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Min–sup-type zero duality gap properties for DC composite optimization problem

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Abstract

In this paper, we present min–sup-type zero duality gap properties for DC composite optimization problem with conic constraints. Using properties of the subdifferentials of involved functions, we introduce some new constraint qualifications. Under these new constraint qualifications, we provide necessary and/or sufficient conditions for the min–sup-type stable zero duality to hold.

Keywords: Zero duality gap property; DC function; Composite function; Constraint qualification; Conic programming

1 Introduction

Let X, Y and Z be real locally convex Hausdorff topological vector spaces with their duals X^*, Y^*, and Z^*, respectively. Let Y and Z be partially ordered by closed convex cones $S \subseteq Y$ and $K \subseteq Z$, respectively. Denote $Y^\bullet := Y \cup \{\infty_Y\}$ and $Z^\bullet := Z \cup \{\infty_Z\}$, where $\{\infty_Y\}$ and $\{\infty_Z\}$ are the greatest elements with respect to the partial orders $\leq_S$ and $\leq_K$, respectively. The following operations are defined on $Y^\bullet$ (resp., $Z^\bullet$): for any $y \in Y$ (resp., Z), $y + \infty = \infty + y = \infty$ and $t\infty = \infty$ for any $t \geq 0$. In this paper, we consider the following DC composite optimization problem:

$$\begin{aligned} &\inf f\bigl(\varphi(x)\bigr) - g(x) \\ &\quad \text{s.t. } x \in C, \quad h(x) \in -S, \end{aligned} \tag{P}$$

where $C \subseteq X$ is a nonempty convex set, $\varphi : X \to Z^\bullet$ is a proper K-convex mapping, $f : Z^\bullet \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is a proper convex K-increasing function with $f(\infty_Z) = +\infty$, $g : X \to \overline{\mathbb{R}}$ is a proper convex function, and $h : X \to Y^\bullet$ is a proper S-convex mapping.

Problem (P) includes several kinds of optimization problems as particular cases. For example, in the case where $X = Z$ and $\varphi = \mathrm{Id}_X$ (the identity operator on X) the problem (P) is reduced to the DC optimization problem (see [4–6, 8, 10, 17] and the references therein)

$$\begin{aligned} &\inf f(x) - g(x) \\ &\quad \text{s.t. } x \in C, \quad h(x) \in -S, \end{aligned} \tag{$\mathbb{P}$}$$

whereas if $g = 0$, then problem (P) is reduced to the convex composite optimization problem (see [2, 7, 9, 14] and the references therein)

$$\begin{aligned} &\inf f(\varphi(x)) \\ &\quad \text{s.t. } x \in C, \quad h(x) \in -S. \end{aligned} \tag{$\mathbf{P}$}$$

In particular, if $X = Z$, $\varphi = \mathrm{Id}_X$, and $g = 0$, then problem (P) is reduced to the classic convex optimization problem (see [1, 3, 11–13] and the references therein)

$$\begin{aligned} &\inf f(x) \\ &\quad \text{s.t. } x \in C, \quad h(x) \in -S. \end{aligned} \tag{$\mathcal{P}$}$$

Following [18], we define the Lagrange dual problem of (P) by

$$\inf_{u^* \in \operatorname{dom} g^*} \sup_{\substack{\lambda \in S^\oplus \\ \beta \in \operatorname{dom} f^*}} \{g^*(u^*) - (\delta_C + \lambda h + \beta\varphi)^*(u^*) - f^*(\beta)\}. \tag{D}$$

As is shown in [18, Remark 4.3], if g is lower semicontinuous (l.s.c.), then the optimal values $v(P)$ and $v(D)$ of problem (P) and (D), respectively, satisfy the so-called weak Lagrange duality, that is, $v(P) \geq v(D)$, but a duality gap may occur, that is, we may have $v(P) > v(D)$. One of the most interesting and challenging problems in convex and nonconvex analysis was giving sufficient conditions that guarantee the zero duality gap property (i.e., $v(P) = v(D)$). Over the years, various criteria have been developed ensuring the zero duality gap property for convex and nonconvex programming problems (see, e.g., [1, 6, 7, 11–13, 18, 20] and the references therein); here let us especially mention some recent papers [1, 11–13] regarding problem ($\mathcal{P}$), [6] regarding problem ($\mathbb{P}$), and [7] regarding problem ($\mathbf{P}$). In particular, the authors in [18] established the inf–sup-type stable zero duality gap property for problem (P) by using the epigraph properties of the infimal convolution of conjugate functions.

Another related and interesting problem is the min–sup-type zero duality property, which corresponds to the situation in which $v(P) = v(D)$ and problem (P) has at least an optimal solution. This problem was considered in [7] for the case where $g = 0$ and in [11] in the case where $\varphi = \mathrm{Id}_X$ and $g = 0$. However, to the best of our knowledge, not many results are known to provide the min–sup-type zero duality property for the DC composite optimization problem (P).

Our main aim in this paper is to give new conditions that completely characterize the min–sup-type zero duality gap properties for problem (P). In general, we do not impose any topological assumption on C, f, φ, and h, that is, we only need to assume that C is convex, f and φ are proper convex, and h is S-convex. Moreover, the results obtained in this paper are either new or proper extensions of some known results in [7, 11].

This paper is organized as follows. The next section contains some necessary notations and preliminary results. In Sect. 3, we provide some new constraint qualifications and give several relationships among them. Using the new constraint qualifications, we establish the min–sup-type zero duality gap properties. In Sect. 4, we give applications to problems ($\mathbb{P}$), ($\mathbf{P}$), and ($\mathcal{P}$).

