 = \min\}. \tag{3.2}$$

System (3.2) always has a unique solution  $x = A^\dagger b$  called the pseudoinverse solution [22]. If *A* is a square nonsingular matrix, then  $A^\dagger = A^{-1}$ . The condition number of *A* is defined by  $\kappa(A) = \|A\|_2 \|A^\dagger\|_2$ . Next we consider the following perturbed general least squares problem:

$$\min_{\tilde{x} \in \tilde{S}} \|\tilde{x}\|_2, \quad \tilde{S} = \{\tilde{x} \in \mathbb{R}^{n_2} \mid \|\tilde{A}\tilde{x} - \tilde{b}\|_2 = \min\}, \tag{3.3}$$

where  $\tilde{A} = A + \partial A$ ,  $\tilde{b} = b + \partial b$ , and  $\tilde{x} = \tilde{A}^\dagger \tilde{b}$  is called the regularized pseudoinverse solution and  $\partial x = \tilde{x} - x$ ,  $\partial \tau = \tilde{\tau} - \tau$  and the residuals are  $\tilde{\tau} = \tilde{b} - \tilde{A}\tilde{x}$ ,  $\tau = b - Ax$ . Also, we take into account the perturbations satisfying  $\text{rank}(\tilde{A}) = \text{rank}(A) = r$  and  $\frac{\|\partial A\|_2}{\|A\|_2} \leq \varepsilon_A$ ,  $\frac{\|\partial b\|_2}{\|b\|_2} \leq \varepsilon_b$ .

#### 3.1 Algorithm for approximate regularized pseudoinverse solution

Let  $\tilde{x}_{m+1,\partial} = \tilde{V}_{m+1} \tilde{b}$  be the approximate regularized pseudoinverse solution of (3.3), where  $\tilde{V}_{m+1}$  is the approximate Moore–Penrose inverse of  $\tilde{A}$  obtained by the PCIM given in (2.4) at the *m* + 1th iteration by using initial approximation  $\tilde{V}_0 = \alpha \tilde{A}^*$  and  $\alpha$  satisfying

$$0 < \alpha < \frac{2}{\tilde{\lambda}_1(\tilde{A}^* \tilde{A})}, \tag{3.4}$$

where

$$\tilde{\lambda}_1(\tilde{A}^* \tilde{A}) \geq \tilde{\lambda}_2(\tilde{A}^* \tilde{A}) \geq \dots \geq \tilde{\lambda}_r(\tilde{A}^* \tilde{A}) > 0 \tag{3.5}$$

are the nonzero eigenvalues of  $\tilde{A}^* \tilde{A}$ . Hence the residual error at the corresponding step is  $\tilde{\tau}_{m+1} = \tilde{b} - \tilde{A} \tilde{x}_{m+1,\partial}$ . Given a predescribed accuracy  $\varepsilon > 0$ , we propose the following algorithm that uses PCIM (2.4) and approximates the regularized pseudoinverse solution  $\tilde{x}_{m+1,\partial}$ .

**Algorithm A** Choose an initial matrix  $\tilde{V}_0 = \alpha \tilde{A}^*$  under the assumption (3.4).

Step 1 Let  $m = 0$ . Evaluate  $\tilde{x}_{0,\partial} = \tilde{V}_0 \tilde{b}$  and  $\tilde{\tau}_0 = \tilde{b} - \tilde{A} x_{0,\partial}$ , then calculate  $\frac{\|\tilde{\tau}_0\|_2}{\|\tilde{b}\|_2}$ .

Do Step 2–Step 6 until  $\frac{\|\tilde{\tau}_m\|_2}{\|\tilde{b}\|_2} \leq \varepsilon$ .

- Step 2 Evaluate  $\tilde{T}_m = I - \tilde{A}\tilde{V}_m$ .
- Step 3 Apply the predictor step  $P$  of PCIM (2.4) to find  $\tilde{V}_{m+\frac{1}{2}}$ .
- Step 4 Evaluate  $\tilde{T}_{m+\frac{1}{2}} = I - \tilde{A}\tilde{V}_{m+\frac{1}{2}}$ .
- Step 5 Apply the corrector step  $C$  of PCIM (2.4) to find  $\tilde{V}_{m+1}$ .
- Step 6 Evaluate  $\tilde{x}_{m+1,\theta} = \tilde{V}_{m+1}\tilde{b}$  and  $\tilde{\tau}_{m+1} = \tilde{b} - \tilde{A}\tilde{x}_{m+1,\theta}$  respectively. Then calculate  $\frac{\|\tilde{\tau}_{m+1}\|_2}{\|\tilde{b}\|_2}$ . Let  $m = m + 1$ .
- Step 7 If  $\tilde{m}$  is the number of iterations performed, then  $\tilde{y}_{\tilde{m}} = \tilde{x}_{\tilde{m},\theta}$  is the approximate regularized pseudoinverse solution satisfying  $\frac{\|\tilde{\tau}_{\tilde{m}}\|_2}{\|\tilde{b}\|_2} \leq \varepsilon$ .

