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Cyclic Brunn–Minkowski inequalities for general width and chord-integrals

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Abstract
In this paper, we establish two cyclic Brunn–Minkowski inequalities for the general *i*th width-integrals and general *i*th chord-integrals, respectively. Our works bring the cyclic inequality and Brunn–Minkowski inequality together.

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1 Introduction and main results

The setting for this paper is the Euclidean n-space $\mathbb{R}^n$. We use S^{n-1} and $V(K)$ to denote the unit sphere and the n-dimensional volume of a body K, respectively. For the standard unit ball B, we write $V(B) = \omega_n$.

If K is a nonempty compact convex set in $\mathbb{R}^n$, then the support function of K, $h_K = h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by (see [1])

$$h(K,x) = \max\{x \cdot y : y \in K\}$$

for $x \in \mathbb{R}^n$, where $x \cdot y$ is the standard inner product of x and y. If K is a compact convex set with nonempty interiors in $\mathbb{R}^n$, then K is called a convex body. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$.

The radial function $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$ of a compact star-shaped (about the origin) set $K \subset \mathbb{R}^n$ is defined by (see [1])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is continuous, then K will be called a star body (about the origin). Let $\mathcal{S}^n_o$ denote the subset of star bodies containing the origin in $\mathbb{R}^n$. Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

The research of width-integrals has a long history. The width-integrals were first considered by Blaschke (see [2]) and were further researched by Hadwiger (see [3]). In 1975, Lutwak [4] gave the ith width-integrals as follows:

For $K \in \mathcal{K}^n$ and any real i, the ith width-integrals $B_i(K)$ of K are defined by

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K,u)^{n-i} \, dS(u). \tag{1.1}$$

Here $b(K,u)$ denotes the half width of K in the direction $u \in S^{n-1}$ which is defined by $b(K,u) = \frac{1}{2}h(K,u) + \frac{1}{2}h(K,-u)$. If there exists a constant $\lambda > 0$ such that $b(K,u) = \lambda b(L,u)$ for all $u \in S^{n-1}$, then K and L are said to have similar width. Further, Lutwak [4] established the following Brunn–Minkowski inequality and cyclic inequality for the ith width-integrals, respectively.

Theorem 1.A *If $K, L \in \mathcal{K}^n$ and real $i < n-1$, then*

$$B_i(K+L)^{\frac{1}{n-i}} \leq B_i(K)^{\frac{1}{n-i}} + B_i(L)^{\frac{1}{n-i}}$$

with equality if and only if K and L have similar width. Here $K + L$ denotes the Minkowski sum of K and L.

Theorem 1.B *If $K \in \mathcal{K}^n$ and reals i, j, k satisfy $i < j < k$, then*

$$B_j(K)^{k-i} \leq B_i(K)^{k-j} B_k(K)^{j-i}$$

with equality if and only if K is of constant width.

Whereafter, Lutwak [5] showed that the mixed width-integral $B(K_1, \dots, K_n)$ of $K_1, \dots, K_n \in \mathcal{K}^n$ was defined by

$$B(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} b(K_1,u) \cdots b(K_n,u) \, dS(u). \tag{1.2}$$

In 2016, based on (1.2), Feng [6] introduced the general mixed width-integrals as follows: For $K_1, \dots, K_n \in \mathcal{K}^n$ and $\tau \in (-1,1)$, the general mixed width-integral $B^{(\tau)}(K_1, \dots, K_n)$ of $K_1, \dots, K_n$ is given by

$$B^{(\tau)}(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} b^{(\tau)}(K_1,u) \cdots b^{(\tau)}(K_n,u) \, dS(u), \tag{1.3}$$

where $b^{\tau}(K,u) = f_1(\tau)h(K,u) + f_2(\tau)h(K,-u)$ and

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \qquad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}. \tag{1.4}$$

Obviously,

$$f_1(\tau) + f_2(\tau) = 1;$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau).$$

Combined with (1.4), the case of $\tau = 0$ in (1.3) is just (1.2). If there exists a constant $\lambda > 0$ such that $b^\tau(K,u) = \lambda b^\tau(L,u)$ for all $u \in S^{n-1}$, then we say convex bodies K and L have similar general width. K and L have joint constant general width means that $b^{(\tau)}(K,u)b^{(\tau)}(L,u)$ is a constant for all $u \in S^{n-1}$.

