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On a more accurate reverse Mulholland-type inequality with parameters

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Abstract

By the use of the weight coefficients, the idea of introducing parameters and Hermite–Hadamard's inequality, a more accurate reverse Mulholland-type inequality with parameters and the equivalent forms are given. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are also considered.

MSC: 26D15

Keywords: Weight coefficient; Mulholland-type inequality; Reverse; Equivalent statement; Hermite–Hadamard's inequality; Parameter

1 Introduction

If $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m, b_n \geq 0, 0<\sum_{m=1}^{\infty} a_m^p<\infty$, and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, then we have the following Hardy–Hilbert's inequality with the best possible constant factor $\pi/\sin(\frac{\pi}{p})$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}. \tag{1}$$

Mulholland's inequality with the same best possible constant factor was provided as follows (cf. [1], Theorem 343, replacing $\frac{a_m}{m}, \frac{b_n}{n}$ by a_m, b_n):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{1}{m^{1-p}}a_m^p\right)^{1/p}\left(\sum_{n=2}^{\infty}\frac{1}{n^{1-q}}b_n^q\right)^{1/q}. \tag{2}$$

If $f(x), g(y) \geq 0, 0<\int_0^{\infty} f^p(x)\,dx<\infty$, and $0<\int_0^{\infty} g^q(y)\,dy<\infty$, then we still have the following Hardy–Hilbert's integral inequality (cf. [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)\,dx\right)^{1/p}\left(\int_0^{\infty}g^q(y)\,dy\right)^{1/q}, \tag{3}$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is the best possible. Inequalities (1), (2), and (3) with their extensions are important in analysis and its applications (cf. [2–12]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351): If $K(t)$ $(t>0)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^{\infty} K(t)t^{s-1}\,dt<\infty$, then we have

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

In the last ten years, some new extensions of (4) with their applications and the reverses were provided by [13–17].

In 2016, by the use of the technique of real analysis, Hong [18] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. The other similar works about Hilbert-type integral inequalities were given by [19–22].

In this paper, following the way of [18], by the use of the weight coefficients, the idea of introducing parameters and Hermite–Hadamard's inequality, a more accurate reverse Mulholland-type inequality with parameters and the equivalent forms are given in Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 2 and Remarks 1–2.

2 Some lemmas

In what follows, we assume that $p < 0\ (0 < q < 1)$, $\frac{1}{p}+\frac{1}{q} = 1$, $\xi, \eta \in [0, \frac{1}{2}]$, $s \in \mathrm{N} = \{1, 2, \dots\}$, $0 < c_1 \leq \cdots \leq c_s$, $0 < \lambda_i < \lambda \leq s$, $\lambda_i \leq 1$ $(i = 1, 2)$, $a_m, b_n \geq 0$, such that

$$0 < \sum_{m=2}^\infty \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p < \infty \quad \text{and}$$
$$0 < \sum_{n=2}^\infty \frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-p}} b_n^q < \infty.$$

For $\gamma = \lambda_1, \lambda - \lambda_2$, we set

$$k_s(\gamma) := \int_0^\infty \frac{t^{\gamma-1}}{\prod_{k=1}^s (t^{\lambda/s}+c_k)}\,dt.$$

By *Example 1 of* [23], *it follows that*

$$k_s(\gamma) = \frac{\pi s}{\lambda \sin(\frac{\pi s\gamma}{\lambda})}\sum_{k=1}^s c_k^{\frac{s\gamma}{\lambda}-1}\prod_{j=1(j\neq k)}^s \frac{1}{c_j-c_k} \in \mathrm{R}_+ = (0,\infty). \tag{5}$$

In particular, for $s = 1$, we have

$$k_1(\gamma) = \int_0^\infty \frac{t^{\gamma-1}}{t^\lambda + c_1}\,dt = \frac{\pi}{\lambda \sin(\frac{\pi\gamma}{\lambda})} c_1^{\frac{\eta}{\lambda}-1};$$

for $s = 2$, we have

$$k_2(\gamma) = \int_0^\infty \frac{t^{\gamma-1}}{(t^{\lambda/2}+c_1)(t^{\lambda/2}+c_2)}\,dt = \frac{2\pi}{\lambda \sin(\frac{2\pi\gamma}{\lambda})}\left(c_1^{\frac{2\gamma}{\lambda}-1} - c_2^{\frac{2\gamma}{\lambda}-1}\right)\frac{1}{c_2-c_1}.$$