**Theorem 3.1** (Theorem 1.4.6 in [22]) *Assume that  $\text{rank}(\tilde{A}) = \text{rank}(A) = r$  and let  $\frac{\|\partial A\|_2}{\|A\|_2} \leq \varepsilon_A$ ,  $\frac{\|\partial b\|_2}{\|b\|_2} \leq \varepsilon_b$ . Then, if  $\eta = \varkappa(A)\varepsilon_A < 1$ , the perturbations  $\partial x$  and  $\partial \tau$  in the least squares solution  $x$  and the residual  $\tau$  satisfy*

$$\|\partial x\|_2 \leq \frac{\varkappa(A)}{1 - \eta} \left( \varepsilon_A \|x\|_2 + \varepsilon_b \frac{\|b\|_2}{\|A\|_2} + \varepsilon_A \varkappa(A) \frac{\|\tau\|_2}{\|A\|_2} \right) + \varepsilon_A \varkappa(A) \|x\|_2, \tag{3.6}$$

$$\|\partial \tau\|_2 \leq \varepsilon_A \|x\|_2 \|A\|_2 + \varepsilon_b \|b\|_2 + \varepsilon_A \varkappa(A) \|\tau\|_2. \tag{3.7}$$

**Theorem 3.2** *Assume that  $\text{rank}(\tilde{A}) = \text{rank}(A) = r$  and let  $\frac{\|\partial A\|_2}{\|A\|_2} \leq \varepsilon_A$ ,  $\frac{\|\partial b\|_2}{\|b\|_2} \leq \varepsilon_b$ . Let  $\tilde{x}_{\tilde{m},\theta} = \tilde{V}_{\tilde{m}}\tilde{b}$  be the approximate regularized pseudoinverse solution (3.3) obtained by Algorithm A. Then, if  $\eta = \varkappa(A)\varepsilon_A < 1$ , the following inequality holds true:*

$$\begin{aligned} \|x - \tilde{x}_{\tilde{m},\theta}\|_2 &\leq \frac{\varkappa(A)}{1 - \eta} \left( \varepsilon_A \|x\|_2 + \varepsilon_b \frac{\|b\|_2}{\|A\|_2} + \varepsilon_A \varkappa(A) \frac{\|\tau\|_2}{\|A\|_2} \right) \\ &\quad + \varepsilon_A \varkappa(A) \|x\|_2 + \frac{\alpha(1 + \varepsilon_b) \|\tilde{R}_0\|_2^{45\tilde{m}} \|\tilde{A}^*\|_2 \|A\|_2 \|x\|_2}{1 - \|\tilde{R}_0\|_2}, \end{aligned} \tag{3.8}$$

where  $x$  is the pseudoinverse solution (3.2),  $\tau = b - Ax$  and  $\tilde{R}_0 = P_{R(\tilde{A})} - \tilde{A}\tilde{V}_0$ .

*Proof* On the basis of Theorem 2.3 and the inequality in (2.21), and from Theorem 3.1, using (3.6), it follows that