2 Notations and preliminary results

The notations used in the present paper are standard (cf.[19]). In particular, we assume that X is a real locally convex Hausdorff topological vector space, X^* denotes the dual space of X, endowed with the weak*-topology $w^*(X^*,X)$. By $\langle x^*,x\rangle$ we denote the value of the functional $x^*\in X^*$ at $x\in X$, that os, $\langle x^*,x\rangle = x^*(x)$. We endow $X^*\times\mathbb{R}$ with the product topology of $w^*(X^*,X)$ and the usual Euclidean topology. Let D be a nonempty subset of X. The closure of D is denoted by $\operatorname{cl} D$. If $D\subseteq X^*$, then $\operatorname{cl} D$ denotes the weak*-closure of D. The positive dual cone $D^\oplus$ and the indicator function δ_D of D are defined, respectively, by

$$D^\oplus := \{x^*\in X^* : \langle x^*,x\rangle \geq 0 \text{ for each } x\in D\},$$

and

$$\delta_D(x) := \begin{cases} 0, & x\in D, \\ +\infty & \text{otherwise.}\end{cases}$$

Let $f: X\to\overline{\mathbb{R}}$ be a proper function. The effective domain, the conjugate function, and the epigraph of f are denoted by $\operatorname{dom} f$, f^*, and $\operatorname{epi} f$, respectively; they are defined respectively by $\operatorname{dom} f := \{x\in X : f(x) < +\infty\}$, $f^*(x^*) := \sup\{\langle x^*,x\rangle - f(x) : x\in X\}$ for $x^*\in X^*$, and $\operatorname{epi} f := \{(x,r)\in X\times\mathbb{R} : f(x)\leq r\}$. It is well known and easy to verify that $\operatorname{epi} f^*$ is weak*-closed. The l.s.c. hull of f, denoted by $\operatorname{cl} f$, is defined by

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f). \tag{1}$$

By [19, Theorem 2.3.1(iv)] we have

$$f^* = (\operatorname{cl} f)^* \quad \text{and} \quad f^{**} := (f^*)^* \leq \operatorname{cl} f \leq f. \tag{2}$$

We can easily see that the following Young–Fenchel inequality holds:

$$f(x) + f^*(x^*) \geq \langle x,x^*\rangle \quad \text{for each pair } (x,x^*)\in X\times X^*. \tag{3}$$

The subdifferential of f at $x\in\operatorname{dom} f$ is defined by

$$\partial f(x) := \{x^*\in X^* : f(x) + \langle x^*, y-x\rangle \leq f(y) \text{ for all } y\in X\}, \tag{4}$$

and for any $\epsilon\geq 0$, the ϵ-subdifferential of f at $x\in\operatorname{dom} f$ is defined by

$$\partial_\epsilon f(x) := \{x^*\in X^* : f(x) + \langle x^*, y-x\rangle \leq f(y) + \epsilon \text{ for all } y\in X\}. \tag{5}$$

Then, for all $\epsilon\geq 0$ and $x\in\operatorname{dom} f$,

$$x^*\in\partial_\epsilon f(x) \quad\Leftrightarrow\quad f(x) + f^*(x^*) \leq \langle x^*,x\rangle + \epsilon \quad\Leftrightarrow\quad (x^*, \epsilon + \langle x^*,x\rangle - f(x)) \in \operatorname{epi} f^*. \tag{6}$$

In particular, we have the following Young's equality:

$$x^*\in\partial f(x) \quad\Leftrightarrow\quad f(x) + f^*(x^*) = \langle x^*,x\rangle. \tag{7}$$

Moreover, by definition we have the following implication:

$$x_0 \text{ is a global minimizer of } \varphi \Leftrightarrow 0 \in \partial\varphi(x_0). \tag{8}$$

3 Min–sup-type zero duality gap property

Recall that a function $\psi : Z \to \overline{\mathbb{R}}$ is said to be K-increasing if for any $x, y \in Z$ such that $y \leq_K x$, we have $\psi(y) \leq \psi(x)$. A function $\phi : X \to Y^\bullet$ is said to be S-convex if for all $x, y \in \operatorname{dom}\phi := \{x \in X : \phi(x) \in Y\}$ and all $t \in [0, 1]$,

$$\phi\bigl(tx + (1-t)y\bigr) \leq_S t\phi(x) + (1-t)\phi(y).$$

Throughout this paper, unless otherwise specified, $C \subseteq X$ is a nonempty convex set, $f : Z \to \overline{\mathbb{R}}$ is a proper convex K-increasing function, $\varphi : X \to Z$ is a proper K-convex mapping, and $h : X \to Y^\bullet$ is a proper S-convex mapping. Set

$$(f \circ \varphi)(x) := \begin{cases} f(\varphi(x)) & \text{if } x \in \operatorname{dom}\varphi, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $f \circ \varphi$ is a proper convex function. For convenience, we write, for each $\lambda \in S^\oplus$,

$$(\lambda h)(x) := \begin{cases} \langle \lambda, h(x)\rangle & \text{if } x \in \operatorname{dom} h, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that h is S-convex if and only if λh is convex for each $\lambda \in S^\oplus$. Let $A := \{x \in C : h(x) \in -S\}$. We always assume that $A \cap \operatorname{dom}(f \circ \varphi - g) \neq \emptyset$. Let $p \in X^*$. Consider the following primal problem with linear perturbation:

$$\begin{aligned} &\inf f\bigl(\varphi(x)\bigr) - g(x) - \langle p, x\rangle \\ &\quad \text{s.t. } x \in C, \quad h(x) \in -S. \end{aligned} \tag{P_p}$$