Lemma 1 *Define the following weight coefficients*:

$$\omega_s(\lambda_2,m) := \ln^{\lambda-\lambda_2}(m-\xi)\sum_{n=2}^{\infty}\frac{\ln^{\lambda_2-1}(n-\eta)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(n-\eta)]}\frac{1}{n-\eta}$$
$$\bigl(m\in \mathrm{N}\backslash\{1\}\bigr), \tag{6}$$
$$\varpi_s(\lambda_1,n) := \ln^{\lambda-\lambda_1}(n-\eta)\sum_{m=2}^{\infty}\frac{\ln^{\lambda_1-1}(m-\xi)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(n-\eta)]}\frac{1}{m-\xi}$$
$$\bigl(n\in \mathrm{N}\backslash\{1\}\bigr). \tag{7}$$

For $\lambda_2 \le 1$, *we have*

$$\omega_s(\lambda_2,m) < k_s(\lambda-\lambda_2)\quad \bigl(m\in \mathrm{N}\backslash\{1\}\bigr); \tag{8}$$

for $\lambda_1 \le 1$, *we have*

$$k_s(\lambda_1)\bigl(1-\theta_s(\lambda_1,n)\bigr) < \varpi_s(\lambda_1,n) < k_s(\lambda_1)\quad \bigl(n\in \mathrm{N}\backslash\{1\}\bigr), \tag{9}$$

where $\theta_s(\lambda_1)$ *is indicated by*

$$\theta_s(\lambda_1,n) := \frac{1}{k_s(\lambda_1)}\int_0^{\frac{\ln(2-\xi)}{\ln(n-\eta)}}\frac{u^{\lambda_1-1}}{\prod_{k=1}^{s}(u^{\lambda/s}+c_k)}\,du = O\biggl(\frac{1}{\ln^{\lambda_1}(n-\eta)}\biggr)\in(0,1). \tag{10}$$

Proof Since for $0<\lambda_2\le 1, 0<\lambda\le s, y>\frac{3}{2}$, we find that

$$(-1)^i\frac{d^i}{dy^i}\ln^{\lambda_2-1}(y-\eta)\ge 0,\qquad (-1)^i\frac{d^i}{dx^i}\frac{1}{y-\eta}>0\quad\text{and}$$
$$(-1)^i\frac{d^i}{dy^i}\frac{1}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(y-\eta)]}>0\quad (i=1,2).$$

It follows that

$$(-1)^i\frac{d^i}{dy^i}\frac{\ln^{\lambda_2-1}(y-\eta)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(y-\eta)]}\frac{1}{y-\eta}>0\quad (i=0,1,2).$$

By Hermite–Hadamard's inequality (cf. [24]), we find

$$\begin{aligned}\omega_s(\lambda_2,m) &< \ln^{\lambda-\lambda_2}(m-\xi)\int_{\frac{3}{2}}^{\infty}\frac{\ln^{\lambda_2-1}(y-\eta)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(y-\eta)]}\frac{1}{y-\eta}\,dy\\ &= \ln^{\lambda-\lambda_2}(m-\xi)\int_{\frac{3}{2}}^{\infty}\frac{\ln^{\lambda_2-\lambda-1}(y-\eta)}{\prod_{k=1}^{s}\{[\frac{\ln(m-\xi)}{\ln(y-\eta)}]^{\lambda/s}+c_k\}}\frac{1}{y-\eta}\,dy.\end{aligned}$$

Setting $u=\frac{\ln(m-\xi)}{\ln(y-\eta)}$, it follows that $du=\frac{-\ln(m-\xi)}{\ln^2(y-\eta)}\frac{1}{y-\eta}\,dy$ and

$$\omega_s(\lambda_2,m) < \int_0^{\frac{\ln(m-\xi)}{\ln(\frac{3}{2}-\eta)}}\frac{u^{(\lambda-\lambda_2)-1}}{\prod_{k=1}^{s}(u^{\lambda/s}+c_k)}\,du < \int_0^{\infty}\frac{u^{(\lambda-\lambda_2)-1}}{\prod_{k=1}^{s}(u^{\lambda/s}+c_k)}\,du = k_s(\lambda-\lambda_2),$$

namely (8) follows.