$$\begin{aligned} \|x - \tilde{x}_{\tilde{m},\theta}\|_2 &= \|x - \tilde{x}_{\tilde{m},\theta} + \tilde{x} - \tilde{x}\|_2 \leq \|x - \tilde{x}\|_2 + \|\tilde{x} - \tilde{x}_{\tilde{m},\theta}\|_2 \\ &= \|x - \tilde{x}\|_2 + \|\tilde{A}^\dagger \tilde{b} - \tilde{V}_{\tilde{m}}\tilde{b}\|_2 \leq \|x - \tilde{x}\|_2 + \|\tilde{A}^\dagger - \tilde{V}_{\tilde{m}}\|_2 \|\tilde{b}\|_2 \\ &\leq \frac{\varkappa(A)}{1 - \eta} \left( \varepsilon_A \|x\|_2 + \varepsilon_b \frac{\|b\|_2}{\|A\|_2} + \varepsilon_A \varkappa(A) \frac{\|\tau\|_2}{\|A\|_2} \right) + \varepsilon_A \varkappa(A) \|x\|_2 \\ &\quad + \frac{\alpha \|\tilde{R}_0\|_2^{45\tilde{m}} \|\tilde{A}^*\|_2}{1 - \|\tilde{R}_0\|_2} \|\tilde{b}\|_2 \\ &\leq \frac{\varkappa(A)}{1 - \eta} \left( \varepsilon_A \|x\|_2 + \varepsilon_b \frac{\|b\|_2}{\|A\|_2} + \varepsilon_A \varkappa(A) \frac{\|\tau\|_2}{\|A\|_2} \right) + \varepsilon_A \varkappa(A) \|x\|_2 \\ &\quad + \frac{\alpha(1 + \varepsilon_b) \|\tilde{R}_0\|_2^{45\tilde{m}} \|\tilde{A}^*\|_2 \|A\|_2 \|x\|_2}{1 - \|\tilde{R}_0\|_2}. \end{aligned} \tag{3.8}$$

### 4 Numerical results

In this section all the computations are performed using a personal computer with properties AMD Ryzen 7 1800X Eight Core Processor 3.60 GHz. Calculations are carried by

Mathematica program in double precision for Example 1 and Example 2 and with 900 precision for Example 3. The tables adopt the following notations:

TCS<sub>AL</sub> is the total solution cost in seconds of Algorithm A.

TCSSP<sub>AL</sub> is the total solution cost in seconds of Algorithm A for successive perturbations.

*Example 1* (One-dimensional image restoration problem (see [18, 23])) In this test problem we consider the first kind Fredholm integral equation

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(\theta, \varphi) f(\varphi) d\varphi = g(\theta), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \tag{4.1}$$

The kernel  $K(\theta, \varphi)$  and the function  $f(\varphi)$  are as follows:

$$K(\theta, \varphi) = \left[ (\cos \theta + \cos \varphi) \frac{\sin \omega}{\omega} \right]^2, \tag{4.2}$$

with

$$\begin{aligned} \omega &= \pi (\sin \theta + \sin \varphi). \\ f(\varphi) &= \exp(-4(\varphi + 0.5)^2) + 2 \exp(-4(\varphi - 0.5)^2). \end{aligned} \tag{4.3}$$

By taking the quadrature nodes  $\varphi_j = -\frac{\pi}{2} + \frac{(j-0.5)\pi}{n_2}$ ,  $j = 1, \dots, n_2$ , and the points  $\theta_i = -\frac{\pi}{2} + \frac{(i-0.5)\pi}{n_1}$ ,  $i = 1, \dots, n_1$ , the discretization of (4.1)–(4.3) gives the discrete ill-posed problem  $Au_h = b$ . The right-hand side vector  $b$  is calculated by multiplying  $A$  with  $u$ , where  $u$  is the trace of  $f(\varphi)$  on the grid points  $\varphi_j, j = 1, \dots, n_2$ . We apply the proposed Algorithm A by using successive perturbations to obtain the regularized pseudoinverse solution for the perturbed systems  $(A + \partial A)y^k = b + \partial^k b$ , where  $\partial A_{i,j} = \begin{cases} \delta^1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$  and  $\partial^k b = (\delta^k, \dots, \delta^k)^T \in R^{n_1}$  for different values of the smoothing parameter  $\delta^1$ , and  $\delta^k = 0.999\delta^{k-1}$ ,  $k = 2, \dots, 6$ . Let  $\tilde{y}_{\tilde{m}_k}^k$ ,  $k = 1, \dots, 6$ , be the approximate regularized pseudoinverse solution of

$$\min_{\tilde{y}^k \in \tilde{S}^k} \|\tilde{y}^k\|_2, \quad \tilde{S}^k = \{ \tilde{y}^k \in R^{n_2} \mid \|(A + \partial A)\tilde{y}^k - (b + \partial^k b)\|_2 = \min \}, \tag{4.4}$$

obtained by the given Algorithm A by performing  $\tilde{m}_k$  iterations for an accuracy of  $\frac{\|\tilde{y}_{\tilde{m}_k}^k\|_2}{\|b\|_2} \leq 5 \times 10^{-7}$  for the corresponding perturbed system. Table 1 presents the total iteration number  $\tilde{m} = \sum_{k=1}^6 \tilde{m}_k$  performed, TCSSP<sub>AL</sub> and the relative  $L_2$  norm of the errors using Algorithm A when  $n_1 = 400$  and  $n_2 = 800$ .

