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Regularity of weak solutions to obstacle problems for nondiagonal quasilinear degenerate elliptic systems

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Abstract

Let $X = \{X_1, \dots, X_m\}$ be a system of smooth real vector fields satisfying Hörmander's rank condition. We consider the interior regularity of weak solutions to an obstacle problem associated with the nonhomogeneous nondiagonal quasilinear degenerate elliptic system

$$X_\alpha^*\big(A_{ij}^{\alpha\beta}(x,u)X_\beta u^j\big) = B_i(x,u,Xu) + X_\alpha^* g_i^\alpha(x,u,Xu).$$

After proving the higher integrability and a Campanato type estimate for the weak solutions to the obstacle problem for the homogeneous nondiagonal quasilinear degenerate elliptic system, the interior Morrey regularity and Hölder continuity of weak solutions to the obstacle problem for the nonhomogeneous system are obtained.

MSC: Primary 35J70; secondary 35B65; 35J50

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\{X_1, \dots, X_m\}$ be a system of smooth real vector fields in $\mathbb{R}^n (m \le n)$, satisfying Hörmander's rank condition [1]:

$$\operatorname{rank}\big(\operatorname{Lie}\{X_1, \dots, X_m\}\big) = n.$$

The main purpose of this paper is to consider the obstacle problem for the nonhomogeneous nondiagonal quasilinear degenerate elliptic system

$$X_\alpha^*\big(A_{ij}^{\alpha\beta}(x,u)X_\beta u^j\big) = B_i(x,u,Xu) + X_\alpha^* g_i^\alpha(x,u,Xu), \tag{1.1}$$

where $i, j = 1, 2, \dots, N$; $\alpha, \beta = 1, 2, \dots, m$, X_α^* is the formal adjoint of X_α, B_i and g_i^α are both Carathéodory functions from $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{mN}$ into $\mathbb{R}$, the coefficients $A_{ij}^{\alpha\beta}(x,u)$ are bounded functions satisfying some assumptions that will be specified later.

Given an obstacle function $\psi = (\psi^1,\dots,\psi^N)$ and a boundary value function $\theta = (\theta^1,\dots,\theta^N)$ with $\theta(x) \ge \psi(x)$ a.e. in Ω, we define

$$\mathfrak{K}_\psi^\theta(\Omega,\mathbb{R}^N) = \{v \in S_X^1(\Omega,\mathbb{R}^N) : v \ge \psi \text{ a.e. in } \Omega, v - \theta \in S_{X,0}^1(\Omega,\mathbb{R}^N)\}.$$

Here we write $\theta(x) \ge \psi(x)$ to mean $\theta^i(x) \ge \psi^i(x)$ for $i = 1,\dots,N$.

A function $u \in \mathfrak{K}_\psi^\theta(\Omega,\mathbb{R}^N)$ is said to be a weak solution to the $\mathfrak{K}_\psi^\theta$-obstacle problem for (1.1) if

$$\int_\Omega A_{ij}^{\alpha\beta}(x,u)X_\beta u^j X_\alpha\phi^i\,dx \ge \int_\Omega B_i(x,u,Xu)\phi^i\,dx + \int_\Omega g_i^\alpha(x,u,Xu)X_\alpha\phi^i\,dx$$

for all $\phi \in C_0^\infty(\Omega,\mathbb{R}^N)$ with $\phi(x) + u(x) \ge \psi(x)$ a.e. $x \in \Omega$.

In the classical Euclidean setting, the interior regularity for solutions of elliptic equations and systems has been extensively investigated. Campanato in [2] and [3] obtained gradient estimates for solutions to linear elliptic equations and systems in divergence form with continuous coefficients. Applying Campanato's approach (see also [4] and [5]), Huang [6] proved the gradient estimates in the generalized Morrey spaces $L_\varphi^{2,\lambda}$ of weak solutions to the linear elliptic systems

$$-D_\alpha\big(a_{ij}^{\alpha\beta}(x)D_\beta u^j\big) = g_i(x) - \operatorname{div} f^i(x), \quad i = 1,2,\dots,N,$$

where the coefficients $a_{ij}^{\alpha\beta}(x) \in L^\infty \cap \mathrm{VMO}$. Similar results for the nonlinear elliptic systems of the form (1.1) with X replaced by the usual gradient $D = (D_1,\dots,D_n)$ were given by Daněček and Viszus in [7, 8], and [9]. For more related papers, we refer readers to [10–12]. Using the techniques that appeared in these papers, the local Morrey regularity and Hölder continuity of weak solutions to the obstacle problems associated with elliptic equations with constant coefficients or continuous coefficients have been obtained in [4, 13, 14], and [15].