Define its dual Lagrange problem by

$$\inf_{u^* \in \operatorname{dom} g^*} \sup_{(\lambda,\beta) \in S^\oplus \times \operatorname{dom} f^*} \bigl\{g^*\bigl(u^*\bigr) - (\beta\varphi + \lambda h + \delta_C)^*\bigl(p + u^*\bigr) - f^*(\beta)\bigr\}. \tag{D_p}$$

In the case where $p = 0$, problem (P_p) and its dual problem (D_p) reduce to problem (P) and problem (D), respectively. Let $v(P_p)$ and $v(D_p)$ denote optimal values of problems (P_p) and (D_p), respectively. As usual, we denote by $S(P_p)$ the solution set of problem (P_p), that is,

$$S(P_p) := \bigl\{x_0 \in A : f\bigl(\varphi(x_0)\bigr) - g(x_0) - \langle p, x_0\rangle = \min_{x \in A}\bigl\{f\bigl(\varphi(x)\bigr) - g(x) - \langle p, x\rangle\bigr\}.$$

As before, we use $S(P)$ to denote $S(P_0)$. As is shown in [6, Example 3.2], the weak duality between (P) and (D) does not necessarily hold in general even in the case φ is an identity operator. To establish the weak duality and the stable weak duality between (P) and (D), the authors in [18] introduce the following condition:

$$\operatorname{epi}(f \circ \varphi - g + \delta_A)^* = \operatorname{epi}(f \circ \varphi - \operatorname{cl} g + \delta_A)^*. \tag{9}$$

We will further need the following lemma, taken from [18, Proposition 4.2].

Lemma 3.1 *Suppose that* (9) *holds. Then, for each* $p \in X^*$, *we have*

$$v(D_p) \leq v(P_p), \tag{10}$$

that is, the stable weak duality holds between (*P*) *and* (*D*).

In particular, if (10) holds for $p = 0$, then we say that the weak duality holds between (*P*) and (*D*). Following [18], if (10) becomes an equality, then we say that the inf–sup-type stable zero duality gap property holds, and if $v(P) = v(D)$, then we say the inf–sup-type zero duality gap property holds. In this section, we mainly study the min–sup-type zero duality gap property, that is, when does the inf–sup-type zero duality gap property holds between (*P*) and (*D*) (assuming that $S(P) \neq \emptyset$). We also study the min–sup-type stable zero duality gap property, that is, when does the following implication hold for any $p \in X^*$:

$$\mathrm{S}(P_p) \neq \emptyset \quad \Rightarrow \quad \mathrm{v}(P_p) = v(D_p).$$

To study the min–sup zero duality gap property of problem (*P*), we introduce some new constraint qualifications. For this purpose, we make use of the subdifferential $\partial\varphi(x)$ for a general proper function (not necessarily convex) $\varphi : X \to \overline{\mathbb{R}}$; see (4). For simplicity, we denote

$$\varLambda(x_0) = \bigcap_{\substack{\epsilon>0 \\ u^* \in \operatorname{dom} g^*}} \left(\bigcup_{\substack{\lambda \in S^{\oplus}, \beta \in \partial_\epsilon(f(\varphi(x_0))) \\ (\lambda h)(x_0) \in [-\epsilon, 0]}} \left\{ \partial_\epsilon(\beta\varphi + \delta_C + \lambda h)(x_0) - u^* \right\} \right)$$

and

$$\varLambda_0(x_0) = \bigcap_{\substack{\epsilon>0 \\ u^* \in \partial g(x_0)}} \left(\bigcup_{\substack{\lambda \in S^{\oplus}, \beta \in \partial_\epsilon(f(\varphi(x_0))) \\ (\lambda h)(x_0) \in [-\epsilon, 0]}} \left\{ \partial_\epsilon(\beta\varphi + \delta_C + \lambda h)(x_0) - u^* \right\} \right),$$

where $x_0 \in \operatorname{dom}(f \circ \varphi - g) \cap A$. By definitions we have $\varLambda(x_0) \subseteq \varLambda_0(x_0)$.

Definition 3.2 We say that the family $\{f, \varphi, g; \delta_A\}$ satisfies

(i) strong-(*ABCQ*) at $x_0 \in \operatorname{dom}(f \circ \varphi - g) \cap A$ if

$$\partial(f \circ \varphi - g + \delta_A)(x_0) \subseteq \varLambda(x_0). \tag{11}$$

(ii) (*ABCQ*) at $x_0 \in \operatorname{dom}(f \circ \varphi - g) \cap A$ if

$$\partial(f \circ \varphi - g + \delta_A)(x_0) \subseteq \varLambda_0(x_0). \tag{12}$$

Moreover, we say the family $\{f,\varphi,g;\delta_A\}$ satisfies strong-($ABCQ$) (resp., the ($ABCQ$)) if strong-($ABCQ$) (resp., ($ABCQ$)) holds at each point $x \in \operatorname{dom}(f\circ\varphi - g)\cap A$.

Remark 3.3
(a) We have the following implication:

$$\text{strong-}(ABCQ) \quad \Rightarrow \quad (ABCQ).$$

(b) Note that, in the particular case $g = 0$, $\operatorname{dom} g^* = \partial g(x_0) = \{0\}$, and hence strong-($ABCQ$) and ($ABCQ$) turn into the following qualification condition:

$$\partial(f\circ\varphi + \delta_A)(x) \subseteq \bigcap_{\epsilon>0}\Bigg(\bigcup_{\substack{\lambda\in S^{\oplus},\beta\in\partial_\epsilon(f(\varphi(x)))\\ (\lambda h)(x)\in[-\epsilon,0]}}\big\{\partial_\epsilon(\beta\varphi + \delta_C + \lambda h)(x)\big\}\Bigg), \tag{13}$$

which was introduced in [7] to study the zero duality gap property for problem (**P**).
(c) In the case where $\varphi = \mathrm{Id}_X$ and $g = 0$, strong-($ABCQ$) and ($ABCQ$) collapse into

$$\overline{(ABCQ)}: \quad \partial(f + \delta_A)(x) \subseteq \bigcap_{\epsilon>0}\Bigg(\partial f(x) + \bigcup_{\substack{\lambda\in S^{\oplus}\\ (\lambda h)(x)\in[-\epsilon,0]}}\big\{\partial_\epsilon(\delta_C + \lambda h)(x)\big\}\Bigg),$$

which was introduced in [7].