In the same way, for $\lambda_1 \le 1$, by Hermite–Hadamard's inequality, we find

$$\varpi_s(\lambda_1, n) < \ln^{\lambda-\lambda_1}(n-\eta) \int_{\frac{3}{2}}^{\infty} \frac{\ln^{\lambda_1-1}(x-\xi)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(x-\xi) + c_k \ln^{\lambda/s}(n-\eta)]} \frac{1}{x-\xi}\, dx.$$

Setting $u = \frac{\ln(x-\xi)}{\ln(n-\eta)}$, it follows that

$$\varpi_s(\lambda_1, n) < \int_{\frac{\ln(\frac{3}{2}-\xi)}{\ln(n-\eta)}}^{\infty} \frac{u^{\lambda_1-1}}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)}\, du \le \int_0^{\infty} \frac{u^{\lambda_1-1}}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)}\, du = k_s(\lambda_i).$$

By the decreasing property, we also find

$$\begin{aligned}
\varpi_s(\lambda_1, n) &> \ln^{\lambda-\lambda_1}(n-\eta) \int_{2}^{\infty} \frac{\ln^{\lambda_1-1}(x-\xi)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(x-\xi) + c_k \ln^{\lambda/s}(n-\eta)]} \frac{1}{x-\xi}\, dx \\
&= \int_{\frac{\ln(2-\xi)}{\ln(n-\eta)}}^{\infty} \frac{u^{\lambda_1-1}}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)}\, du = k_s(\lambda_1)\big[1 - \theta_s(\lambda_1, n)\big] > 0, \\
0 < \theta_s(\lambda_1, n) &\le \frac{1}{k_s(\lambda_1)} \int_0^{\frac{\ln(2-\xi)}{\ln(n-\eta)}} \frac{u^{\lambda_1-1}}{c_1^s}\, du = \frac{1}{\lambda_1 k_s(\lambda_1) c_1^s} \left[\frac{\ln(2-\xi)}{\ln(n-\eta)}\right]^{\lambda_1}.
\end{aligned}$$

Hence, (9) and (10) follow. □

Lemma 2 *We have the following inequality*:

$$\begin{aligned}
I &:= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{a_m b_n}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]} \\
&> k_s^{\frac{1}{p}}(\lambda-\lambda_2) k_s^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{n=2}^{\infty} \big(1-\theta_s(\lambda_1, n)\big) \frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right\}^{\frac{1}{q}}.
\end{aligned} \tag{11}$$

Proof By reverse Hölder's inequality (cf. [24]), we obtain

$$\begin{aligned}
I &:= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{1}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]} \left[\frac{\ln^{(\lambda_2-1)p}(n-\xi)}{(n-\eta)^{1/p}} \frac{\ln^{(1-\lambda_1)/q}(m-\xi)}{(m-\xi)^{-1/q}} a_m \right] \\
&\quad \times \left[\frac{\ln^{(\lambda_1-1)/q}(m-\xi)}{(m-\xi)^{1/q}} \frac{\ln^{(1-\lambda_2)/p}(n-\eta)}{(n-\eta)^{-1/p}} b_n \right] \\
&\ge \left\{ \sum_{m=2}^{\infty} \left[\ln^{\lambda_1}(m-\xi) \sum_{n=2}^{\infty} \frac{\ln^{\lambda_2-1}(n-\eta)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]} \frac{1}{n-\eta} \right] \right. \\
&\quad \left. \times \frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{n=2}^{\infty} \left[\ln^{\lambda_2}(n-\eta) \sum_{m=2}^{\infty} \frac{\ln^{\lambda_1-1}(m-\xi)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]} \frac{1}{m-\xi} \right] \right.
\end{aligned}$$

$$\times \frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \Bigg\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{m=2}^{\infty} \omega_s(\lambda_2, m) \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=2}^{\infty} \varpi_s(\lambda_1, n) \frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right\}^{\frac{1}{q}}.$$

Then, by (8) and (9), we have (11). □

Remark 1 By (11), for $\lambda_1 + \lambda_2 = \lambda$, we find

$$\omega_s(\lambda_2, m) = \ln^{\lambda_1}(m-\xi) \sum_{n=2}^{\infty} \frac{\ln^{\lambda_2-1}(n-\eta)}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]} \frac{1}{n-\eta} \quad \bigl(m \in \mathrm{N}\backslash\{1\}\bigr),$$

$$0 < \sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p < \infty, \qquad 0 < \sum_{n=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-p}} b_n^q < \infty,$$

and the following inequality:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]}$$
$$> k_s(\lambda_1) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right]^{\frac{1}{p}}$$
$$\times \left[\sum_{n=2}^{\infty} \bigl(1-\theta_s(\lambda_1, n)\bigr) \frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{12}$$

In particular, for $\xi = \eta = 0$, we have $\tilde{\theta}_s(\lambda_1, n) = O(\frac{1}{\ln^{\lambda_1} n}) \in (0,1)$, and

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{\prod_{k=1}^{s}(\ln^{\lambda/s} m + c_k \ln^{\lambda/s} n)}$$
$$> k_s(\lambda_1) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \bigl(1-\tilde{\theta}_s(\lambda_1, n)\bigr) \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{13}$$

Hence, (12) is a more accurate extension of (13).

