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Estimates of upper bound for a kth order differentiable functions involving Riemann–Liouville integrals via higher order strongly h-preinvex functions

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Abstract

In this paper the notion of higher order strongly h-preinvex functions is presented, which unifies several known classes of preinvexity. An identity related to the kth order differentiable functions and Riemann–Liouville integrals is established. The identity is then used to obtain some estimates of upper bound for the kth order differentiable functions involving Riemann–Liouville integrals via higher order strongly h-preinvex functions.

MSC: 26A51; 26D15; 52A01

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1 Introduction

We start by briefly summarizing the concepts on generalized convex functions which are related to the contents of this paper.

A set $K \subset \mathbb{R}$ is said to be convex if

$$(1-t)x + ty \in K, \quad \forall x, y \in K, t \in [0,1].$$

A function $f : K \to \mathbb{R}$ is said to be convex if

$$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y), \quad \forall x, y \in K, t \in [0,1].$$

For decades, the convexity properties of sets and functions have played the vital role in modern analysis, and they are particularly associated with the theory of inequalities. Applications of convexity in different fields of pure and applied sciences such as optimization theory, numerical analysis, economics etc. make this research topic more fascinating. Numerous generalizations of convexity have been proposed in the literature. For details, see [7]. Hanson [13] gave the notion of differentiable invex functions in connection with their special global optimum behavior. Craven [6] introduced the term "invex" for calling this class of functions due to their property described as "invariance by convexity". Invex sets were first defined by Mititelu [19].

Let K_η be a non-empty set in $\mathbb{R}$, and suppose that $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous bifunction.

Definition 1.1 ([19]) A set K_η is said to be invex if

$$x + t\eta(y,x) \in K_\eta, \quad \forall x,y \in K_\eta, t \in [0,1].$$

Note that the invexity property of sets reduces to convexity if we take $\eta(y,x) = y - x$. Thus, every convex set is also an invex set with respect to $\eta(y,x) = y - x$, but the converse is not necessarily true.

Weir and Mond [31] introduced the notion of preinvex functions as follows.

Definition 1.2 ([31]) A function $f : K_\eta \to \mathbb{R}$ is said to be preinvex if

$$f\bigl(x + t\eta(y,x)\bigr) \le (1-t)f(x) + tf(y), \quad \forall x,y \in K_\eta, t \in [0,1].$$

For recent studies on preinvex functions, interested readers are referred to [4, 13, 19, 20, 22–25].

Varosanec [30] introduced the class of h-convex functions. The idea at the time of introduction was to unify some generalizations of classical convexity, such as Breckner type of s-convex functions [5], Q-functions [12], and P-functions [11]. Now we know that this class also generalizes some other classes of classical convex functions. For details, see [9]. h-convex functions have received special attention by many researchers and, consequently, a considerable amount of research papers have been uniquely devoted to the study of this class. Noor et al. [25] extended this concept using the invexity property of sets and defined the notion of h-preinvex functions. They have observed that it contains several new and known classes of convexity.

Mohan and Neogy [20] discussed a very famous condition C.

Condition C Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, we say that the bifunction $\eta(\cdot,\cdot)$ satisfies condition C if, for any $x,y \in \mathbb{R}^n$,

(1) $\eta(x, x + t\eta(y,x)) = -t\eta(y,x)$

(2) $\eta(y, x + t\eta(y,x)) = (1-t)\eta(y,x)$

for all $t \in [0,1]$.

Note that, for any $x,y \in \mathbb{R}^n$, $t_1, t_2 \in [0,1]$ and from condition C we can deduce

$$\eta\bigl(x + t_2\eta(y,x), x + t_1\eta(y,x)\bigr) = (t_2 - t_1)\eta(y,x).$$

Karamardian [14] and Polyak [26] independently introduced strongly convex functions. Strong convexity is just a strengthening property of convexity.

Definition 1.3 A function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a strongly convex function with modulus $\mu > 0$ if

$$f\bigl((1-t)x + ty\bigr) \le (1-t)f(x) + tf(y) - \mu t(1-t)\|y - x\|^2, \quad \forall x,y \in X, t \in [0,1].$$

Karamardian [14] noticed that every differentiable function is strongly convex if and only if its gradient map is strongly monotone. Lin et al. [16] introduced higher order strongly convex functions to simplify the study of mathematical programs with equilibrium constraints. For some investigations on strongly convex functions, see [1–3, 17, 18, 21, 24, 26].

Definition 1.4 ([16]) A function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a strongly convex function with order $\sigma > 0$ and modulus $\mu > 0$ if

$$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y) - \mu t(1-t)\|y-x\|^{\sigma}, \quad \forall x, y \in X, t \in [0,1].$$

Observe that when $\sigma = 2$, then the above definition reduces to strong convexity in the classical sense. Lin et al. [16] have also shown that the higher order strong convexity of a function is equivalent to higher order strong monotonicity of the gradient map of the function.

Theory of convexity has also a great impact on theory of inequalities. One cannot deny the important role of convexity in the development of inequality theory. Many famous results in inequalities are due to the convexity property of functions. Hermite–Hadamard's inequality, which is very simple in nature yet powerful, is one of the most extensively studied results for convex functions. This result provides us a necessary and sufficient condition for a function to be convex. It reads as follows.

Theorem 1.5 *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

For more and interesting details on Hermite–Hadamard's inequalities, its generalizations and applications, see [8, 10].

Sarikaya et al. [27] utilized the concepts of fractional calculus and obtained a new variant of Hermite–Hadamard's inequality via convex functions. In that paper authors also obtained a new fractional integral identity. This particular paper of Sarikaya et al. [27] has opened a new dimension of research. Since then many researchers have extensively utilized the concepts of fractional calculus and obtained numerous new and novel refinements of inequalities via convex functions and their generalizations. For more information, see [27–29, 32].

The aim of this article is to introduce the notion of higher order strongly h-preinvex functions. We also show that the class of higher order strongly h-preinvex functions unifies several other classes of strong preinvexity. We present an integral identity related to the kth order differentiable functions. Then, using this auxiliary result, we establish our main results that are some estimates of upper bound for kth order differentiable function via higher order strongly h-preinvex functions.

2 Higher order strongly h-preinvex functions

In this section, we introduce the higher order strongly h-preinvex functions and associated notions.