The study of interior regularity for degenerate elliptic equations and systems has attracted much attention (see, e.g., [16–19], etc.). Di Fazio and Fanciullo in [16] pointed out that the local gradient estimates in [6] still hold true for the diagonal degenerate elliptic systems. The Morrey and Campanato regularities for weak solutions to the nondiagonal degenerate elliptic systems were established by Dong and Niu [19]. Another method of the so-called A-harmonic approximation for proving partial optimal Hölder regularity for weak solutions to nonlinear elliptic or subelliptic systems can be found in [20–23] and the references therein.

Since the degenerate obstacle problem is an important topic in various branches of the applied sciences, such as mechanical engineering and robotics, mathematical finance, image reconstruction and neurophysiology, a large amount of work has been devoted to the study of regularity for solutions to the relevant problems (see, for example, [24–29]). Gianazza and Marchi in [28] proved a Wiener criterion and an estimate on the modulus of continuity for weak solutions to the obstacle problem for a quasilinear degenerate elliptic equation constructed by Hörmander vector fields. Marchi in [29] derived the Hölder continuity of the horizontal gradient of weak solutions to the double obstacle problem for subelliptic equations in the Heisenberg group. Du and Li in [25, 26] proved the global higher integrability and interior regularity for subelliptic obstacle problems.

However, as far as we know, there is no result concerning the problem of regularity for solutions to the obstacle problem related to the nondiagonal degenerate elliptic systems. In this paper, we try to fill this gap. The aim of this paper is to establish the Hölder regularity results for weak solutions to the $\mathfrak{K}_\psi^\theta$-obstacle problem for (1.1). In order to state our results, we make the following hypotheses:

(H1) The coefficients $A_{ij}^{\alpha\beta}(x,u) = a^{\alpha\beta}(x)\delta_{ij} + B_{ij}^{\alpha\beta}(x,u)$, where $a^{\alpha\beta}(x) \in L^\infty(\Omega) \cap \mathrm{VMO}(\Omega)$ satisfying the strong ellipticity condition

$$a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \nu|\xi|^2, \quad \text{a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^m \tag{1.2}$$

for some $\nu > 0$, and $B_{ij}^{\alpha\beta}$ are measurable and there exists a constant $0 < \delta < \frac{1}{2}$ small enough such that, for any $(x,u) \in \Omega \times \mathbb{R}^N$,

$$\left|B_{ij}^{\alpha\beta}(x,u)\right| \leq \delta\nu;$$

(H2) For any $(x,u,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{mN}$,

$$\begin{aligned} &\left|B_i(x,u,\xi)\right| \leq f_i(x) + L|\xi|^{\gamma_0}, \\ &\left|g_i^\alpha(x,u,\xi)\right| \leq f_i^\alpha(x) + L|\xi|^{\gamma}, \end{aligned}$$

where $1 \leq \gamma_0 < 1/q_0$, $q_0 = Q/(Q+2)$, $0 \leq \gamma < 1$, $L > 0$ is a constant, and

$$f_i \in L_X^{2q_0,\lambda q_0}(\Omega), \qquad f_i^\alpha \in L_X^{2,\lambda}(\Omega), \quad 0 < \lambda < Q.$$

Here Q is the homogeneous dimension relative to Ω and in the sequel we set $f = (f_i)$, $\tilde{f} = (f_i^\alpha)$.

Now we state our main results.

Theorem 1.1 *Suppose that* (H1) *and* (H2) *hold and* $X\psi \in L_X^{\sigma,\lambda}(\Omega,\mathbb{R}^{mN})$, $\sigma > 2$. *Let* $u \in S_{X,\mathrm{loc}}^1(\Omega,\mathbb{R}^N)$ *be a weak solution to the* $\mathfrak{K}_\psi^\theta$*-obstacle problem for system* (1.1), *then* $Xu \in L_{X,\mathrm{loc}}^{2,\lambda}(\Omega,\mathbb{R}^{mN})$.

Theorem 1.2 *Suppose that* (H1) *and* (H2) *hold and* $X\psi \in L_X^{\sigma,\lambda}(\Omega,\mathbb{R}^{mN})$, $\sigma > 2$. *If* $u \in S_{X,\mathrm{loc}}^1(\Omega,\mathbb{R}^N)$ *is a weak solution to the* $\mathfrak{K}_\psi^\theta$*-obstacle problem for* (1.1) *and* $0 < \lambda < 2$, *then* $u \in C_X^{0,\alpha}(\Omega,\mathbb{R}^N)$ *with* $\alpha = 1 - \frac{\lambda}{2}$.