Taking $K_1 = \cdots = K_{n-i} = K, K_{n-i+1} = \cdots = K_n = B$ in (1.3) and allowing i to be any real, the general ith width-integrals $B_i^{(\tau)}(K)$ of $K \in \mathcal{K}^n$ were given by (see [6])

$$B_i^{(\tau)}(K) = \frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{n-i}\,dS(u). \tag{1.5}$$

From (1.1), (1.4), and (1.5), we easily see that if $\tau = 0$, then $B_i^{(0)}(K) = B_i(K)$.

In 2006, motivated by Lutwak's ith width-integrals and together with the notion of radial function, Li, Yuan, and Leng [7] gave the ith chord-integrals as follows: For $K \in \mathcal{S}_o^n$ and i is any real, the ith chord-integrals $C_i(K)$ of K are defined by

$$C_i(K) = \frac{1}{n}\int_{S^{n-1}} c(K,u)^{n-i}\,dS(u). \tag{1.6}$$

Here $c(K,u)$ denotes the half chord of K in the direction u and $c(K,u) = \frac{1}{2}\rho(K,u) + \frac{1}{2}\rho(K,-u)$. If there exists a constant $\lambda > 0$ such that $c(K,u) = \lambda c(L,u)$ for all $u \in \mathcal{S}^{n-1}$, then we say that K and L have similar chord.

For the ith chord-integrals, the authors [7] proved the following Brunn–Minkowski inequality and cyclic inequality.

Theorem 1.C *For $K, L \in \mathcal{S}_o^n$ and real $i \neq n$. If $i < n-1$, then*

$$C_i(K \tilde{+} L)^{\frac{1}{n-i}} \leq C_i(K)^{\frac{1}{n-i}} + C_i(L)^{\frac{1}{n-i}};$$

if $i > n-1$, then

$$C_i(K \tilde{+} L)^{\frac{1}{n-i}} \geq C_i(K)^{\frac{1}{n-i}} + C_i(L)^{\frac{1}{n-i}}.$$

In each inequality, equality holds if and only if K and L are dilates. Here $K \tilde{+} L$ denotes the radial sum of K and L.

Theorem 1.D *If $K \in \mathcal{S}_o^n$ and reals i, j, k satisfy $i < j < k$, then*

$$C_j(K)^{k-i} \leq C_i(K)^{k-j} C_k(K)^{j-i},$$

with equality if and only if K is of constant chord.

The mixed chord-integrals of star bodies were defined by Lu (see [8]): For $K_1, \ldots, K_n \in \mathcal{S}_o^n$, the mixed chord-integrals $C(K_1, \ldots, K_n)$ of $K_1, \ldots, K_n$ are defined by

$$C(K_1, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} c(K_1,u)\cdots c(K_n,u)\,dS(u).$$

Recently, Feng and Wang [9] gave the general mixed chord-integrals $C^{(\tau)}(K_1,\ldots,K_n)$ of $K_1,\ldots,K_n \in \mathcal{S}_o^n$ defined by

$$C^{(\tau)}(K_1,\ldots,K_n) = \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K_1,u)\cdots c^{(\tau)}(K_n,u)\,dS(u), \tag{1.7}$$

where $c^{(\tau)}(K,u) = f_1(\tau)\rho(K,u) + f_2(\tau)\rho(K,-u)$ and functions $f_1(\tau)$, $f_2(\tau)$ satisfy (1.4). By (1.4), let $\tau = 0$ in (1.7), this is just Lu's mixed chord-integrals $C(K_1,\ldots,K_n)$. Star bodies K and L are said to have similar general chord mean that there exist constants $\lambda,\mu > 0$ such that $\lambda c^{(\tau)}(K,u) = \mu c^{(\tau)}(L,u)$ for all $u \in S^{n-1}$. If the product $c^{(\tau)}(K,u)c^{(\tau)}(L,u)$ is constant for all $u \in S^{n-1}$, then they are said to have joint constant general chord.