Lemma 3 *For* $0 < \varepsilon < q\lambda_2$, *we have*

$$L := \sum_{n=2}^{\infty} O\left(\frac{1}{\ln^{\lambda_1+\varepsilon+1}(n-\eta)} \right) \frac{1}{n-\eta} = O(1). \tag{14}$$

Proof There exist constants $m, M > 0$ such that

$$0 < m \sum_{n=2}^{\infty} \frac{1}{\ln^{\lambda_1+\varepsilon+1}(n-\eta)} \frac{1}{n-\eta} \le L \le M \left[\frac{(2-\eta)^{-1}}{\ln^{\lambda_1+\varepsilon+1}(2-\eta)} + \sum_{n=3}^{\infty} \frac{1}{\ln^{\lambda_1+\varepsilon+1}(n-\eta)} \frac{1}{n-\eta} \right].$$

By Hermite–Hadamard's inequality, it follows that

$$
\begin{aligned}
0 < L &\le M\left[\frac{(2-\eta)^{-1}}{\ln^{\lambda_1+\varepsilon+1}(2-\eta)} + \int_{\frac{5}{2}}^{\infty}\frac{1}{\ln^{\lambda_1+\varepsilon+1}(y-\eta)}\frac{1}{y-\eta}\,dy\right] \\
&= M\left[\frac{(2-\eta)^{-1}}{\ln^{\lambda_1+\varepsilon+1}(2-\eta)} + \frac{1}{\lambda_1+\varepsilon}\ln^{-\lambda_1-\varepsilon}\left(\frac{5}{2}-\eta\right)\right] \\
&\le M\left[\frac{(2-\eta)^{-1}}{\ln^{\lambda_1+q\lambda_2+1}(2-\eta)} + \frac{1}{\lambda_1}\ln^{-\lambda_1-q\lambda_2}\left(\frac{5}{2}-\eta\right)\right] < \infty.
\end{aligned}
$$

Hence, (14) follows. □

Lemma 4 *The constant factor $k_s(\lambda_1)$ in (12) is the best possible.*

Proof For $0 < \varepsilon < q\lambda_2$, we set

$$
\tilde{a}_m := \frac{\ln^{\lambda_1-\frac{\varepsilon}{p}-1}(m-\xi)}{m-\xi}, \qquad \tilde{b}_n := \frac{\ln^{\lambda_2-\frac{\varepsilon}{q}-1}(n-\eta)}{n-\eta} \quad \bigl(m,n \in \mathrm{N}\backslash\{1\}\bigr).
$$

If there exists a constant $M \ge k_s(\lambda_1)$ such that (12) is valid when replacing $k_s(\lambda_1)$ by M, then, in particular, we have

$$
\begin{aligned}
\tilde{I} &:= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi) + c_k\ln^{\lambda/s}(n-\eta)]} \\
&> M\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}\tilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-p}}\tilde{b}_n^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

In view of (10) and (14), we obtain

$$
\begin{aligned}
\tilde{I} &> M\left\{\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}\frac{\ln^{p\lambda_1-\varepsilon-p}(m-\xi)}{(m-\xi)^p}\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}\frac{\ln^{q\lambda_2-\varepsilon-q}(n-\eta)}{(n-\eta)^q}\right\}^{\frac{1}{q}} \\
&= M\left[\frac{\ln^{-\varepsilon-1}(2-\xi)}{2-\xi} + \sum_{m=3}^{\infty}\frac{\ln^{-\varepsilon-1}(m-\xi)}{m-\xi}\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=2}^{\infty}\frac{\ln^{-\varepsilon-1}(n-\eta)}{n-\eta} - \sum_{n=2}^{\infty}O\left(\frac{1}{\ln^{\lambda_1}(n-\eta)}\right)\frac{\ln^{-\varepsilon-1}(n-\eta)}{n-\eta}\right]^{\frac{1}{q}} \\
&> M\left[\frac{\ln^{-\varepsilon-1}(2-\xi)}{2-\xi} + \int_2^{\infty}\frac{\ln^{-\varepsilon-1}(x-\xi)}{x-\xi}\,dx\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_2^{\infty}\frac{\ln^{-\varepsilon-1}(y-\eta)}{y-\eta}\,dy - \sum_{n=2}^{\infty}O\left(\frac{1}{\ln^{\lambda_1+\varepsilon+1}(n-\eta)}\right)\frac{1}{n-\eta}\right]^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon}\left[\frac{\varepsilon\ln^{-\varepsilon-1}(2-\xi)}{2-\xi} + \ln^{-\varepsilon}(2-\xi)\right]^{\frac{1}{p}}\bigl[\ln^{-\varepsilon}(2-\eta) - \varepsilon O(1)\bigr]^{\frac{1}{q}}.
\end{aligned}
$$