Given two proper functions $h_1, h_2 : X \to \overline{\mathbb{R}}$, their infimal convolution is

$$h_1 \mathbin{\square} h_2 : X \to \mathbb{R}\cup\{\pm\infty\}, \qquad (h_1 \mathbin{\square} h_2)(x) := \inf_{z\in X}\big\{h_1(z) + h_2(x-z)\big\}.$$

Recall that the authors in [18] introduced the qualification condition ($DCCQ$)

$$\operatorname{epi}(f\circ\varphi - g + \delta_A)^* = \bigcap_{u^*\in\operatorname{dom} g^*}\operatorname{epi}\big(\digamma \mathbin{\square} \delta_C^* \mathbin{\square} h^\diamond\big) - \big(u^*, g^*(u^*)\big) \tag{14}$$

to study the inf–sup-type zero duality gap property for problem (P), where the functions $h^\diamond : X^* \to \overline{\mathbb{R}}$ and $\digamma : X^* \to \overline{\mathbb{R}}$ are defined respectively by

$$h^\diamond(x^*) = \inf_{\lambda\in S^{\oplus}}(\lambda h)^*(x^*) \quad \text{for } x^*\in X^*$$

and

$$\digamma(x^*) = \inf_{\beta\in\operatorname{dom} f^*}\big(\beta\varphi - f^*(\beta)\big)^*(x^*) \quad \text{for } x^*\in X^*.$$

The following proposition on relationship between ($DCCQ$) and ($ABCQ$) is an extension of [7, Proposition 3.8] from $g = 0$ to the general g.

Proposition 3.4 *We have the following implication*:

$$(DCCQ) \quad \Rightarrow \quad (ABCQ).$$

Proof Suppose that condition ($DCCQ$) holds. Let $x_0 \in \operatorname{dom}(f \circ \varphi - g) \cap A$ and $p \in \partial(f \circ \varphi - g + \delta_A)(x_0)$. By (6) we have

$$\begin{aligned}\bigl(p, \langle p, x_0\rangle - (f \circ \varphi - g + \delta_A)(x_0)\bigr) &\in \operatorname{epi}(f \circ \varphi - g + \delta_A)^* \\ &= \bigcap_{u^* \in \operatorname{dom} g^*} \operatorname{epi}\bigl(F \,\square\, \delta_C^* \,\square\, h^\diamond\bigr) - \bigl(u^*, g^*(u^*)\bigr).\end{aligned}$$

Let $u^* \in \partial g(x_0)$. Then

$$\bigl(p + u^*, \langle p, x_0\rangle - (f \circ \varphi - g + \delta_A)(x_0) + g^*(u^*)\bigr) \in \operatorname{epi}\bigl(F \,\square\, \delta_C^* \,\square\, h^\diamond\bigr).$$

It follows that

$$\begin{aligned}\bigl(F \,\square\, \delta_C^* \,\square\, h^\diamond\bigr)(p + u^*) &\le \langle p, x_0\rangle - (f \circ \varphi - g + \delta_A)(x_0) + g^*(u^*) \\ &= \langle p, x_0\rangle - (f \circ \varphi + \delta_A)(x_0) + g^*(u^*) + g(x_0) \\ &= \langle p + u^*, x_0\rangle - (f \circ \varphi + \delta_A)(x_0),\end{aligned}$$

where the last equality holds by (7). This implies that

$$\inf_{x_1^*, x_2^* \in X^*} \bigl\{F(x_1^*) + \delta_C^*(x_2^*) + h^\diamond(p + u^* - x_1^* - x_2^*)\bigr\} \le \langle p + u^*, x_0\rangle - f\bigl(\varphi(x_0)\bigr). \tag{15}$$

Let $\epsilon > 0$. Then by (15) there exist $\overline{x}_1^*, \overline{x}_2^* \in X^*$ such that

$$F(\overline{x}_1^*) + \delta_C^*(\overline{x}_2^*) + h^\diamond(p + u^* - \overline{x}_1^* - \overline{x}_2^*) \le \langle p + u^*, x_0\rangle - f\bigl(\varphi(x_0)\bigr) + \frac{\epsilon}{3}, \tag{16}$$

whereas by definitions there exist $\overline{\beta} \in \operatorname{dom} f^*$ and $\overline{\lambda} \in S^\oplus$ such that

$$(\overline{\beta}\varphi)^*(\overline{x}_1^*) + f^*(\overline{\beta}) \le F(\overline{x}_1^*) + \frac{\epsilon}{3} \tag{17}$$

and

$$(\overline{\lambda} h)^*(p + u^* - \overline{x}_1^* - \overline{x}_2^*) \le h^\diamond(p + u^* - \overline{x}_1^* - \overline{x}_2^*) + \frac{\epsilon}{3}.$$

Combining this with (16) and (17), we have

$$(\overline{\beta}\varphi)^*(\overline{x}_1^*) + f^*(\overline{\beta}) + \delta_C^*(\overline{x}_2^*) + (\overline{\lambda} h)^*(p + u^* - \overline{x}_1^* - \overline{x}_2^*) \le \langle p + u^*, x_0\rangle - f\bigl(\varphi(x_0)\bigr) + \epsilon. \tag{18}$$