Definition 2.1 Let $h:(0,1)\to\mathbb{R}$ be a real function. A function $f:X\subset\mathbb{R}^n\to\mathbb{R}$ is said to be a higher order strongly h-preinvex function with order $\sigma>0$ and modulus $\mu\geq 0$ if

$$f\big(x+t\eta(y,x)\big)\leq h(1-t)f(x)+h(t)f(y)-\mu t(1-t)\big\|\eta(y,x)\big\|^{\sigma},\quad \forall x,y\in X,t\in[0,1].$$

Higher order strongly preinvex function is defined as follows.

Definition 2.2 ([2]) A function $f:X\subset\mathbb{R}^n\to\mathbb{R}$ is said to be a higher order strongly preinvex function with order $\sigma>0$ and modulus $\mu\geq 0$ if

$$f\big(x+t\eta(y,x)\big)\leq (1-t)f(x)+tf(y)-\mu t(1-t)\big\|\eta(y,x)\big\|^{\sigma},\quad \forall x,y\in X,t\in[0,1].$$

Higher order strongly s-preinvex function of Breckner type is defined as follows.

Definition 2.3 A function $f:X\subset\mathbb{R}^n\to\mathbb{R}$ is said to be a higher order strongly s-preinvex function of Breckner type with order $\sigma>0$, modulus $\mu\geq 0$, and $s\in(0,1]$ if

$$f\big(x+t\eta(y,x)\big)\leq (1-t)^s f(x)+t^s f(y)-\mu t(1-t)\big\|\eta(y,x)\big\|^{\sigma},\quad \forall x,y\in X,t\in[0,1].$$

Higher order strongly s-preinvex function of Godunova–Levin type is defined as follows.

Definition 2.4 A function $f:X\subset\mathbb{R}^n\to\mathbb{R}$ is said to be a higher order strongly s-preinvex function of Godunova–Levin type with order $\sigma>0$, modulus $\mu\geq 0$, and $s\in(0,1)$ if

$$f\big(x+t\eta(y,x)\big)\leq \frac{1}{(1-t)^s}f(x)+\frac{1}{t^s}f(y)-\mu t(1-t)\big\|\eta(y,x)\big\|^{\sigma},\quad \forall x,y\in X,t\in(0,1).$$

Higher order strongly Q-preinvex function is defined as follows.

Definition 2.5 A function $f:X\subset\mathbb{R}^n\to\mathbb{R}$ is said to be a higher order strongly Q-preinvex function with order $\sigma>0$ and modulus $\mu\geq 0$ if

$$f\big(x+t\eta(y,x)\big)\leq \frac{1}{1-t}f(x)+\frac{1}{t}f(y)-\mu t(1-t)\big\|\eta(y,x)\big\|^{\sigma},\quad \forall x,y\in X,t\in(0,1).$$

Higher order strongly P-preinvex function is defined as follows.

Definition 2.6 A function $f:X\subset\mathbb{R}^n\to\mathbb{R}$ is said to be higher order strongly P-preinvex function with order $\sigma>0$ and modulus $\mu\geq 0$ if

$$f\big(x+t\eta(y,x)\big)\leq f(x)+f(y)-\mu t(1-t)\big\|\eta(y,x)\big\|^{\sigma},\quad \forall x,y\in X,t\in(0,1).$$

Remark 2.7 Note that if we take $h(t)=t,t^s,t^{-s},t^{-1}$ and $h(t)=1$, then Definition 2.1 reduces to Definitions 2.2, 2.3, 2.4, 2.5 and Definition 2.6 respectively. Also it is obvious that if we take $\eta(y,x)=y-x$ in Definition 2.1, then we have the class of higher order strongly h-convex functions. It is worth to mention here that to the best of our knowledge this class is also new in the literature. Similarly, for different suitable choices of the function $h(\cdot)$, we have other classes of higher order strong convexity. Thus the class of higher order strongly h-preinvex functions is a quite unifying one.

3 Auxiliary result

In this section we shall prove an auxiliary result which plays an important role in dealing with subsequent results. Before we start to describe the auxiliary result, let us recall the classic definition of Riemann–Liouville fractional integrals.

Definition 3.1 ([15]) Let $f \in L[c,d]$, where $c \geq 0$. The Riemann–Liouville integrals $J^{\nu}_{c+}f$ and $J^{\nu}_{d-}f$, of order $\nu > 0$, are defined by

$$J^{\nu}_{c+}f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\,\mathrm{d}t \quad \text{for } x > c$$

and

$$J^{\nu}_{d-}f(x) = \frac{1}{\Gamma(\nu)} \int_x^d (t-x)^{\nu-1} f(t)\,\mathrm{d}t \quad \text{for } x < d,$$

respectively. Here, $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1}\,\mathrm{d}t$ is the gamma function. We also make the convention

$$J^0_{c+}f(x) = J^0_{d-}f(x) = f(x).$$

Some of our calculations need beta and hypergeometric functions, which are respectively defined as

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

The integral form of the hypergeometric function is

$${}_2f_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\,\mathrm{d}t$$

for $|z| < 1$, $c > y > 0$.

Lemma 3.2 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable. If $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $0 < \alpha$, and $n \in \mathbb{N}$, then*

$$\begin{aligned}&\mathcal{H}(k,n,\alpha,a,b)(f)\\&\quad= \int_0^1 (1-t)^{\alpha+k-1}\left[f^{(k)}\left(a + \frac{1-t}{n+1}\eta(b,a)\right) + f^{(k)}\left(a + \frac{n+t}{n+1}\eta(b,a)\right)\right]\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}&\mathcal{H}(k,n,\alpha,a,b)(f)\\&\quad= \left(\frac{n+1}{\eta(b,a)}\right)^{k+\alpha} \Gamma(\alpha+k)\big[J^{\alpha}_{(a+\frac{n}{n+1}\eta(b,a))^+}f\big(a+\eta(b,a)\big) + (-1)^k J^{\alpha}_{(a+\frac{1}{n+1}\eta(b,a))^-}f(a)\big]\end{aligned}$$

$$-\sum_{j=1}^{k}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)}\left(\frac{n+1}{\eta(b,a)}\right)^{j}$$
$$\times\left[f^{(k-j)}\left(a+\frac{n}{n+1}\eta(b,a)\right)+(-1)^{j}f^{(k-j)}\left(a+\frac{1}{n+1}\eta(b,a)\right)\right]. \tag{3.1}$$

Proof Let

$$\begin{aligned}
&\int_0^1(1-t)^{\alpha+k-1}\left[f^{(k)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)+f^{(k)}\left(a+\frac{n+t}{n+1}\eta(b,a)\right)\right]\mathrm{d}t\\
&=\int_0^1(1-t)^{\alpha+k-1}f^{(k)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\mathrm{d}t\\
&\quad+\int_0^1(1-t)^{\alpha+k-1}f^{(k)}\left(a+\frac{n+t}{n+1}\eta(b,a)\right)\mathrm{d}t\\
&=I_1+I_2.
\end{aligned}$$