By (8), setting $\hat{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q} \in (0,\lambda)(\hat{\lambda}_2 \le 1, \hat{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q})$, we find

$$\begin{aligned}\tilde{I} &= \sum_{m=2}^{\infty}\left\{\ln^{(\lambda_1+\frac{\varepsilon}{q})}(m-\xi)\sum_{n=2}^{\infty}\frac{1}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(n-\eta)]}\frac{\ln^{(\lambda_2-\frac{\varepsilon}{q})-1}(n-\eta)}{n-\eta}\right\}\\ &\quad\times\frac{\ln^{-\varepsilon-1}(m-\xi)}{m-\xi}\\ &= \sum_{m=2}^{\infty}\omega_s(\hat{\lambda}_2,m)\frac{\ln^{-\varepsilon-1}(m-\xi)}{m-\xi}\le k_s(\hat{\lambda}_1)\left[\frac{\ln^{-1-\varepsilon}(2-\xi)}{2-\xi}+\sum_{m=3}^{\infty}\frac{\ln^{-1-\varepsilon}(m-\xi)}{m-\xi}\right]\\ &\le k_s(\hat{\lambda}_1)\left[\frac{\ln^{-1-\varepsilon}(2-\xi)}{2-\xi}+\int_2^{\infty}\frac{\ln^{-1-\varepsilon}(x-\xi)}{x-\xi}\,dx\right]\\ &= \frac{1}{\varepsilon}k_s(\hat{\lambda}_1)\left[\frac{\varepsilon\ln^{-1-\varepsilon}(2-\xi)}{2-\xi}+\ln^{-\varepsilon}(2-\xi)\right].\end{aligned}$$

Then we have

$$\begin{aligned}&k_s(\hat{\lambda}_1)\left[\frac{\varepsilon\ln^{-1-\varepsilon}(2-\xi)}{2-\xi}+\ln^{-\varepsilon}(2-\xi)\right]\\ &\quad\ge\varepsilon\tilde{I}>M\left[\frac{\varepsilon\ln^{-\varepsilon-1}(2-\xi)}{2-\xi}+\ln^{-\varepsilon}(2-\xi)\right]^{\frac{1}{p}}\left[\ln^{-\varepsilon}(2-\eta)-\varepsilon O(1)\right]^{\frac{1}{q}}.\end{aligned}$$

For $\varepsilon\to 0^+$, we find $k_s(\lambda_1)\ge M$. Hence, $M=k_s(\lambda_1)$ is the best possible constant factor of (12).

Setting $\tilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \tilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, we find

$$\tilde{\lambda}_1+\tilde{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda,$$

and we can rewrite (11) as follows:

$$\begin{aligned}I&>k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\tilde{\lambda}_1)-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q(1-\tilde{\lambda}_2)-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{15}$$

□

Lemma 5 *If $\lambda\in(\lambda_1+(1-q)\lambda_2,(1-p)\lambda_1+\lambda_2)$, the constant factor $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ in (15) is the best possible, then we have $\lambda=\lambda_1+\lambda_2$.*

Proof For $\lambda_1+(1-q)\lambda_2<\lambda\le\lambda_1+\lambda_2$, we obtain

$$\begin{aligned}&\tilde{\lambda}_1\ge\frac{\lambda-\lambda_2}{p}+\frac{\lambda-\lambda_2}{q}=\lambda-\lambda_2>0,\qquad\tilde{\lambda}_1=\frac{\lambda}{p}-\frac{\lambda_2}{p}+\frac{\lambda_1}{q}<\lambda,\\ &0<\tilde{\lambda}_1<\lambda, 0<\tilde{\lambda}_2=\lambda-\tilde{\lambda}_1<\lambda;\end{aligned}$$

for $\lambda_1 + \lambda_2 < \lambda < (1-p)\lambda_1 + \lambda_2$, we still obtain

$$\tilde{\lambda}_2 \geq \frac{\lambda_2}{q} + \frac{\lambda_2}{p} = \lambda_2 > 0, \qquad \tilde{\lambda}_2 = \frac{\lambda}{q} - \frac{\lambda_1}{q} + \frac{\lambda_2}{p} < \lambda,$$
$$0 < \tilde{\lambda}_2 < \lambda, 0 < \tilde{\lambda}_1 = \lambda - \tilde{\lambda}_2 < \lambda.$$

Hence, we have $\tilde{\lambda}_i \in (0,\lambda)$ $(i = 1,2)$, and then $k_s(\tilde{\lambda}_1) \in \mathrm{R}_+$.