Noting $\delta_C(x_0) = 0$ and $(\overline{\lambda} h)(x_0) \le 0$, it follows from (18) and the Young–Fenchel inequality (3) that

$$\begin{aligned}0 &\le f^*(\overline{\beta}) + f\bigl(\varphi(x_0)\bigr) - (\overline{\beta}\varphi)(x_0) \\ &\le \langle p + u^*, x_0\rangle - (\overline{\beta}\varphi)^*(\overline{x}_1^*) - (\overline{\beta}\varphi)(x_0) - \delta_C^*(\overline{x}_2^*) - (\overline{\lambda} h)^*(p + u^* - \overline{x}_1^* - \overline{x}_2^*) + \epsilon \\ &\le \langle p + u^*, x_0\rangle - \langle \overline{x}_1^*, x_0\rangle - \langle \overline{x}_2^*, x_0\rangle + \delta_C(x_0) - \langle p + u^* - \overline{x}_1^* - \overline{x}_2^*, x_0\rangle + (\overline{\lambda} h)(x_0) + \epsilon \\ &\le \epsilon.\end{aligned}$$

This, together with (6), implies that $\overline{\beta} \in \partial_\epsilon f(\varphi(x_0))$ and $(\overline{\lambda}h)(x_0) \in [-\epsilon, 0]$. Moreover, by (18) and (3) we get that, for each $x \in X$,

$$\begin{aligned}(\overline{\beta}\varphi)(x_0) + (\overline{\lambda}h)(x_0) - \langle p + u^*, x_0\rangle &\leq f^*(\overline{\beta}) + f\big(\varphi(x_0)\big) + (\overline{\lambda}h)(x_0) - \langle p + u^*, x_0\rangle \\ &\leq -(\overline{\beta}\varphi)^*\big(\overline{x}_1^*\big) - \delta_C^*\big(\overline{x}_2^*\big) - (\overline{\lambda}h)^*\big(p + u^* - \overline{x}_1^* - \overline{x}_2^*\big) + \epsilon \\ &\leq (\overline{\beta}\varphi)(x) + \delta_C(x) + (\overline{\lambda}h)(x) - \langle p + u^*, x\rangle + \epsilon.\end{aligned}$$

This yields $p + u^* \in \partial_\epsilon(\overline{\beta}\varphi + \delta_C + \overline{\lambda}h)(x_0)$, and hence $p \in \Lambda_0(x_0)$. Therefore the result holds, and the proof is complete. □

The following theorem gives a sufficient condition and a necessary condition to ensure the min–sup-type stable zero duality gap property for problem (*P*).

Theorem 3.5 *Suppose that* (9) *holds. Let* $x_0 \in \mathrm{dom}(f \circ \varphi - g) \cap A$. *Consider the following statements*:

(i) *The family* $\{f, \varphi, g; \delta_A\}$ *satisfies strong-(ABCQ) at* x_0.
(ii) *For each* $p \in X^*$ *such that* $x_0 \in S(P_p)$, $v(P_p) = v(D_p)$.
(iii) *The family* $\{f, \varphi, g; \delta_A\}$ *satisfies (ABCQ) at* x_0.

Then (i) ⇒ (ii) ⇒ (iii).

Proof (i) ⇒ (ii). Suppose that (i) holds. Let $p \in X^*$ be such that $x_0 \in S(P_p)$, that is,

$$f\big(\varphi(x_0)\big) - g(x_0) - \langle p, x_0\rangle = \inf_{x\in A}\big\{f\big(\varphi(x)\big) - g(x) - \langle p, x\rangle\big\}.$$

Then by (8) $p \in \partial(f \circ \varphi - g + \delta_A)(x_0)$, and hence $p \in \Lambda(x_0)$ by strong-(*ABCQ*). Let $\epsilon > 0$ and $u^* \in \mathrm{dom}\, g^*$. Then there exist $\overline{\beta} \in \partial_\epsilon(f(\varphi(x_0)))$ and $\overline{\lambda} \in S^\oplus$ with $(\overline{\lambda}h)(x_0) \in [-\epsilon, 0]$ such that $p + u^* \in \partial_\epsilon(\overline{\beta}\varphi + \delta_C + \overline{\lambda}h)(x_0)$. This implies that, for each $x \in X$,

$$\begin{aligned}\langle p + u^*, x\rangle &\leq (\overline{\beta}\varphi + \delta_C + \overline{\lambda}h)(x) - (\overline{\beta}\varphi + \delta_C + \overline{\lambda}h)(x_0) + \epsilon + \langle p, x_0\rangle + \langle u^*, x_0\rangle \\ &\leq (\overline{\beta}\varphi + \delta_C + \overline{\lambda}h)(x) - (\overline{\beta}\varphi + \delta_C + \overline{\lambda}h - g)(x_0) + \langle p, x_0\rangle + g^*\big(u^*\big) + \epsilon,\end{aligned}$$

where the inequality holds since $g^*(u^*) + g(x_0) \geq \langle u^*, x_0\rangle$ by (3). Thus, for each $x \in C$,

$$(\overline{\beta}\varphi)(x_0) + (\overline{\lambda}h)(x_0) - g(x_0) - \langle p, x_0\rangle \leq (\overline{\beta}\varphi)(x) + (\overline{\lambda}h)(x) + g^*\big(u^*\big) - \langle p + u^*, x\rangle + \epsilon. \tag{19}$$