We first calculate I_1, integration by parts gives

$$\begin{aligned}
I_1&=\int_0^1(1-t)^{\alpha+k-1}f^{(k)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\mathrm{d}t\\
&=\left[-\frac{n+1}{\eta(b,a)}(1-t)^{\alpha+k-1}f^{(k-1)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\right]\Bigg|_0^1\\
&\quad-\frac{(\alpha+k-1)(n+1)}{\eta(b,a)}\int_0^1(1-t)^{\alpha+k-2}f^{(k-1)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\mathrm{d}t.
\end{aligned}$$

Using integration by parts again, we have

$$\begin{aligned}
I_1&=\frac{n+1}{\eta(b,a)}f^{(k-1)}\left(a+\frac{1}{n+1}\eta(b,a)\right)-\frac{(\alpha+k-1)(n+1)^2}{\eta^2(b,a)}f^{(k-2)}\left(a+\frac{1}{n+1}\eta(b,a)\right)\\
&\quad+\frac{(\alpha+k-1)(\alpha+k-2)(n+1)^2}{\eta^2(b,a)}\int_0^1(1-t)^{\alpha+k-3}f^{(k-2)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\mathrm{d}t.
\end{aligned}$$

Repeat the integration by parts successively and obtain, after k integrations,

$$\begin{aligned}
I_1&=\sum_{j=1}^{k}\frac{(-1)^{j-1}}{\alpha+k}\left(\frac{n+1}{\eta(b,a)}\right)^{j}f^{(k-j)}\left(a+\frac{1}{n+1}\eta(b,a)\right)\prod_{p=0}^{j-1}(\alpha+k-p)\\
&\quad+\frac{(-1)^{k}}{\alpha+k}\left(\frac{n+1}{\eta(b,a)}\right)^{k}\left[\prod_{p=0}^{k}(\alpha+k-p)\right]\int_0^1(1-t)^{\alpha-1}f\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\mathrm{d}t\\
&=\sum_{j=1}^{k}\frac{(-1)^{j-1}}{\alpha+k}\left(\frac{n+1}{\eta(b,a)}\right)^{j}f^{(k-j)}\left(a+\frac{1}{n+1}\eta(b,a)\right)\prod_{p=0}^{j-1}(\alpha+k-p)\\
&\quad+(-1)^{k}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\left(\frac{n+1}{\eta(b,a)}\right)^{k}\int_0^1(1-t)^{\alpha-1}f\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} (-1)^{j-1} \left(\frac{n+1}{\eta(b,a)}\right)^{j} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)} f^{(k-j)}\left(a + \frac{1}{n+1}\eta(b,a)\right) \\
&\quad + (-1)^{k} \Gamma(\alpha+k) \left(\frac{n+1}{\eta(b,a)}\right)^{\alpha+k} J^{\alpha}_{(a+\frac{1}{n+1}\eta(b,a))^-} f(a).
\end{aligned}
$$

A similar method as above is used to compute I_2, we get

$$
\begin{aligned}
I_2 &= \int_0^1 (1-t)^{\alpha+k-1} f^{(k)}\left(a + \frac{n+t}{n+1}\eta(b,a)\right) \mathrm{d}t \\
&= -\frac{n+1}{\eta(b,a)} f^{(k-1)}\left(a + \frac{n}{n+1}\eta(b,a)\right) - (\alpha+k-1)\left(\frac{n+1}{\eta(b,a)}\right)^{2} f^{(k-2)}\left(a + \frac{n}{n+1}\eta(b,a)\right) \\
&\quad - (\alpha+k-1)(\alpha+k-2)\left(\frac{n+1}{\eta(b,a)}\right)^{3} f^{(k-3)}\left(a + \frac{n}{n+1}\eta(b,a)\right) \\
&\quad + (\alpha+k-1)(\alpha+k-2)(\alpha+k-3)\left(\frac{n+1}{\eta(b,a)}\right)^{3} \\
&\quad \times \int_0^1 (1-t)^{\alpha+k-4} f^{(k-3)}\left(a + \frac{n+t}{n+1}\eta(b,a)\right) \mathrm{d}t \\
&\vdots \\
&= -\sum_{j=1}^{k} \frac{\prod_{p=0}^{j-1}(\alpha+k-p)}{\alpha+k} \left(\frac{n+1}{\eta(b,a)}\right)^{j} f^{(k-j)}\left(a + \frac{n}{n+1}\eta(b,a)\right) \\
&\quad + \frac{\prod_{p=0}^{k}(\alpha+k-p)}{\alpha+k} \left(\frac{n+1}{\eta(b,a)}\right)^{k} \int_0^1 (1-t)^{\alpha-1} f\left(a + \frac{n+t}{n+1}\eta(b,a)\right) \mathrm{d}t \\
&= -\sum_{j=1}^{k} \left(\frac{n+1}{\eta(b,a)}\right)^{j} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)} f^{(k-j)}\left(a + \frac{n}{n+1}\eta(b,a)\right) \\
&\quad + \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \left(\frac{n+1}{\eta(b,a)}\right)^{k} \int_0^1 (1-t)^{\alpha-1} f\left(a + \frac{n+t}{n+1}\eta(b,a)\right) \mathrm{d}t \\
&= -\sum_{j=1}^{k} \left(\frac{n+1}{\eta(b,a)}\right)^{j} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)} f^{(k-j)}\left(a + \frac{n}{n+1}\eta(b,a)\right) \\
&\quad + \Gamma(\alpha+k) \left(\frac{n+1}{\eta(b,a)}\right)^{\alpha+k} J^{\alpha}_{(a+\frac{n}{n+1}\eta(b,a))^+} f\big(a+\eta(b,a)\big).
\end{aligned}
$$

Now, by summing up I_1 and I_2, we have

$$
\begin{aligned}
I_1 + I_2 &= \sum_{j=1}^{k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)} \left(\frac{n+1}{\eta(b,a)}\right)^{j} \left[(-1)^{j-1} f^{(k-j)}\left(a + \frac{1}{n+1}\eta(b,a)\right)\right. \\
&\quad \left. - f^{(k-j)}\left(a + \frac{n}{n+1}\eta(b,a)\right)\right] \\
&\quad + \Gamma(\alpha+k) \left(\frac{n+1}{\eta(b,a)}\right)^{\alpha+k} \left[J^{\alpha}_{(a+\frac{n}{n+1}\eta(b,a))^+} f\big(a+\eta(b,a)\big) + (-1)^{k} J^{\alpha}_{(a+\frac{1}{n+1}\eta(b,a))^-} f(a)\right].
\end{aligned}
$$

After convenient arrangement, we deduce the identity described in Lemma 3.2. □

4 Main results

We consider some estimates of upper bound for the function $\mathcal{H}(k,n,\alpha,a,b)(f)$ via higher order strongly h-preinvex functions, our main results are stated in the following theorems.