Taking $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = B$ in (1.7) and allowing i to be any real, the general ith chord-integral $C_i^{(\tau)}(K)$ of $K \in \mathcal{S}_o^n$ was given by (see [9])

$$C_i^{(\tau)}(K) = \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{n-i}\,dS(u). \tag{1.8}$$

Obviously, (1.4), (1.6), and (1.8) give $C_i^{(0)}(K) = C_i(K)$.

In this paper, based on Theorems 1.A–1.B and Theorems 1.C–1.D, we respectively establish two cyclic Brunn–Minkowski inequalities for general ith width-integrals and general ith chord-integrals by using Zhao's ideas (see [10] and [11]). Our works bring the cyclic inequality and Brunn–Minkowski inequality together. Our main results can be stated as follows.

Theorem 1.1 *Let $K,L \in \mathcal{K}^n$, $\tau \in (-1,1)$, and i, j, k all be reals. If $j < n-1$ and $i \le j < k$, then*

$$B_j^{(\tau)}(K+L)^{\frac{k-i}{n-j}} \le B_i^{(\tau)}(K)^{\frac{k-j}{n-j}} B_k^{(\tau)}(K)^{\frac{j-i}{n-j}} + B_i^{(\tau)}(L)^{\frac{k-j}{n-j}} B_k^{(\tau)}(L)^{\frac{j-i}{n-j}} \tag{1.9}$$

with equality if and only if K and L have similar general width. If $n-1 < j < n$ and $j \le i < k$ or $j > n$ and $i \le j < k$, then inequality (1.9) is reversed.

Theorem 1.2 *Let $K,L \in \mathcal{S}_o^n$, $\tau \in (-1,1)$, and i, j, k all be reals. If $j < n-1$ and $i \le j < k$, then*

$$C_j^{(\tau)}(K \tilde{+} L)^{\frac{k-i}{n-j}} \le C_i^{(\tau)}(K)^{\frac{k-j}{n-j}} C_k^{(\tau)}(K)^{\frac{j-i}{n-j}} + C_i^{(\tau)}(L)^{\frac{k-j}{n-j}} C_k^{(\tau)}(L)^{\frac{j-i}{n-j}} \tag{1.10}$$

with equality if and only if K and L have similar general chord. If $n-1 < j < n$ and $j \le i < k$ or $j > n$ and $i \le j < k$, then inequality (1.10) is reversed.

Remark 1.1 Let $i = j$ in Theorem 1.1 and Theorem 1.2, we may obtain the following Brunn–Minkowski inequalities for general ith width-integrals and general ith chord-integrals, respectively.

Corollary 1.1 *Let $K,L \in \mathcal{K}^n$, real $i \ne n$, and $\tau \in (-1,1)$. If $i < n-1$, then*

$$B_i^{(\tau)}(K+L)^{\frac{1}{n-i}} \le B_i^{(\tau)}(K)^{\frac{1}{n-i}} + B_i^{(\tau)}(L)^{\frac{1}{n-i}} \tag{1.11}$$

with equality if and only if K and L have similar general width. If $i > n-1$, then inequality (1.11) is reversed.

Corollary 1.2 *Let $K, L \in \mathcal{S}_o^n$, real $i \neq n$, and $\tau \in (-1,1)$. If $i < n-1$, then*

$$C_i^{(\tau)}(K \tilde{+} L)^{\frac{1}{n-i}} \leq C_i^{(\tau)}(K)^{\frac{1}{n-i}} + C_i^{(\tau)}(L)^{\frac{1}{n-i}}$$

with equality holds if and only if K and L have similar general chord. If $i > n-1$, then the above inequality is reversed.

Obviously, if $\tau = 0$, then inequality (1.11) yields Theorem 1.A, Corollary 1.2 gives Theorem 1.C, respectively.