If the constant factor $k_s^{\frac{1}{p}}(\lambda - \lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ in (15) is the best possible, then, in view of (12), the unique best possible constant factor must be the form of $k_s(\tilde{\lambda}_1)$, namely

$$k_s(\tilde{\lambda}_1) = k_s^{\frac{1}{p}}(\lambda - \lambda_2)k_s^{\frac{1}{q}}(\lambda_1).$$

By reverse Hölder's inequality, we find

$$\begin{aligned} k_s(\tilde{\lambda}_1) &= k_\lambda\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}\right) \\ &= \int_0^\infty \frac{1}{\prod_{k=1}^s (u^{\lambda/s} + c_k)} u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} - 1}\,du = \int_0^\infty \frac{1}{\prod_{k=1}^s (u^{\lambda/s} + c_k)} \bigl(u^{\frac{\lambda-\lambda_2-1}{p}}\bigr)\bigl(u^{\frac{\lambda_1-1}{q}}\bigr)\,du \\ &\geq \left(\int_0^\infty \frac{1}{\prod_{k=1}^s (u^{\lambda/s} + c_k)} u^{\lambda-\lambda_2-1}\,du\right)^{\frac{1}{p}} \left(\int_0^\infty \frac{1}{\prod_{k=1}^s (u^{\lambda/s} + c_k)} u^{\lambda_1-1}\,du\right)^{\frac{1}{q}} \\ &= k_s^{\frac{1}{p}}(\lambda - \lambda_2)k_s^{\frac{1}{q}}(\lambda_1). \end{aligned} \tag{16}$$

We conclude that (16) keeps the form of equality if and only if there exist constants A and B such that they are not all zero and (cf. [24])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \quad \text{a.e. in } \mathrm{R}_+ = (0,\infty).$$

Assuming that $A \neq 0$ (otherwise, $B = A = 0$), it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{A}{B}$ a.e. in R_+, and then $\lambda - \lambda_2 - \lambda_1 = 0$, namely $\lambda = \lambda_1 + \lambda_2$. □

3 Main results and particular cases

Theorem 1 *Inequality* (11) *is equivalent to the following inequalities*:

$$\begin{aligned} J &:= \left\{\sum_{n=2}^\infty \frac{\ln^{p(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})-1}(n-\eta)}{(1-\theta_s(\lambda_1,n))^{p-1}(n-\eta)} \left[\sum_{m=2}^\infty \frac{a_m}{\prod_{k=1}^s [\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]}\right]^p\right\}^{\frac{1}{p}} \\ &> k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^\infty \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p\right\}^{\frac{1}{p}}, \end{aligned} \tag{17}$$

$$\begin{aligned} J_1 &:= \left\{\sum_{m=2}^\infty \frac{\ln^{q(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1}(m-\xi)}{m-\xi} \left\{\sum_{n=2}^\infty \frac{b_n}{\prod_{k=1}^s [\ln^{\lambda/s}(m-\xi) + c_k \ln^{\lambda/s}(n-\eta)]}\right\}^q\right\}^{\frac{1}{q}} \\ &> k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left\{\sum_{n=2}^\infty \bigl(1-\theta_s(\lambda_1,n)\bigr) \frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{18}$$

If the constant factor in (11) *is the best possible, then so is the constant factor in* (17) *and* (18).

Proof Suppose that (17) is valid. By Hölder's inequality, we have

$$\begin{aligned} I &= \sum_{n=2}^{\infty}\left\{\frac{\ln^{\frac{-1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}(n-\eta)}{(1-\theta_s(\lambda_1,n))^{1/q}(n-\eta)^{1/p}}\sum_{m=2}^{\infty}\frac{a_m}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(n-\eta)]}\right\} \\ &\quad\times\left\{\bigl(1-\theta_s(\lambda_1,n)\bigr)^{\frac{1}{q}}\frac{\ln^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}(n-\eta)}{(n-\eta)^{-1/p}}b_n\right\} \\ &\geq J\left\{\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{19}$$

Then, by (17), we obtain (11). On the other hand, assuming that (11) is valid, we set

$$b_n := \frac{\ln^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}(n-\eta)}{(1-\theta_s(\lambda_1,n))^{p-1}(n-\eta)}\left\{\sum_{m=2}^{\infty}\frac{a_m}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(n-\eta)]}\right\}^{p-1},$$
$$n\in\mathbf{N}\backslash\{1\}.$$

If $J=0$, then (17) is naturally valid; if $J=\infty$, then it is impossible that makes (17) valid, namely $J<\infty$. Suppose that $0<J<\infty$. By (11), we have

$$\begin{aligned} &\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q \\ &\quad= J^p = I > k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^{\infty}\frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right\}^{\frac{1}{p}} \\ &\qquad\times\left\{\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right\}^{\frac{1}{q}}, \\ J &= \left\{\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right\}^{\frac{1}{p}} \\ &> k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^{\infty}\frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right\}^{\frac{1}{p}}, \end{aligned}$$

namely (17) follows. Hence, inequality (11) is equivalent to (17).