Theorem 4.1 *Suppose that $k \in \mathbb{N}$, the function $f: I \to \mathbb{R}$ is of kth order differentiable, $a, a+\eta(b,a) \in I$, $0 < \eta(b,a)$, $0 < \alpha$, and $n \in \mathbb{N}$. If $|f^{(k)}|$ is a higher order strongly h-preinvex function, then*

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&\quad \le \phi(\alpha,k,n,t)\left[\left|f^{(k)}(a)\right| + \left|f^{(k)}(b)\right|\right] \\
&\qquad - \frac{2\mu}{(n+1)^2}\left(\frac{n(\alpha+k+2)+1}{(\alpha+k+1)(\alpha+k+2)}\right)\left\|\eta(b,a)\right\|^{\sigma},
\end{aligned}$$

where

$$\phi(\alpha,k,n,t) := \int_0^1 (1-t)^{\alpha+k-1}\left[h\left(\frac{n+t}{n+1}\right) + h\left(\frac{1-t}{n+1}\right)\right]\mathrm{d}t. \tag{4.1}$$

Proof Utilizing Lemma 3.2, the property of the modulus and the fact that $|f^{(k)}|$ is a higher order strongly h-preinvex function, we have

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&\quad = \left|\int_0^1 (1-t)^{\alpha+k-1}\left[f^{(k)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right) + f^{(k)}\left(a+\frac{n+t}{n+1}\eta(b,a)\right)\right]\mathrm{d}t\right| \\
&\quad \le \left|\int_0^1 (1-t)^{\alpha+k-1} f^{(k)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\mathrm{d}t\right| \\
&\qquad + \left|\int_0^1 (1-t)^{\alpha+k-1} f^{(k)}\left(a+\frac{n+t}{n+1}\eta(b,a)\right)\mathrm{d}t\right| \\
&\quad \le \int_0^1 (1-t)^{\alpha+k-1}\left[h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(a)\right| + h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(b)\right|\right. \\
&\qquad \left. - \frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^{\sigma}\right]\mathrm{d}t \\
&\qquad + \int_0^1 (1-t)^{\alpha+k-1}\left[h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(a)\right| + h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(b)\right|\right. \\
&\qquad \left. - \frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^{\sigma}\right]\mathrm{d}t \\
&\quad = \phi(\alpha,k,n,t)\left[\left|f^{(k)}(a)\right| + \left|f^{(k)}(b)\right|\right] \\
&\qquad - \frac{2\mu}{(n+1)^2}\left(\frac{n(\alpha+k+2)+1}{(\alpha+k+1)(\alpha+k+2)}\right)\left\|\eta(b,a)\right\|^{\sigma}.
\end{aligned}$$

This completes the proof of Theorem 4.1. □

We now discuss some special cases which can be deduced directly from Theorem 4.1.

I. If we take $h(t) = t$, then we have the result for higher order strongly preinvex functions.

Corollary 4.2 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, and $n \in \mathbb{N}$. If $|f^{(k)}|$ is a higher order strongly preinvex function, then*

$$\begin{aligned}\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \le \phi^{*}(\alpha,k,n)\left[\left|f^{(k)}(a)\right| + \left|f^{(k)}(b)\right|\right] \\ - \frac{2\mu}{(n+1)^2}\left(\frac{n(\alpha+k+2)+1}{(\alpha+k+1)(\alpha+k+2)}\right)\left\|\eta(b,a)\right\|^{\sigma},\end{aligned}$$

where

$$\phi^{*}(\alpha,k,n) := \frac{1}{\alpha+k}.$$

II. If we put $h(t) = t^s$, then we have the result for Breckner type of higher order strongly s-preinvex functions.

Corollary 4.3 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, and $n \in \mathbb{N}$. If $|f^{(k)}|$ is Breckner type of a higher order strongly s-preinvex function, then*

$$\begin{aligned}&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\ &\quad\le \phi^{**}(\alpha,k,n)\left[\left|f^{(k)}(a)\right| + \left|f^{(k)}(b)\right|\right] \\ &\qquad - \frac{2\mu}{(n+1)^2}\left(\frac{n(\alpha+k+2)+1}{(\alpha+k+1)(\alpha+k+2)}\right)\left\|\eta(b,a)\right\|^{\sigma},\end{aligned}$$

where

$$\begin{aligned}\phi^{**}(\alpha,k,n) &:= \int_0^1 (1-t)^{\alpha+k-1}\left[\left(\frac{n+t}{n+1}\right)^s + \left(\frac{1-t}{n+1}\right)^s\right] \mathrm{d}t \\ &:= \frac{1}{(n+1)^s}\left[n^s B(1,\alpha+k)\,{}_2f_1\left(-s,1;\alpha+k+1;-\frac{1}{n}\right) + \frac{1}{\alpha+k+s}\right].\end{aligned}$$

III. If we put $h(t) = t^{-s}$, then we have the result for Godunova–Levin type of higher order strongly s-preinvex functions.

Corollary 4.4 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, and $n \in \mathbb{N}$. If $|f^{(k)}|$ is Godunova–Levin type of a higher order strongly s-preinvex function, then*

$$\begin{aligned}&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\ &\quad\le \phi^{***}(\alpha,k,n)\left[\left|f^{(k)}(a)\right| + \left|f^{(k)}(b)\right|\right] \\ &\qquad - \frac{2\mu}{(n+1)^2}\left(\frac{n(\alpha+k+2)+1}{(\alpha+k+1)(\alpha+k+2)}\right)\left\|\eta(b,a)\right\|^{\sigma},\end{aligned}$$

where

$$\phi^{***}(\alpha,k,n) := \int_0^1 (1-t)^{\alpha+k-1}\left[\left(\frac{n+t}{n+1}\right)^{-s} + \left(\frac{1-t}{n+1}\right)^{-s}\right]\mathrm{d}t$$
$$:= (n+1)^s\left[n^{-s}B(1,\alpha+k)_2f_1\left(s,1;\alpha+k+1;-\frac{1}{n}\right) + \frac{1}{\alpha+k-s}\right].$$