Remark 1.2 If K and L are nonempty compact convex sets, then Theorem 1.1 also is true. From this, take $L = \{o\}$ in Theorem 1.1. Since $K + \{o\} = K$ and notice that $B_i^\tau(\{o\}) = 0$, thus by inequality (1.9) we have the following.

Corollary 1.3 *If $K \in \mathcal{K}^n$, $\tau \in (-1,1)$, and $i < j < k$, then*

$$B_j^{(\tau)}(K)^{k-i} \leq B_i^{(\tau)}(K)^{k-j} B_k^{(\tau)}(K)^{j-i}$$

with equality if and only if K is of constant width.

Because of $\{o\} \in \mathcal{S}_o^n$, hence let $L = \{o\}$ in Theorem 1.2, we may obtain the following.

Corollary 1.4 *If $K \in \mathcal{S}_o^n$, $\tau \in (-1,1)$, and $i < j < k$, then*

$$C_j^{(\tau)}(K)^{k-i} \leq C_i^{(\tau)}(K)^{k-j} C_k^{(\tau)}(K)^{j-i}$$

with equality if and only if K is of constant chord.

If $\tau = 0$, then Corollary 1.3 and Corollary 1.4 respectively give Theorem 1.B and Theorem 1.D.

Our works belong to the asymmetric Brunn–Minkowski theory, which has its starting point in the theory of valuations in connection with isoperimetric and analytic inequalities. As an important research object in convex geometry, asymmetric Brunn–Minkowski theory has gotten rich development, readers can refer to [12–19].

2 Preliminaries

For nonempty compact convex bodies K and L, $\lambda, \mu \geq 0$ (not both zero), the Minkowski combination $\lambda K + \mu L$ of K and L is defined by (see [1])

$$h(\lambda K + \mu L, \cdot) = \lambda h(K, \cdot) + \mu h(L, \cdot), \tag{2.1}$$

where "+" and "λK" are called Minkowski addition and Minkowski scalar multiplication, respectively. When $\lambda = \mu = 1$, the $K + L$ is called Minkowski sum.

For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero), the radial combination $\lambda \circ K \tilde{+} \mu \circ L$ of K, L is given by (see [1, 20])

$$\rho(\lambda \circ K \tilde{+} \mu \circ L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot), \tag{2.2}$$

where "$\tilde{+}$" and "$\lambda \circ K$" are radial addition and radial scalar multiplication, respectively. When $\lambda = \mu = 1$, the $K \tilde{+} L$ is radial sum of K and L.

3 Proofs of theorems

In this part, we give the proofs of Theorem 1.1 and Theorem 1.2. First, we give the following lemmas.

Lemma 3.1 *Let $f \in L^p(E)$, $g \in L^q(E)$, real number $p, q \neq 0$, and $\frac{1}{p} + \frac{1}{q} = 1$, if $p > 1$, then*

$$\left(\int_E \left|f(x)\right|^p dx\right)^{\frac{1}{p}} \left(\int_E \left|g(x)\right|^q dx\right)^{\frac{1}{q}} \geq \int_E \left|f(x)g(x)\right| dx \tag{3.1}$$

with equality if and only if there exists constants c_1 and c_2 such that $c_1|f(x)|^p = c_2|g(x)|^q$. The inequality is reversed if $p < 0$ or $0 < p < 1$. Here $L^p(E)$ denotes all function sets defined on a measurable set E in L_p spaces.

Lemma 3.2 *Let $f, g \in L^p(E)$, if real number $p \neq 0$ and $p > 1$, then*

$$\left(\int_E \left|f(x)\right|^p dx\right)^{\frac{1}{p}} + \left(\int_E \left|g(x)\right|^p dx\right)^{\frac{1}{p}} \geq \left(\int_E \left|f(x) + g(x)\right|^p\right)^{\frac{1}{p}} dx \tag{3.2}$$

with equality if and only if there exists constants c_1 and c_2 such that $c_1|f(x)|^p = c_2|g(x)|^q$. The inequality is reversed if $p < 0$ or $0 < p < 1$.