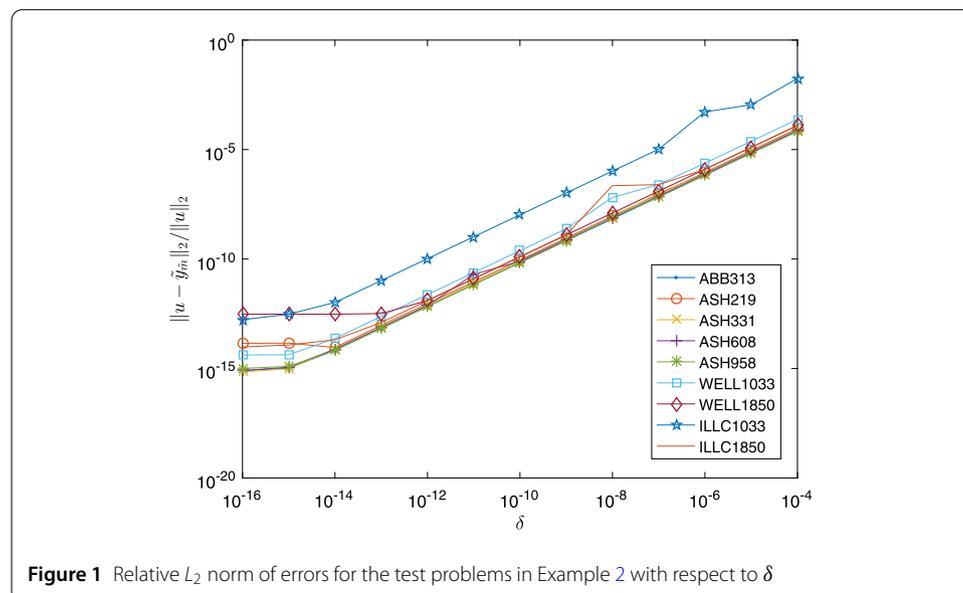
**Table 1** TCSSP<sub>AL</sub>, iteration numbers, relative  $L_2$  norm of the errors obtained by Algorithm A for Example 1

$\delta^1$	TCSSP <sub>AL</sub>	$\tilde{m}$	$\frac{\ u - \tilde{y}_{\tilde{m}_1}^1\ _2}{\ u\ _2}$	$\frac{\ u - \tilde{y}_{\tilde{m}_2}^2\ _2}{\ u\ _2}$	$\frac{\ u - \tilde{y}_{\tilde{m}_3}^3\ _2}{\ u\ _2}$	$\frac{\ u - \tilde{y}_{\tilde{m}_4}^4\ _2}{\ u\ _2}$	$\frac{\ u - \tilde{y}_{\tilde{m}_5}^5\ _2}{\ u\ _2}$	$\frac{\ u - \tilde{y}_{\tilde{m}_6}^6\ _2}{\ u\ _2}$
$10^{-3}$	0.45	10	1.7423E-02	6.2443E-02	1.1128E-01	1.3823E-01	2.4639E-02	3.6325E-03
$10^{-4}$	0.45	10	9.0878E-03	1.4484E-02	2.0965E-02	3.3861E-03	4.1878E-03	1.0131E-04
$10^{-6}$	0.45	10	8.2943E-03	3.1554E-03	3.2967E-04	3.2244E-04	1.1128E-01	1.3823E-01
$10^{-8}$	0.45	10	8.2620E-03	2.8571E-03	2.8571E-03	2.8571E-03	2.8571E-03	2.8571E-03
$10^{-10}$	0.45	10	8.2616E-03	2.8499E-03	2.8499E-03	2.8499E-03	2.8499E-03	2.8499E-03