IV. If we choose $h(t)=1$, then we have the result for higher order strongly P-preinvex functions.

Corollary 4.5 *Suppose that $k\in\mathbb{N}$, the function $f: I\to\mathbb{R}$ is of kth order differentiable, $a, a+\eta(b,a)\in I$, $0<\eta(b,a)$, and $n\in\mathbb{N}$. If $|f^{(k)}|$ is a higher order strongly P-preinvex function, then*

$$\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right|$$
$$\le \frac{2}{\alpha+k}\left[\left|f^{(k)}(a)\right| + \left|f^{(k)}(b)\right|\right] - \frac{2\mu}{(n+1)^2}\left(\frac{n(\alpha+k+2)+1}{(\alpha+k+1)(\alpha+k+2)}\right)\left\|\eta(b,a)\right\|^{\sigma}.$$

Theorem 4.6 *Suppose that $k\in\mathbb{N}$, the function $f: I\to\mathbb{R}$ is of kth order differentiable, $a, a+\eta(b,a)\in I$, $0<\eta(b,a)$, $0<\alpha$, and $n\in\mathbb{N}$. If $|f^{(k)}|^q$ where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$ is a higher order strongly h-preinvex function, then*

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&\le \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}}\Bigg\{\Bigg[\int_0^1\left(h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(a)\right|^q\right. \\
&\quad \left.+ h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(b)\right|^q\right)\mathrm{d}t \\
&\quad - \frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^{\sigma}\Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg[\int_0^1\left(h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(a)\right|^q + h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(b)\right|^q\right)\mathrm{d}t \\
&\quad - \frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^{\sigma}\Bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Proof Utilizing Lemma 3.2, Hölder's inequality, and the fact that $|f^{(k)}|^q$ is a higher order strongly h-preinvex function, we have

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&= \left|\int_0^1 (1-t)^{\alpha+k-1}\left[f^{(k)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right) + f^{(k)}\left(a+\frac{n+t}{n+1}\eta(b,a)\right)\right]\mathrm{d}t\right| \\
&\le \left(\int_0^1 (1-t)^{p(\alpha+k-1)}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(k)}\left(a+\frac{1-t}{n+1}\eta(b,a)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (1-t)^{p(\alpha+k-1)}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(k)}\left(a+\frac{n+t}{n+1}\eta(b,a)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}} \left[\int_0^1 \left(h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(a)\right|^q + h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(b)\right|^q \right.\right.\\
&\quad \left.\left. - \frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^{\sigma}\right)\mathrm{d}t\right]^{\frac{1}{q}}\\
&\quad + \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}} \left[\int_0^1 \left(h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(a)\right|^q + h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(b)\right|^q \right.\right.\\
&\quad \left.\left. - \frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^{\sigma}\right)\mathrm{d}t\right]^{\frac{1}{q}}\\
&= \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}} \left\{\left[\int_0^1 \left(h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(a)\right|^q \right.\right.\right.\\
&\quad \left. + h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(b)\right|^q\right)\mathrm{d}t\\
&\quad \left. - \frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^{\sigma}\right]^{\frac{1}{q}}\\
&\quad + \left[\int_0^1 \left(h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(a)\right|^q + h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(b)\right|^q\right)\mathrm{d}t \right.\\
&\quad \left.\left. - \frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^{\sigma}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

The proof of Theorem 4.6 is completed. □

In the following, we give four corollaries that follow from the special cases of Theorem 4.6.

I. If we choose $h(t) = t$, then we have the result for higher order strongly preinvex functions.

Corollary 4.7 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$ is a higher order strongly preinvex function, then*

$$
\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right|\\
&\quad \leq \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}} \left(\frac{1}{2(n+1)}\right)^{\frac{1}{q}}\\
&\quad\quad \times \left\{\left[(2n+1)\left|f^{(k)}(a)\right|^q + \left|f^{(k)}(b)\right|^q - \frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^{\sigma}\right]^{\frac{1}{q}}\right.\\
&\quad\quad \left. + \left[\left|f^{(k)}(a)\right|^q + (2n+1)\left|f^{(k)}(b)\right|^q - \frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^{\sigma}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

II. If we take $h(t) = t^s$, then we have the result for Breckner type of higher order strongly s-preinvex functions.

Corollary 4.8 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$ is Breckner type of a* *higher order strongly s-preinvex function, then*

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&\leq \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{(n+1)^s}\right)^{\frac{1}{q}} \\
&\times \left\{\left[\left(\frac{(1+n)^{s+1}-n^{s+1}}{s+1}\right)\left|f^{(k)}(a)\right|^q+\left(\frac{1}{s+1}\right)\left|f^{(k)}(b)\right|^q\right.\right. \\
&\left.-\frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}} \\
&+\left[\left(\frac{1}{s+1}\right)\left|f^{(k)}(a)\right|^q+\left(\frac{(1+n)^{s+1}-n^{s+1}}{s+1}\right)\left|f^{(k)}(b)\right|^q\right. \\
&\left.\left.-\frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

III. If we take $h(t)=t^{-s}$, then we have the result for Godunova–Levin type of higher order strongly s-preinvex functions.

Corollary 4.9 *Suppose that* $k\in\mathbb{N}$, *the function* $f:I\to\mathbb{R}$ *is of* k*th order differentiable,* $a, a+\eta(b,a)\in I$, $0<\eta(b,a)$, *and* $n\in\mathbb{N}$. *If* $|f^{(k)}|^q$ *where* $\frac{1}{p}+\frac{1}{q}=1$, $q>1$ *is Godunova–Levin type of a higher order strongly s-preinvex function, then*

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&\leq \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}}(n+1)^{\frac{s}{q}} \\
&\times \left\{\left[\left(\frac{(1+n)^{1-s}-n^{1-s}}{1-s}\right)\left|f^{(k)}(a)\right|^q+\left(\frac{1}{1-s}\right)\left|f^{(k)}(b)\right|^q\right.\right. \\
&\left.-\frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}} \\
&+\left[\left(\frac{1}{1-s}\right)\left|f^{(k)}(a)\right|^q+\left(\frac{(1+n)^{1-s}-n^{1-s}}{1-s}\right)\left|f^{(k)}(b)\right|^q\right. \\
&\left.\left.-\frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