Proof of Theorem 1.1 If $j < n-1$, i.e., $n-j > 1$, then from (1.5), (2.1), and (3.2), we have

$$\begin{aligned}
B_j^{(\tau)}(K+L)^{\frac{1}{n-j}} &= \left[\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K+L,u)^{n-j}\, dS(u)\right]^{\frac{1}{n-j}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \big(f_1(\tau)h(K+L,u) + f_2(\tau)h(K+L,-u)\big)^{n-j}\, dS(u)\right]^{\frac{1}{n-j}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \big(f_1(\tau)h(K,u) + f_1(\tau)h(L,u) \right. \\
&\qquad \left. + f_2(\tau)h(K,-u) + f_2(\tau)h(L,-u)\big)^{n-j}\, dS(u)\right]^{\frac{1}{n-j}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \big(b^{(\tau)}(K,u) + b^{(\tau)}(L,u)\big)^{n-j}\, dS(u)\right]^{\frac{1}{n-j}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \big(b^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{(k-i)(n-j)}} b^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{(k-i)(n-j)}} \right. \\
&\qquad \left. + b^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{(k-i)(n-j)}} b^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{(k-i)(n-j)}}\big)^{n-j}\, dS(u)\right]^{\frac{1}{n-j}} \\
&\leq \left[\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}} b^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\, dS(u)\right]^{\frac{1}{n-j}} \\
&\qquad + \left[\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{k-i}} b^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{k-i}}\, dS(u)\right]^{\frac{1}{n-j}}.
\end{aligned} \tag{3.3}$$

In the inequality on the right above, notice that $i < j < k$ means $\frac{k-i}{k-j} > 1$, thus by (3.1) we obtain

$$\begin{aligned}
&\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}} b^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)\\
&\quad\le \left[\frac{1}{n}\int_{S^{n-1}} \left(b^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}}\right)^{\frac{k-i}{k-j}} dS(u)\right]^{\frac{k-j}{k-i}}\\
&\qquad\times \left[\frac{1}{n}\int_{S^{n-1}} \left(b^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\right)^{\frac{k-i}{j-i}} dS(u)\right]^{\frac{j-i}{k-i}}\\
&\quad= B_i^{(\tau)}(K)^{\frac{k-j}{k-i}} B_k^{(\tau)}(K)^{\frac{j-i}{k-i}}.
\end{aligned} \tag{3.4}$$

From this, for $j < n-1$, we can get the following inequality by (3.4):

$$\begin{aligned}
&\left[\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}} b^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)\right]^{\frac{1}{n-j}}\\
&\quad\le B_i^{(\tau)}(K)^{\frac{k-j}{(k-i)(n-j)}} B_k^{(\tau)}(K)^{\frac{j-i}{(k-i)(n-j)}}.
\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}
&\left[\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{k-i}} b^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)\right]^{\frac{1}{n-j}}\\
&\quad\le B_i^{(\tau)}(L)^{\frac{k-j}{(k-i)(n-j)}} B_k^{(\tau)}(L)^{\frac{j-i}{(k-i)(n-j)}}.
\end{aligned} \tag{3.6}$$

Hence, by (3.3), (3.5), and (3.6) we have

$$B_j^{(\tau)}(K+L)^{\frac{1}{n-j}} \le B_i^{(\tau)}(K)^{\frac{k-j}{(k-i)(n-j)}} B_k^{(\tau)}(K)^{\frac{j-i}{(k-i)(n-j)}} + B_i^{(\tau)}(L)^{\frac{k-j}{(k-i)(n-j)}} B_k^{(\tau)}(L)^{\frac{j-i}{(k-i)(n-j)}}.$$

This yields inequality (1.9).

If $n-1 < j < n$, then $0 < n-j < 1$, this gives (3.3) is reversed. But $j < i < k$ means $0 < \frac{k-i}{k-j} < 1$, thus (3.4) is reversed. Hence inequality (3.5) is reversed. Similarly, inequality (3.6) is also reversed. From this, we know that inequality (1.9) is reversed.