Suppose that (18) is valid. By Hölder's inequality, we have

$$\begin{aligned} I &= \sum_{m=2}^{\infty}\left\{\frac{\ln^{\frac{1}{q}-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}(m-\xi)}{(m-\xi)^{-1/q}}a_m\right\} \\ &\quad\times\left\{\frac{\ln^{\frac{-1}{q}+(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}(m-\xi)}{(m-\xi)^{1/q}}\sum_{n=2}^{\infty}\frac{b_n}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(n-\eta)]}\right\} \end{aligned}$$

$$\geq \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}} J_1. \tag{20}$$

Then, by (18), we obtain (11). On the other hand, assuming that (11) is valid, we set

$$a_m := \frac{\ln^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}(m-\xi)}{m-\xi} \left\{ \sum_{n=2}^{\infty} \frac{b_n}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k \ln^{\lambda/s}(n-\eta)]} \right\}^{q-1},$$

$$m \in \mathrm{N}\backslash\{1\}.$$

If $J_1 = 0$, then (18) is naturally valid; if $J_1 = \infty$, then it is impossible that makes (18) valid, namely $J_1 < \infty$. Suppose that $0 < J_1 < \infty$. By (11), we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \\
&\quad = J_1^q = I > k_s^{\frac{1}{p}}(\lambda-\lambda_2) k_s^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}} \\
&\qquad \times \left\{ \sum_{n=2}^{\infty} (1-\theta_s(\lambda_1, n)) \frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right\}^{\frac{1}{q}}, \\
J_1 &= \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{q}} \\
&> k_s^{\frac{1}{p}}(\lambda-\lambda_2) k_s^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{n=2}^{\infty} (1-\theta_s(\lambda_1, n)) \frac{\ln^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right\}^{\frac{1}{q}},
\end{aligned}$$

namely (18) follows. Hence, inequality (11) is equivalent to (17) and (18).

If the constant factor in (11) is the best possible, then so is the constant factor in (17) and (18). Otherwise, by (19) (or (20)), we would reach a contradiction that the constant factor in (11) is not the best possible. □

Theorem 2 *If* $\lambda \in (\lambda_1 + (1-q)\lambda_2, (1-p)\lambda_1 + \lambda_2)$, *then the following statements* (i), (ii), (iii), *and* (iv) *are equivalent*:

(i) $k_s^{\frac{1}{p}}(\lambda-\lambda_2) k_s^{\frac{1}{q}}(\lambda_1)$ *is independent of* p, q;

(ii) $k_s^{\frac{1}{p}}(\lambda-\lambda_2) k_s^{\frac{1}{q}}(\lambda_1)$ *is expressed by a single integral*;

(iii) $k_s^{\frac{1}{p}}(\lambda-\lambda_2) k_s^{\frac{1}{q}}(\lambda_1)$ *in* (10) *is the best possible constant factor*;

(iv) $\lambda = \lambda_1 + \lambda_2$.

If statement (iv) *follows, namely* $\lambda = \lambda_1 + \lambda_2$, *then we have* (12) *and the following equivalent inequalities with the best possible constant factor* $k_s(\lambda_1)$:

$$\begin{aligned}
&\left\{ \sum_{n=2}^{\infty} \frac{\ln^{p\lambda_2-1}(n-\eta)}{(1-\theta_s(\lambda_1, n))^{p-1}(n-\eta)} \left[\sum_{m=2}^{\infty} \frac{a_m}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k \ln^{\lambda/s}(n-\eta)]} \right]^p \right\}^{\frac{1}{p}} \\
&\quad > k_s(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}},
\end{aligned} \tag{21}$$

$$\left\{\sum_{m=2}^{\infty}\frac{\ln^{q\lambda_1-1}(m-\xi)}{m-\xi}\left\{\sum_{n=2}^{\infty}\frac{b_n}{\prod_{k=1}^{s}[\ln^{\lambda/s}(m-\xi)+c_k\ln^{\lambda/s}(n-\eta)]}\right\}^q\right\}^{\frac{1}{q}}$$
$$> k_s(\lambda_1)\left[\sum_{n=2}^{\infty}\bigl(1-\theta_s(\lambda_1,n)\bigr)\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{22}$$