IV. If we put $h(t)=1$, then we have the result for higher order strongly P-preinvex functions.

Corollary 4.10 *Suppose that* $k\in\mathbb{N}$, *the function* $f:I\to\mathbb{R}$ *is of* k*th order differentiable,* $a, a+\eta(b,a)\in I$, $0<\eta(b,a)$, *and* $n\in\mathbb{N}$. *If* $|f^{(k)}|^q$ *where* $\frac{1}{p}+\frac{1}{q}=1$, $q>1$ *is a higher order strongly P-preinvex function, then*

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&\leq 2\left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}}\left[\left|f^{(k)}(a)\right|^q+\left|f^{(k)}(b)\right|^q-\frac{\mu(3n+1)}{6(n+1)^2}\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4.11 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $0 < \alpha$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is a higher order strongly h-preinvex function, then*

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&\leq \Bigg[\psi_1(\alpha,k,n)\left|f^{(k)}(a)\right|^q + \psi_2(\alpha,k,n)\left|f^{(k)}(b)\right|^q \\
&\quad - \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^\sigma \Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg[\psi_2(\alpha,k,n)\left|f^{(k)}(a)\right|^q + \psi_1(\alpha,k,n)\left|f^{(k)}(b)\right|^q \\
&\quad - \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^\sigma \Bigg]^{\frac{1}{q}},
\end{aligned}$$

where

$$\psi_1(\alpha,k,n) := \int_0^1 (1-t)^{q(\alpha+k-1)} h\left(\frac{n+t}{n+1}\right) \mathrm{d}t \tag{4.2}$$

and

$$\psi_2(\alpha,k,n) := \int_0^1 (1-t)^{q(\alpha+k-1)} h\left(\frac{1-t}{n+1}\right) \mathrm{d}t. \tag{4.3}$$

Proof Utilizing Lemma 3.2, Hölder's inequality, and the fact that $|f^{(k)}|$ is a higher order strongly h-preinvex function, we have

$$\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right| \\
&= \left|\int_0^1 (1-t)^{\alpha+k-1}\left[f^{(k)}\left(a + \frac{1-t}{n+1}\eta(b,a)\right) + f^{(k)}\left(a + \frac{n+t}{n+1}\eta(b,a)\right)\right] \mathrm{d}t\right| \\
&\leq \left(\int_0^1 1\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^{q(\alpha+k-1)}\left|f^{(k)}\left(a + \frac{1-t}{n+1}\eta(b,a)\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 1\,\mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^{q(\alpha+k-1)}\left|f^{(k)}\left(a + \frac{n+t}{n+1}\eta(b,a)\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \Bigg[\int_0^1 (1-t)^{q(\alpha+k-1)}\Bigg[h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(a)\right|^q + h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(b)\right|^q \\
&\quad - \frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^\sigma\Bigg] \mathrm{d}t\Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg[\int_0^1 (1-t)^{q(\alpha+k-1)}\Bigg[h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(a)\right|^q + h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(b)\right|^q \\
&\quad - \frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^\sigma\Bigg] \mathrm{d}t\Bigg]^{\frac{1}{q}} \\
&= \Bigg[\psi_1(\alpha,k,n)\left|f^{(k)}(a)\right|^q + \psi_2(\alpha,k,n)\left|f^{(k)}(b)\right|^q
\end{aligned}$$

$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}}$$
$$+ \Bigg[\psi_2(\alpha,k,n)\left|f^{(k)}(a)\right|^q + \psi_1(\alpha,k,n)\left|f^{(k)}(b)\right|^q$$
$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}}.$$

This completes the proof of Theorem 4.11. □

We next discuss some special cases of Theorem 4.11.

I. If we put $h(t) = t$, then we have the result for higher order strongly preinvex functions.

Corollary 4.12 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is a higher order strongly preinvex function, then*

$$\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right|$$
$$\leq \Bigg[\psi_1^*(\alpha,k,n)\left|f^{(k)}(a)\right|^q + \psi_2^*(\alpha,k,n)\left|f^{(k)}(b)\right|^q$$
$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}}$$
$$+ \Bigg[\psi_2^*(\alpha,k,n)\left|f^{(k)}(a)\right|^q + \psi_1^*(\alpha,k,n)\left|f^{(k)}(b)\right|^q$$
$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}},$$

where

$$\psi_1^*(\alpha,k,n) := \frac{1}{q(\alpha+k-1)+1} - \frac{1}{(n+1)[q(\alpha+k-1)+2]}$$

and

$$\psi_2^*(\alpha,k,n) := \frac{1}{(n+1)[q(\alpha+k-1)+2]}.$$

II. If we take $h(t) = t^s$, then we have the result for Breckner type of higher order strongly s-preinvex functions.

Corollary 4.13 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is Breckner type of a higher order strongly s-preinvex function, then*

$$\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right|$$
$$\leq \Bigg[\psi_1^{**}(\alpha,k,n)\left|f^{(k)}(a)\right|^q + \psi_2^{**}(\alpha,k,n)\left|f^{(k)}(b)\right|^q$$

$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}}$$
$$+ \Bigg[\psi_2^{**}(\alpha,k,n)\big|f^{(k)}(a)\big|^q + \psi_1^{**}(\alpha,k,n)\big|f^{(k)}(b)\big|^q$$
$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}},$$

where

$$\psi_1^{**}(\alpha,k,n) := \frac{1}{(n+1)^s}\int_0^1 (1-t)^{q(\alpha+k-1)}(n+t)^s \,\mathrm{d}t$$
$$:= \frac{n^s}{(n+1)^s} B\big(1, q(\alpha+k-1)+1\big)_2 f_1\left(-s, 1; q(\alpha+k-1)+2; -\frac{1}{n}\right)$$

and

$$\psi_2^{**}(\alpha,k,n) := \frac{1}{(n+1)^s}\cdot\frac{1}{q(\alpha+k-1)+s+1}.$$

III. If we take $h(t) = t^{-s}$, then we have the result for Godunova–Levin type of higher order strongly s-preinvex functions.