If $j > n$, then $n-j < 0$ implies that (3.3) is reversed. But by $n-j < 0$ and (3.4), we see inequality (3.5) is reversed (inequality (3.6) is also reversed). These yield that inequality (1.9) is reversed.

For $i = j$, by (3.3) (or its reverse) we easily get inequality (1.9) (or its reverse).

According to the equality conditions of inequalities (3.1) and (3.2), we see that equality holds in (1.9) (or its reverse) if and only if K and L have similar general width. □

Proof of Theorem 1.2 For $j < n-1$ and $i \le j < k$, since $n-j > 1$, thus by (1.8), (2.2), and (3.2) we get

$$\begin{aligned}
C_j^{(\tau)}(K\tilde{+}L)^{\frac{1}{n-j}} &= \left[\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K\tilde{+}L,u)^{n-j}\,dS(u)\right]^{\frac{1}{n-j}}\\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(f_1(\tau)\rho(K\tilde{+}L,u)+f_2(\tau)\rho(K\tilde{+}L,-u)\right)^{n-j}\,dS(u)\right]^{\frac{1}{n-j}}
\end{aligned}$$

$$= \left[\frac{1}{n}\int_{S^{n-1}} \big(f_1(\tau)\rho(K,u) + f_1(\tau)\rho(L,u) + f_2(\tau)\rho(K,-u) + f_2(\tau)\rho(L,-u)\big)^{n-j}\,dS(u)\right]^{\frac{1}{n-j}}$$

$$= \left[\frac{1}{n}\int_{S^{n-1}} \big(c^{(\tau)}(K,u) + c^{(\tau)}(L,u)\big)^{n-j}\,dS(u)\right]^{\frac{1}{n-j}}$$

$$= \left[\frac{1}{n}\int_{S^{n-1}} \big(c^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{(k-i)(n-j)}} c^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{(k-i)(n-j)}} + c^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{(k-i)(n-j)}} c^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{(k-i)(n-j)}}\big)^{n-j}\,dS(u)\right]^{\frac{1}{n-j}}$$

$$\le \left[\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}} c^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)\right]^{\frac{1}{n-j}} + \left[\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{k-i}} c^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)\right]^{\frac{1}{n-j}}. \tag{3.7}$$

On the right-hand side of the above inequality, when $i < j < k$, i.e., $\frac{k-i}{k-j} > 1$, by (3.1) we have

$$\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}} c^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)$$
$$\le \left[\frac{1}{n}\int_{S^{n-1}} \big(c^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}}\big)^{\frac{k-i}{k-j}}\,dS(u)\right]^{\frac{k-j}{k-i}} \times \left[\frac{1}{n}\int_{S^{n-1}} \big(c^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\big)^{\frac{k-i}{j-i}}\,dS(u)\right]^{\frac{j-i}{k-i}}$$
$$= C_i^{(\tau)}(K)^{\frac{k-j}{k-i}} C_k^{(\tau)}(K)^{\frac{j-i}{k-i}}. \tag{3.8}$$

Hence, for $j < n-1$,

$$\left[\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}} c^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)\right]^{\frac{1}{n-j}} \le C_i^{(\tau)}(K)^{\frac{k-j}{(k-i)(n-j)}} C_k^{(\tau)}(K)^{\frac{j-i}{(k-i)(n-j)}}. \tag{3.9}$$

Similarly,

$$\left[\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{k-i}} c^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{k-i}}\,dS(u)\right]^{\frac{1}{n-j}} \le C_i^{(\tau)}(L)^{\frac{k-j}{(k-i)(n-j)}} C_k^{(\tau)}(L)^{\frac{j-i}{(k-i)(n-j)}}. \tag{3.10}$$

According to (3.7), (3.9), and (3.10), we see that

$$C_j^{(\tau)}(K \tilde{+} L)^{\frac{1}{n-j}} \le C_i^{(\tau)}(K)^{\frac{k-j}{(k-i)(n-j)}} C_k^{(\tau)}(K)^{\frac{j-i}{(k-i)(n-j)}} + C_i^{(\tau)}(L)^{\frac{k-j}{(k-i)(n-j)}} C_k^{(\tau)}(L)^{\frac{j-i}{(k-i)(n-j)}}.$$

This gives inequality (1.10).