Noting that $\overline{\beta} \in \partial_\epsilon(f(\varphi(x_0)))$, it follows from (6) that

$$f\big(\varphi(x_0)\big) + f^*(\overline{\beta}) \leq (\overline{\beta}\varphi)(x_0) + \epsilon.$$

This, together with (19) and the fact $(\overline{\lambda}h)(x_0) \in [-\epsilon, 0]$, implies that, for each $x \in C$,

$$\begin{aligned}&f\big(\varphi(x_0)\big) - g(x_0) - \langle p, x_0\rangle \\ &\quad\leq (\overline{\beta}\varphi)(x_0) - f^*(\overline{\beta}) - g(x_0) - \langle p, x_0\rangle + \epsilon \\ &\quad\leq (\overline{\beta}\varphi)(x) + (\overline{\lambda}h)(x) + g^*\big(u^*\big) - \langle p + u^*, x\rangle - f^*(\overline{\beta}) - (\overline{\lambda}h)(x_0) + 2\epsilon \\ &\quad\leq (\overline{\beta}\varphi)(x) + (\overline{\lambda}h)(x) + g^*\big(u^*\big) - \langle p + u^*, x\rangle - f^*(\overline{\beta}) + 3\epsilon.\end{aligned}$$

Consequently, we get that

$$\begin{aligned} & g^*(u^*) - (\overline{\beta}\varphi + \overline{\lambda}h + \delta_C)^*(p + u^*) - f^*(\overline{\beta}) \\ & \quad = \inf_{x \in C} \{ (\overline{\beta}\varphi)(x) + (\overline{\lambda}h)(x) + g^*(u^*) - f^*(\overline{\beta}) - \langle p + u^*, x \rangle \} \\ & \quad \geq f(\varphi(x_0)) - g(x_0) - \langle p, x_0 \rangle - 3\epsilon. \end{aligned}$$

This means that

$$v(D_p) \geq f(\varphi(x_0)) - g(x_0) - \langle p, x_0 \rangle - 3\epsilon.$$

Letting $\epsilon \to 0$, we have that

$$v(D_p) \geq f(\varphi(x_0)) - g(x_0) - \langle p, x_0 \rangle = v(P_p). \tag{20}$$

This, together with Lemma 3.1, implies that $v(D_p) = v(P_p)$.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. Let $p \in \partial(f \circ \varphi - g + \delta_A)(x_0)$. Then by (8) we see that $x_0 \in S(P_p)$. This implies that

$$f(\varphi(x_0)) - g(x_0) - \langle p, x_0 \rangle = \min_{x \in A} \{ f(\varphi(x)) - g(x) - \langle p, x \rangle \},$$

and hence, by (ii),

$$f(\varphi(x_0)) - g(x_0) - \langle p, x_0 \rangle = v(D_p). \tag{21}$$

We will further show that $p \in \Lambda_0(x_0)$. For this purpose, let $\epsilon > 0$ and $u^* \in \partial g(x_0)$. It follows from (21) that there exists $(\overline{\lambda}, \overline{\beta}) \in S^{\oplus} \times \operatorname{dom} f^*$ such that, for each $x \in X$,

$$\begin{aligned} f(\varphi(x_0)) - g(x_0) - \langle p, x_0 \rangle & \leq g^*(u^*) - (\overline{\beta}\varphi + \overline{\lambda}h + \delta_C)^*(p + u^*) - f^*(\overline{\beta}) + \epsilon \\ & \leq (\overline{\beta}\varphi + \overline{\lambda}h + \delta_C)(x) - \langle p + u^*, x \rangle + g^*(u^*) - f^*(\overline{\beta}) + \epsilon, \end{aligned} \tag{22}$$

where the last inequality holds by (3). Note that $u^* \in \partial g(x_0)$. It follows that $g^*(u^*) + g(x_0) = \langle u^*, x_0 \rangle$. Combining this with (22), we have that, for each $x \in C$,

$$f(\varphi(x_0)) - \langle p, x_0 \rangle \leq (\overline{\beta}\varphi + \overline{\lambda}h)(x) - \langle p + u^*, x \rangle + \langle u^*, x_0 \rangle - f^*(\overline{\beta}) + \epsilon,$$

that is,

$$f(\varphi(x_0)) - \langle p + u^*, x_0 \rangle \leq (\overline{\beta}\varphi)(x) + (\overline{\lambda}h)(x) - \langle p + u^*, x \rangle - f^*(\overline{\beta}) + \epsilon. \tag{23}$$

Letting $x = x_0$ and noting that $(\overline{\lambda}h)(x_0) \leq 0$, we see from (23) and (3) that

$$0 \geq (\overline{\lambda}h)(x_0) \geq f(\varphi(x_0)) + f^*(\overline{\beta}) - (\overline{\beta}\varphi)(x_0) - \epsilon \geq -\epsilon.$$

This implies that $(\overline{\lambda}h)(x_0) \in [-\epsilon, 0]$ and

$$f(\varphi(x_0)) + f^*(\overline{\beta}) - (\overline{\beta}\varphi)(x_0) \leq \epsilon.$$

Thus by (6) $\overline{\beta} \in \partial_\epsilon f(\varphi(x_0))$. Moreover, by (23) and (3) we can see that, for each $x \in C$,

$$\begin{aligned}(\overline{\beta}\varphi)(x_0) + (\overline{\lambda}h)(x_0) - \langle p + u^*, x_0\rangle &\leq f(\varphi(x_0)) + f^*(\overline{\beta}) + (\overline{\lambda}h)(x_0) - \langle p + u^*, x_0\rangle \\ &\leq (\overline{\beta}\varphi)(x) + (\overline{\lambda}h)(x) - \langle p + u^*, x\rangle + \epsilon.\end{aligned}$$

This, together with (6), implies that $p + u^* \in \partial_\epsilon(\overline{\beta}\varphi + \delta_C + \overline{\lambda}h)(x_0)$, and hence $p \in \Lambda_0(x_0)$. Therefore (12) holds, and the proof is complete. □

In [18, Theorem 4.5] the authors showed that condition ($DCCQ$) implies that the inf–sup-type stable zero duality gap property holds for problem (P). Thus by definition we easily to see that the following corollary holds.