Corollary 4.14 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a+\eta(b,a) \in I$, $0 < \eta(b,a)$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is Godunova–Levin type of a higher order strongly s-preinvex function, then*

$$\big|\mathcal{H}(k,n,\alpha,a,b)(f)\big|$$
$$\leq \Bigg[\psi_1^{***}(\alpha,k,n)\big|f^{(k)}(a)\big|^q + \psi_2^{***}(\alpha,k,n)\big|f^{(k)}(b)\big|^q$$
$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}}$$
$$+ \Bigg[\psi_2^{***}(\alpha,k,n)\big|f^{(k)}(a)\big|^q + \psi_1^{***}(\alpha,k,n)\big|f^{(k)}(b)\big|^q$$
$$- \frac{\mu}{n+1}\left(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\right) \| \eta(b,a) \|^{\sigma} \Bigg]^{\frac{1}{q}},$$

where

$$\psi_1^{***}(\alpha,k,n) := (n+1)^s \int_0^1 (1-t)^{q(\alpha+k-1)}(n+t)^{-s} \,\mathrm{d}t$$
$$:= \frac{(n+1)^s}{n^s} B\big(1, q(\alpha+k-1)+1\big)_2 f_1\left(s, 1; q(\alpha+k-1)+2; -\frac{1}{n}\right)$$

and

$$\psi_2^{***}(\alpha,k,n) := \frac{(n+1)^s}{q(\alpha+k-1)-s+1}.$$

IV. If we choose $h(t) = 1$, then we have the result for higher order strongly P-preinvex functions.

Corollary 4.15 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is a higher order strongly P-preinvex function, then*

$$\begin{aligned}
&\bigl|\mathcal{H}(k,n,\alpha,a,b)(f)\bigr| \\
&\quad\le \frac{2}{(q(\alpha+k-1)+1)^{\frac{1}{q}}}\biggl[\bigl|f^{(k)}(a)\bigr|^q + \bigl|f^{(k)}(b)\bigr|^q \\
&\qquad - \frac{\mu(q(\alpha+k-1)+1)}{n+1}\biggl(\frac{1}{q(\alpha+k-1)+2} - \frac{1}{(n+1)(q(\alpha+k-1)+3)}\biggr) \parallel \eta(b,a) \parallel^{\sigma}\biggr]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4.16 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $0 < \alpha$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is a higher order strongly h-preinvex function, then*

$$\begin{aligned}
&\bigl|\mathcal{H}(k,n,\alpha,a,b)(f)\bigr| \\
&\quad\le \biggl(\frac{1}{\alpha+k}\biggr)^{1-\frac{1}{q}}\biggl\{\biggl[\theta_1(\alpha,k,n)\bigl|f^{(k)}(a)\bigr|^q + \theta_2(\alpha,k,n)\bigl|f^{(k)}(b)\bigr|^q \\
&\qquad - \frac{\mu}{(n+1)^2}\biggl(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\biggr)\bigl\|\eta(b,a)\bigr\|^{\sigma}\biggr]^{\frac{1}{q}} \\
&\qquad + \biggl[\theta_2(\alpha,k,n)\bigl|f^{(k)}(a)\bigr|^q + \theta_1(\alpha,k,n)\bigl|f^{(k)}(b)\bigr|^q \\
&\qquad - \frac{\mu}{(n+1)^2}\biggl(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\biggr)\bigl\|\eta(b,a)\bigr\|^{\sigma}\biggr]^{\frac{1}{q}}\biggr\},
\end{aligned}$$

where

$$\theta_1(\alpha,k,n) := \int_0^1 (1-t)^{\alpha+k-1} h\biggl(\frac{n+t}{n+1}\biggr)\,\mathrm{d}t \tag{4.4}$$

and

$$\theta_2(\alpha,k,n) := \int_0^1 (1-t)^{\alpha+k-1} h\biggl(\frac{1-t}{n+1}\biggr)\,\mathrm{d}t. \tag{4.5}$$

Proof Utilizing Lemma 3.2, power mean inequality, and the fact that $|f^{(k)}|$ is a higher order strongly h-preinvex function, we have

$$\begin{aligned}
&\bigl|\mathcal{H}(k,n,\alpha,a,b)(f)\bigr| \\
&\quad= \biggl|\int_0^1 (1-t)^{\alpha+k-1}\biggl[f^{(k)}\biggl(a + \frac{1-t}{n+1}\eta(b,a)\biggr) + f^{(k)}\biggl(a + \frac{n+t}{n+1}\eta(b,a)\biggr)\biggr]\,\mathrm{d}t\biggr| \\
&\quad\le \biggl(\int_0^1 (1-t)^{\alpha+k-1}\,\mathrm{d}t\biggr)^{1-\frac{1}{q}}\biggl(\int_0^1 (1-t)^{\alpha+k-1}\biggl|f^{(k)}\biggl(a + \frac{1-t}{n+1}\eta(b,a)\biggr)\biggr|^q\,\mathrm{d}t\biggr)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 (1-t)^{\alpha+k-1}\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{\alpha+k-1}\left|f^{(k)}\left(a+\frac{n+t}{n+1}\eta(b,a)\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\\
&\leq\left(\frac{1}{\alpha+k}\right)^{1-\frac{1}{q}}\left\{\left[\int_0^1 (1-t)^{\alpha+k-1}\left[h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(a)\right|^q\right.\right.\right.\\
&+h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(b)\right|^q\\
&\left.\left.-\frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^\sigma\right]\mathrm{d}t\right]^{\frac{1}{q}}\\
&+\left[\int_0^1 (1-t)^{\alpha+k-1}\left[h\left(\frac{1-t}{n+1}\right)\left|f^{(k)}(a)\right|^q+h\left(\frac{n+t}{n+1}\right)\left|f^{(k)}(b)\right|^q\right.\right.\\
&\left.\left.\left.-\frac{\mu}{(n+1)^2}(n+t)(1-t)\left\|\eta(b,a)\right\|^\sigma\right]\mathrm{d}t\right]^{\frac{1}{q}}\right\}\\
&=\left(\frac{1}{\alpha+k}\right)^{1-\frac{1}{q}}\left\{\left[\theta_1(\alpha,k,n)\left|f^{(k)}(a)\right|^q+\theta_2(\alpha,k,n)\left|f^{(k)}(b)\right|^q\right.\right.\\
&\left.-\frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}\\
&+\left[\theta_2(\alpha,k,n)\left|f^{(k)}(a)\right|^q+\theta_1(\alpha,k,n)\left|f^{(k)}(b)\right|^q\right.\\
&\left.\left.-\frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

The proof of Theorem 4.16 is complete. □

In the following we discuss some special cases of Theorem 4.16.