Proof (i)⇒(ii). By (i), we have

$$k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)=\lim_{q\to 1^+}k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)=k_s(\lambda_1),$$

namely $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ is expressed by a single integral

$$k_s(\lambda_1)=\int_0^{\infty}\frac{1}{\prod_{k=1}^{s}(u^{\lambda/s}+c_k)}u^{\lambda_1-1}\,du.$$

(ii)⇒(iv). If $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)$ is expressed by a convergent single integral $k_s(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (16) keeps the form of equality. In view of the proof of Lemma 5, it follows that $\lambda=\lambda_1+\lambda_2$.

(iv)⇒(i). If $\lambda=\lambda_1+\lambda_2$, then $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)=k_s(\lambda_1)$, which is independent of p,q. Hence, it follows that (i) ⇔ (ii) ⇔ (iv).

(iii)⇒(iv). By Lemma 5, we have $\lambda=\lambda_1+\lambda_2$.

(iv)⇒(iii). By Lemma 4, for $\lambda=\lambda_1+\lambda_2$, $k_s^{\frac{1}{p}}(\lambda-\lambda_2)k_s^{\frac{1}{q}}(\lambda_1)(=k_s(\lambda_1))$ is the best possible constant factor of (11). Therefore, we have (iii) ⇔ (iv).

Hence, statements (i), (ii), (iii), and (iv) are equivalent. □

Remark 2 For $\lambda=1,\lambda_1=\lambda_2=\frac{1}{2}$,

$$\hat{k}_s\left(\frac{1}{2}\right):=\int_0^{\infty}\frac{t^{-1/2}}{\prod_{k=1}^{s}(t^{1/s}+c_k)}\,dt=\frac{\pi s}{\sin(\frac{\pi s}{2})}\sum_{k=1}^{s}c_k^{\frac{s}{2}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j-c_k},$$
$$\hat{\theta}_s\left(\frac{1}{2},n\right):=\frac{1}{k_s(\frac{1}{2})}\int_0^{\frac{\ln(2-\xi)}{\ln(n-\eta)}}\frac{u^{-1/2}}{\prod_{k=1}^{s}(u^{1/s}+c_k)}\,du=O\left(\frac{1}{\ln^{1/2}(n-\eta)}\right)\in(0,1),$$

in (12), (21), and (22), we have the following equivalent inequalities with the best possible constant factor $\hat{k}_s(\frac{1}{2})$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_mb_n}{\prod_{k=1}^{s}[\ln^{1/s}(m-\xi)+c_k\ln^{1/s}(n-\eta)]}$$
$$>\hat{k}_s\left(\frac{1}{2}\right)\left[\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\left(1-\hat{\theta}_s\left(\frac{1}{2},n\right)\right)\frac{\ln^{\frac{q}{2}-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right]^{\frac{1}{q}}, \tag{23}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}(n-\eta)}{(1-\hat{\theta}_s(\frac{1}{2},n))^{p-1}(n-\eta)}\left[\sum_{m=2}^{\infty}\frac{a_m}{\prod_{k=1}^{s}[\ln^{1/s}(m-\xi)+c_k\ln^{1/s}(n-\eta)]}\right]^p\right\}^{\frac{1}{p}}$$
$$>\hat{k}_s\left(\frac{1}{2}\right)\left\{\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right\}^{\frac{1}{p}}, \tag{24}$$

$$\left\{\sum_{m=2}^{\infty}\frac{\ln^{\frac{q}{2}-1}(m-\xi)}{m-\xi}\left\{\sum_{n=2}^{\infty}\frac{b_n}{\prod_{k=1}^{s}[\ln^{1/s}(m-\xi)+c_k\ln^{1/s}(n-\eta)]}\right\}^{q}\right\}^{\frac{1}{q}}$$
$$>\hat{k}_s\left(\frac{1}{2}\right)\left\{\sum_{n=2}^{\infty}\left(1-\hat{\theta}_s\left(\frac{1}{2},n\right)\right)\frac{\ln^{\frac{q}{2}-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right\}^{\frac{1}{q}}. \quad (25)$$

4 Conclusions

In this paper, by the use of the weight coefficients, the idea of introducing parameters and Hermite–Hadamard's inequality, a more accurate reverse Mulholland-type inequality with parameters and the equivalent forms are given in Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 2 and Remarks 1–2. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. LH and HL participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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