Corollary 3.6 *Suppose that* (9) *holds. If the family* $\{f, \varphi, g; \delta_A\}$ *satisfies condition* ($DCCQ$), *then for each* $p \in X^*$ *such that* $x_0 \in S(P_p)$, $v(P_p) = v(D_p)$.

Remark 3.7 Let $\phi : X \to [-\infty, +\infty]$ be an extended real-valued function. Recall from [16] (see also [15, p. 90]) that the Fréchet subdifferential of ϕ at a point x_0 with $|\phi(x_0)| < \infty$ is defined by

$$\widehat{\partial}\phi(x_0) := \left\{x^* \in X^* : \liminf_{x \to x_0} \frac{\phi(x) - \phi(x_0) - \langle x^*, x - x_0\rangle}{\| x - x_0 \|} \geq 0\right\}.$$

By using Fréchet subdifferential properties we can give a new constraint qualification

$$\widehat{\partial}(f \circ \varphi - g + \delta_A)(x_0) \subseteq \bigcap_{\substack{\epsilon > 0 \\ u^* \in \operatorname{dom} g^*}} \left(\bigcup_{\substack{\lambda \in S^{\oplus}, \beta \in \partial_\epsilon(f(\varphi(x_0))) \\ (\lambda h)(x_0) \in [-\epsilon, 0]}} \left\{\partial_\epsilon(\beta\varphi + \delta_C + \lambda h)(x_0) - u^*\right\}\right), \tag{24}$$

where $x_0 \in \operatorname{dom}(f \circ \varphi - g) \cap A$. Similarly to the proof of implication (i) ⇒ (ii) in Theorem 3.5, we see the min–sup-type stable zero duality gap property also holds under condition (24).

Taking $p = 0$ in Theorem 3.5(ii), we get the following corollary.

Corollary 3.8 *Suppose that* (9) *holds. Let* $x_0 \in \operatorname{dom}(f \circ \varphi - g) \cap A \cap S(P)$. *If the family* $\{f, \varphi, g; \delta_A\}$ *satisfies strong-*($ABCQ$) *at* x_0, *then*

$$\min_{x \in A}\{f(\varphi(x)) - g(x)\} = \inf_{u \in \operatorname{dom} g^*} \sup_{(\lambda,\beta) \in S^{\oplus} \times \operatorname{dom} f^*} \{g^*(u^*) - (\beta\varphi + \delta_C + \lambda h)^*(u^*) - f^*(\beta)\}.$$

4 Applications

4.1 Application to DC programming

Let $X = Z$ and $\varphi = \mathrm{Id}_X$. Then problem (P_p) turns into the problem

$$\begin{aligned}&\inf f(x) - g(x) - \langle p, x\rangle \\ &\quad \text{s.t. } x \in C, h(x) \in -S,\end{aligned} \tag{$\mathbb{P}_p$}$$

and problem (D_p) reduces to the problem

$$\inf_{u^*\in\operatorname{dom} g^*}\sup_{(\lambda,\beta)\in S^{\oplus}\times\operatorname{dom} f^*}\bigl\{g^*(u^*)-f^*(x^*)-(\delta_C+\lambda h)^*(p+u^*-x^*)\bigr\}, \qquad (\mathbb{D}_p)$$

where $p\in X^*$. Moreover, we see that the conditions strong-($ABCQ$) and ($ABCQ$) are recast into

$$\partial(f-g+\delta_A)(x)\subseteq\bigcap_{\substack{\epsilon>0\\ u^*\in\operatorname{dom} g^*}}\Biggl(\bigcup_{\substack{\lambda\in S^{\oplus}\\ (\lambda h)(x)\in[-\epsilon,0]}}\bigl\{\partial f(x)+\partial_\epsilon(\delta_C+\lambda h)(x)-u^*\bigr\}\Biggr) \qquad (25)$$

and

$$\partial(f-g+\delta_A)(x)\subseteq\bigcap_{\substack{\epsilon>0\\ u^*\in\partial g(x_0)}}\Biggl(\bigcup_{\substack{\lambda\in S^{\oplus}\\ (\lambda h)(x)\in[-\epsilon,0]}}\bigl\{\partial f(x)+\partial_\epsilon(\delta_C+\lambda h)(x)-u^*\bigr\}\Biggr), \qquad (26)$$

respectively. As before, we use $S(\mathbb{P}_p)$ to denote the solution set of problem ($\mathbb{P}_p$), that is,

$$S(\mathbb{P}_p):=\Bigl\{x_0\in A: f(x_0)-g(x_0)-\langle p,x_0\rangle=\min_{x\in A}\bigl\{f(x)-g(x)-\langle p,x\rangle\bigr\}\Bigr\}.$$

Thus by Theorem 3.5 we get the following result.