**Example 2** (Test problems from Harwell–Boeing collection) The experiments are carried out on nine of the matrices from the Harwell–Boeing test collection (Duff, Grimes, and Lewis, see [22]). The numerical values of the ABB and ASH matrices are random numbers uniformly distributed in  $[-1, 1]$ , while WELL and ILLC have their original values. The WELL and ILLC matrices have the same nonzero structure but different numerical values. A set of consistent least squares problems is defined by taking the exact solution to be  $u = (1, \dots, 1)^T \in R^{n_2}$ , and the right-hand side  $b$  is obtained by multiplying the coefficient matrix  $A$  by  $u$ . We apply the proposed Algorithm A using the dual version of PCIM for obtaining the regularized pseudoinverse solution for each test problem, where  $\partial A_{i,j} = \begin{cases} \delta & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$  and  $\partial b = (\delta, \dots, \delta)^T \in R^{n_1}$  for different values of the smoothing parameter  $\delta$ . Let  $\tilde{y}_{\tilde{m}}$  be the approximate regularized pseudoinverse solution obtained by the given Algorithm A using the dual version of PCIM by performing  $\tilde{m}$  iterations for an accuracy of  $\frac{\|u - \tilde{y}_{\tilde{m}}\|_2}{\|b\|_2} \leq \varepsilon$ . Table 2 presents the test problems, description of the problems, and the size of the matrices,  $TCS_{AL}$ , iteration numbers  $\tilde{m}$  performed, and the relative  $L_2$  norm of the errors obtained for the test problems in Example 2 when  $\delta = \varepsilon = 5 \times 10^{-15}$ . Figure 1 illustrates the relative  $L_2$  norm of errors for the test problems in Example 2 with respect to  $\delta$ . The obtained numerical results justify the theoretical results in Theorem 3.2.

**Table 2**  $TCS_{AL}$ , iteration numbers, and relative  $L_2$  norm of the errors obtained by Algorithm A for the test problems of Example 2

Test problem	Description	Size ( $n_1 \times n_2$ )	$TCS_{AL}$	$\tilde{m}$	$\frac{\ u - \tilde{y}_{\tilde{m}}\ _2}{\ u\ _2}$
ABB313	Sudan survey	313 × 176	0.02	2	3.4674E-15
ASH219	Geodesy problem	219 × 85	0.01	3	4.5483E-15
ASH331	Geodesy problem	331 × 104	0.02	3	3.8656E-15
ASH608	Geodesy problem	608 × 188	0.05	3	3.8984E-15
ASH958	Geodesy problem	958 × 292	0.06	3	3.4699E-15
WELL1033	Gravity-meter	1033 × 320	0.09	5	1.2690E-14
WELL1850	Gravity-meter	1850 × 712	0.77	5	7.0907E-15
ILLC1033	Gravity-meter	1033 × 320	0.16	7	7.9173E-13
ILLC1850	Gravity-meter	1850 × 712	1.07	8	4.7693E-14



**Figure 1** Relative  $L_2$  norm of errors for the test problems in Example 2 with respect to  $\delta$

**Table 3** Second norm errors and the estimated order of convergence of PCIM for Example 3

$\ V_1 - A^\dagger\ _2$	$\ V_2 - A^\dagger\ _2$	$\ V_3 - A^\dagger\ _2$	$\tilde{p}$
0.11199	3.51011E-20	7.43666E-853	47.16

*Example 3* (A benchmark example [17]) With this example we aim to numerically compute the theoretical order of convergence of the proposed PCIM (2.4) using the known exact Moore–Penrose inverse. In this example  $A$  and  $A^\dagger$  are

$$A = \begin{pmatrix} 1 & 0 & 0 & -6 \\ 2 & 6 & 0 & -6 \\ 7 & 8 & 9 & -6 \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} \frac{28}{1931} & \frac{-143}{3862} & \frac{84}{1931} \\ \frac{-653}{3862} & \frac{1335}{7724} & \frac{-14}{1931} \\ \frac{57}{1931} & \frac{-249}{1931} & \frac{171}{1931} \\ \frac{-1903}{11,586} & \frac{-143}{23,172} & \frac{14}{1931} \end{pmatrix},$$

and we use

$$\tilde{p} = \ln\left(\frac{\|V_{m+1} - A^\dagger\|_2}{\|V_m - A^\dagger\|_2}\right) / \ln\left(\frac{\|V_m - A^\dagger\|_2}{\|V_{m-1} - A^\dagger\|_2}\right)$$

to estimate the order of convergence  $p$ . The maximum norm of the errors and the calculated order of convergence  $\tilde{p}$  are presented in Table 3. The fourth column of Table 3 shows that the numerically calculated order of convergence  $\tilde{p}$  of PCIM (2.4) is at least 45.

### 5 Conclusion

The PCIM with convergence order  $p = 45$  is proposed for computing the Moore–Penrose inverse of a nonzero matrix  $A \in \mathbb{C}^{n_1 \times n_2}$ . Also we give the Algorithm A which uses the PCIM and approximates the regularized pseudoinverse solution of general least squares problem. The economical computational efficiency and stability of Algorithm A are useful for the numerical regularized pseudoinverse solution of difficult problems such as the first kind Fredholm integral equations.

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#### Availability of data and materials

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#### Competing interests

The authors declare that they have no competing interest.

#### Authors' contributions

The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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