For $n-1<j<n$ and $j\leq i<k$, we easily obtain that (3.7) and (3.8) are reversed, thus inequalities (3.9) and (3.10) both are reversed. So, we can get that inequality (1.10) is reversed.

For $j>n$ and $i\leq j<k$, we know that inequality (3.7) is reversed, notice that inequality (3.8) still holds, thus by (3.8) and $n-j<0$, inequality (3.9) is reversed. Similarly, inequality (3.10) is also reversed. Therefore, inequality (1.10) is reversed.

When $i=j$, by (3.7) (or its reverse) we easily get inequality (1.10) (or its reverse).

The equality conditions of Lemma 3.1 and Lemma 3.2 show that equality holds in (1.10) (or its reverse) if and only if K and L have similar general chord. □

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References

1. Gardner, R.J.: Geometric Tomography, 2nd edn. Encyclopedia Math. Appl. Cambridge University Press, Cambridge (2006)
2. Blaschke, W.: Vorlesungen über integral geometric I, II. Teubner, Leipzig (1936, 1937). Reprint: Chelsea, New York (1949)
3. Hadwiger, H.: Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Springer, Berlin (1957)
4. Lutwak, E.: Width-integrals of convex bodies. Proc. Am. Math. Soc. **53**, 435–439 (1975)
5. Lutwak, E.: Mixed width-integrals of convex bodies. Isr. J. Math. **28**(3), 249–253 (1977)
6. Feng, Y.B.: General mixed width-integral of convex bodies. J. Nonlinear Sci. Appl. **9**, 4226–4234 (2016)
7. Li, Y.H., Yuan, J., Leng, G.S.: Chord-integrals of star bodies. J. Shanghai Univ. Nat. Sci. **12**(4), 394–398 (2006)
8. Lu, F.H.: Mixed chord-integrals of star bodies. J. Korean Math. Soc. **47**, 277–288 (2010)
9. Feng, Y.B., Wang, W.D.: General mixed chord-integrals of star bodies. Rocky Mt. J. Math. **46**, 1499–1518 (2016)
10. Zhao, C.J.: Dual cyclic Brunn–Minkowski inequalities. Bull. Belg. Math. Soc. Simon Stevin **22**, 391–401 (2015)
11. Zhao, C.J.: Cyclic Brunn–Minkowski inequalities for p-affine surface area. Quaest. Math. **40**(4), 467–476 (2017)
12. Feng, Y.B., Wang, W.D.: General L_p-harmonic Blaschke bodies. Proc. Indian Acad. Sci. Math. Sci. **124**, 109–119 (2014)
13. Feng, Y.B., Wang, W.D., Lu, F.H.: Some inequalities on general L_p-centroid bodies. Math. Inequal. Appl. **18**, 39–49 (2015)
14. Haberl, C., Schuster, F.E.: Asymmetric affine L_p Sobolev inequalities. J. Funct. Anal. **257**, 641–658 (2009)
15. Li, Z.F., Wang, W.D.: General L_p-mixed chord integrals of star bodies. J. Inequal. Appl. **2016**, 58 (2016)
16. Ludwig, M.: Minkowski areas and valuations. J. Differ. Geom. **86**, 133–162 (2010)
17. Wang, W.D., Feng, Y.B.: A general L_p-version of Petty's affine projection inequality. Taiwan. J. Math. **17**, 517–528 (2013)
18. Wang, W.D., Li, Y.N.: General L_p-intersection bodies. Taiwan. J. Math. **19**, 1247–1259 (2015)
19. Wang, W.D., Ma, T.Y.: Asymmetric L_p-difference bodies. Proc. Am. Math. Soc. **142**, 2517–2527 (2014)
20. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory, 2nd edn. Encyclopedia Math. Appl. Cambridge University Press, Cambridge (2014)