I. If we put $h(t)=t$, then we have the result for higher order strongly preinvex functions.

Corollary 4.17 *Suppose that $k\in\mathbb{N}$, the function $f:I\to\mathbb{R}$ is of kth order differentiable, $a, a+\eta(b,a)\in I$, $0<\eta(b,a)$, $q>1$, and $n\in\mathbb{N}$. If $|f^{(k)}|^q$ is a higher order strongly preinvex function, then*

$$
\begin{aligned}
&\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right|\\
&\leq\left(\frac{1}{\alpha+k}\right)^{1-\frac{1}{q}}\left\{\left[\theta_1^*(\alpha,k,n)\left|f^{(k)}(a)\right|^q+\theta_2^*(\alpha,k,n)\left|f^{(k)}(b)\right|^q\right.\right.\\
&\left.-\frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}\\
&+\left[\theta_2^*(\alpha,k,n)\left|f^{(k)}(a)\right|^q+\theta_1^*(\alpha,k,n)\left|f^{(k)}(b)\right|^q\right.\\
&\left.\left.-\frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$\theta_1^*(\alpha,k,n):=\frac{n(\alpha+k+1)+1}{(n+1)(\alpha+k)(\alpha+k+1)}$$

and

$$\theta_2^*(\alpha,k,n) := \frac{1}{(n+1)(\alpha+k+1)}.$$

II. If we choose $h(t) = t^s$, then we have the result for Breckner type of higher order strongly s-preinvex functions.

Corollary 4.18 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is Breckner type of a higher order strongly s-preinvex function, then*

$$\begin{aligned}
&\bigl|\mathcal{H}(k,n,\alpha,a,b)(f)\bigr| \\
&\quad\le \left(\frac{1}{\alpha+k}\right)^{1-\frac{1}{q}} \left\{\left[\theta_1^{**}(\alpha,k,n)\bigl|f^{(k)}(a)\bigr|^q + \theta_2^{**}(\alpha,k,n)\bigl|f^{(k)}(b)\bigr|^q \right.\right.\\
&\qquad \left. - \frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\bigl\|\eta(b,a)\bigr\|^\sigma\right]^{\frac{1}{q}} \\
&\qquad + \left[\theta_2^{**}(\alpha,k,n)\bigl|f^{(k)}(a)\bigr|^q + \theta_1^{**}(\alpha,k,n)\bigl|f^{(k)}(b)\bigr|^q \right.\\
&\qquad \left.\left. - \frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\bigl\|\eta(b,a)\bigr\|^\sigma\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\begin{aligned}
\theta_1^{**}(\alpha,k,n) &:= \frac{1}{(n+1)^s}\int_0^1 (1-t)^{\alpha+k-1}(n+t)^s\,\mathrm{d}t, \\
&:= \frac{n^s}{(n+1)^s} B(1,\alpha+k)_2 f_1\left(-s,1;\alpha+k+1;-\frac{1}{n}\right)
\end{aligned}$$

and

$$\theta_2^{**}(\alpha,k,n) := \int_0^1 (1-t)^{\alpha+k-1}\left(\frac{1-t}{n+1}\right)^s \mathrm{d}t = \frac{1}{(n+1)^s}\cdot\frac{1}{\alpha+k+s}.$$

III. If we choose $h(t) = t^{-s}$, then we have the result for Godunova–Levin type of higher order strongly s-preinvex functions.

Corollary 4.19 *Suppose that $k \in \mathbb{N}$, the function $f : I \to \mathbb{R}$ is of kth order differentiable, $a, a + \eta(b,a) \in I$, $0 < \eta(b,a)$, $q > 1$, and $n \in \mathbb{N}$. If $|f^{(k)}|^q$ is Godunova–Levin type of a higher order strongly s-preinvex function, then*

$$\begin{aligned}
&\bigl|\mathcal{H}(k,n,\alpha,a,b)(f)\bigr| \\
&\quad\le \left(\frac{1}{\alpha+k}\right)^{1-\frac{1}{q}} \left\{\left[\theta_1^{***}(\alpha,k,n)\bigl|f^{(k)}(a)\bigr|^q + \theta_2^{***}(\alpha,k,n)\bigl|f^{(k)}(b)\bigr|^q \right.\right.\\
&\qquad \left.\left. - \frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\bigl\|\eta(b,a)\bigr\|^\sigma\right]^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left[\theta_2^{***}(\alpha,k,n)\left|f^{(k)}(a)\right|^q+\theta_1^{***}(\alpha,k,n)\left|f^{(k)}(b)\right|^q\right.$$
$$\left.\left.-\frac{\mu}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}\right\},$$

where

$$\theta_1^{***}(\alpha,k,n):=(n+1)^s\int_0^1(1-t)^{\alpha+k-1}(n+t)^{-s}\,\mathrm{d}t,$$
$$:=\frac{(n+1)^s}{n^s}B(1,\alpha+k){}_2f_1\left(s,1;\alpha+k+1;-\frac{1}{n}\right)$$

and

$$\theta_2^{***}(\alpha,k,n):=\int_0^1(1-t)^{\alpha+k-1}\left(\frac{1-t}{n+1}\right)^{-s}\mathrm{d}t=\frac{(n+1)^s}{\alpha+k-s}.$$

IV. If we take $h(t)=1$, then we have the result for higher order strongly P-preinvex functions.

Corollary 4.20 *Suppose that $k\in\mathbb{N}$, the function $f:I\to\mathbb{R}$ is of kth order differentiable, $a,a+\eta(b,a)\in I$, $0<\eta(b,a)$, $q>1$, and $n\in\mathbb{N}$. If $|f^{(k)}|^q$ is a higher order strongly P-preinvex function, then*

$$\left|\mathcal{H}(k,n,\alpha,a,b)(f)\right|$$
$$\leq\frac{2}{\alpha+k}\left[\left|f^{(k)}(a)\right|^q+\left|f^{(k)}(b)\right|^q-\frac{\mu(k+\alpha)}{(n+1)^2}\left(\frac{n(k+\alpha+2)+1}{(k+\alpha+1)(k+\alpha+2)}\right)\left\|\eta(b,a)\right\|^\sigma\right]^{\frac{1}{q}}.$$

5 Conclusion

In this paper, we introduce the notion of higher order strongly h-preinvex functions. As special cases we deduce some other types of higher order strongly preinvex functions. We prove an identity related to the kth order differentiable functions and Riemann–Liouville integrals. Utilizing the identity, we obtained some estimates of upper bound for the kth order differentiable functions involving Riemann–Liouville integrals via higher order strongly h-preinvex functions. We also discussed several special cases of the main results. We would like to point out here that all the results obtained in this paper continued to hold for strongly preinvex functions, indeed, which can be observed by the special case of $\sigma=2$ and $h(t)=t$. We hope that the ideas and techniques of this paper will inspire interested readers.

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Availability of data and materials
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The authors declare that they have no competing interests.

Authors' contributions
SW and MUA finished the proofs of the main results and the writing work. MVM, MAN, and ST gave lots of advice on the proofs of the main results and the writing work. All authors read and approved the final manuscript.

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