Theorem 4.1 *Let $x_0\in\operatorname{dom}(f-g)\cap A$. Suppose that*

$$\operatorname{epi}(f-g+\delta_A)^*=\operatorname{epi}(f-\operatorname{cl} g+\delta_A)^*. \qquad (27)$$

Consider the following statements.
(i) *Inclusion* (25) *holds.*
(ii) *For each $p\in X^*$ such that $x_0\in S(\mathbb{P}_p)$, we have the following equality*:

$$\min_{x\in A}\bigl\{f(x)-g(x)-\langle p,x\rangle\bigr\}=\inf_{u^*\in\operatorname{dom} g^*}\sup_{(\lambda,x^*)\in S^{\oplus}\times X^*}\bigl\{g^*(u^*)-f^*(x^*)-(\delta_C+\lambda h)^*(p+u^*-x^*)\bigr\}.$$

(iii) *Inclusion* (26) *holds.*
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

The proof of the following theorem is almost similar to that of Theorem 3.5, so we omit it here.

Theorem 4.2 *Let $x_0\in\operatorname{dom}(f-g)\cap A$. Suppose that* (27) *holds. Consider the following statements.*
(i) *The following inclusion holds*:

$$\partial(f-g+\delta_A)(x_0)\subseteq\bigcap_{\substack{\epsilon>0\\ u^*\in\operatorname{dom} g^*}}\Biggl(\bigcup_{\substack{\lambda\in S^{\oplus}\\ (\lambda h)(x_0)\in[-\epsilon,0]}}\bigl\{\partial_\epsilon(f+\delta_C+\lambda h)(x_0)-u^*\bigr\}\Biggr).$$

(ii) *For each* $p \in X^*$ *such that* $x_0 \in S(\mathbb{P}_p)$, *we have the following equality*:

$$\min_{x\in A}\{f(x) - g(x) - \langle p,x\rangle\} = \inf_{u^*\in \operatorname{dom} g^*} \sup_{\lambda\in S^\oplus} \{g^*(u^*) - (f + \delta_C + \lambda h)^*(p + u^*)\}.$$

(iii) *The following inclusion holds*:

$$\partial(f - g + \delta_A)(x_0) \subseteq \bigcap_{\substack{\epsilon>0\\ u^*\in\partial g(x_0)}} \Bigg(\bigcup_{\substack{\lambda\in S^\oplus\\ (\lambda h)(x_0)\in[-\epsilon,0]}} \{\partial_\epsilon(f + \delta_C + \lambda h)(x_0) - u^*\}\Bigg).$$

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

4.2 Application to composite optimization problem

In the case where $g = 0$, problem (P_p) is reduced to the composite optimization problem

$$\begin{aligned} &\inf f(\varphi(x)) - \langle p,x\rangle \\ &\quad \text{s.t. } x \in C, \quad h(x) \in -S. \end{aligned} \tag{$\mathbf{P}_p$}$$

Noting that $g^* = \delta_{\{0\}}$, it follows that problem (D_p) reduces to the problem

$$\sup_{(\lambda,\beta)\in S^\oplus\times \operatorname{dom} f^*} \{-f^*(\beta) - (\beta\varphi + \delta_C + \lambda h)^*(p)\}. \tag{$\mathbf{D}_p$}$$

Thus by Theorem 3.5 and Remark 3.3(b) we straightforwardly get the following corollary, which was given in [7, Theorem 4.5].

Theorem 4.3 *Let* $x_0 \in \operatorname{dom} f(\varphi(x)) \cap A$. *Then* (13) *holds at* x_0 *if and only if for each* $p \in X^*$ *with* $x_0 \in S(\mathbf{P}_p)$, *problem* (**P**) *has a stable zero duality gap property, that is,*

$$\min_{x\in A}\{f(\varphi(x)) - \langle p,x\rangle\} = \sup_{\lambda\in S^\oplus} \sup_{\beta\in \operatorname{dom} f^*} \{-f^*(\beta) - (\beta\varphi + \delta_C + \lambda h)^*(p)\},$$

where $S(\mathbf{P}_p) := \{x_0 \in A : f(\varphi(x_0)) - \langle p,x_0\rangle = \min_{x\in A}\{f(\varphi(x)) - \langle p,x\rangle\}\}$.

4.3 Application to conic programming

Let $X = Z$, $\varphi = \mathrm{Id}_X$, and $g = 0$. Then problem (P_p) is reduced to the classical convex conical programming problem

$$\begin{aligned} &\inf f(x) - \langle p,x\rangle \\ &\quad \text{s.t. } x \in C, \quad h(x) \in -S, \end{aligned} \tag{$\mathcal{P}_p$}$$

and problem (D_p) reduces to the following Fenchel–Lagrange dual problem

$$\sup_{\lambda\in S^\oplus} \sup_{x^*\in X^*} \{-f^*(x^*) - (\delta_C + \lambda h)^*(p - x^*)\}. \tag{$\mathcal{D}_p$}$$

As mentioned in Remark 3.3(c), the conditions strong-($ABCQ$) and ($ABCQ$) are collapsed into condition $\overline{(ABCQ)}$. Thus by Theorem 3.5 we straightforwardly get the following corollary, which was given in [7, Theorem 5.3].

Theorem 4.4 *Let $x_0 \in \operatorname{dom} f \cap A$. The condition $\overline{(ABCQ)}$ holds at x_0 if and only if for each $p \in X^*$ with $x_0 \in S(\mathcal{P}_p)$, problem $(\mathcal{P})$ has a stable zero gap property, that is,*

$$\min_{x\in A}\bigl\{f(x) - \langle p, x\rangle\bigr\} = \sup_{\lambda\in S^{\oplus}} \sup_{x^*\in X^*} \bigl\{-f^*\bigl(x^*\bigr) - (\delta_C + \lambda h)^*\bigl(p - x^*\bigr)\bigr\},$$

where $S(\mathcal{P}_p) = \{x_0 \in A : f(x_0) - \langle p, x_0\rangle = \min_{x\in A}\{f(x) - \langle p, x\rangle\}\}$.

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