We apply the idea in [13] for proving the Hölder continuity of weak solutions to the obstacle problem for elliptic equation with continuous coefficients to the obstacle problem associated with a nondiagonal degenerate elliptic system with VMO coefficients. Inspired by the way in [16] and [19], we divide (1.1) into a nondiagonal homogeneous system and a nondiagonal nonhomogeneous system and then consider respectively the corresponding obstacle problem to prove our main results.

The paper is organized as follows. In Sect. 2, we present some concepts and results related to Carnot–Carathéodory spaces that will be used in our proof. In Sect. 3, we first prove the higher integrability for gradients of weak solutions to the $\mathfrak{K}_\psi^\theta$-obstacle problem for the homogeneous system (see (3.1) below) by constructing suitable test functions and using the Gehring lemma on the metric measure space. Based on this result, a Campanato type estimate for gradients of weak solutions to the $\mathfrak{K}_{\psi}^{\theta}$-obstacle problem for (3.1) is obtained. Section 4 is devoted to the proofs of our main results. We prove Theorem 1.1 by applying the Campanato type estimate established in Sect. 3 and an iteration lemma (Lemma 2.7 below). Theorem 1.2 is a direct consequence of Theorem 1.1 and the integral characterization of Hölder continuous functions.

2 Preliminaries

Let

$$X_\alpha = \sum_{k=1}^{n} b_{\alpha k}\frac{\partial}{\partial x_k}, \quad b_{\alpha k} \in C^\infty, \alpha = 1,2,\ldots,m,$$

be a family of vector fields in $\mathbb{R}^n$ satisfying the Hörmander's condition. We consider X_α as a first order differential operator acting on $u \in \operatorname{Lip}(\mathbb{R}^n)$ defined by

$$X_\alpha u(x) = \langle X_\alpha(x), \nabla u(x)\rangle, \quad \alpha = 1,2,\ldots,m.$$

We denote by $Xu = (X_1u,\ldots,X_mu)$ the gradient of u with respect to the system $X = \{X_1,\ldots,X_m\}$, and hence

$$|Xu(x)| = \left(\sum_{\alpha=1}^{m}|X_\alpha u(x)|^2\right)^{\frac{1}{2}}.$$

An absolutely continuous curve $\gamma : [a,b] \to \mathbb{R}^n$ is said to be admissible for the family X if there exist functions $c_\alpha(t)$, $a \le t \le b$, satisfying

$$\sum_{\alpha=1}^{m} c_\alpha(t)^2 \le 1 \quad \text{and} \quad \gamma'(t) = \sum_{\alpha=1}^{m} c_\alpha(t)X_\alpha\bigl(\gamma(t)\bigr), \quad \text{a.e. } t \in [a,b].$$

The Carnot–Carathéodory distance $d(x,y)$ generated by X is defined by

$$d(x,y) = \inf\bigl\{T > 0 : \text{there is an admissible curve } \gamma, \gamma(0) = x, \gamma(T) = y\bigr\}.$$

Following the accessibility theorem of Chow [30], the distance d is a metric and therefore $(\mathbb{R}^n, d)$ is a metric space which is called Carnot–Carathéodory space associated with the system X. The metric ball is denoted by

$$B_R(x_0) = B(x_0,R) = \bigl\{x \in \mathbb{R}^n : d(x,x_0) < R\bigr\}.$$

If $\sigma > 0$ and $B = B(x_0,R)$, we will write σB to indicate $B(x_0,\sigma R)$.

Theorem 2.1 ([31, 32]) *For every compact set $K \subset \Omega$, there exist constants $C_1, C_2 > 0$ and $0 < \lambda < 1$ such that*

$$C_1|x-y| \le d(x,y) \le C_2|x-y|^\lambda$$

for every $x, y \in K$. Moreover, there are $R_d > 0$ and $C_d \geq 1$ such that

$$\left|B(x,2R)\right| \leq C_d \left|B(x,R)\right| \tag{2.1}$$

whenever $x \in K$ and $R \leq R_d$.

Here, $|B(x,R)|$ denotes the Lebesgue measure of $B(x,R)$. The best constant C_d in (2.1) is called the doubling constant. We say that $Q = \log_2 C_d$ is the homogeneous dimension relative to Ω. As a consequence of doubling condition (2.1), we have

$$|B_{tR}| \geq Ct^Q |B_R|, \quad \forall R \leq R_d, t \in (0,1), \tag{2.2}$$

where $C = C_d^{-2}$.

Now we introduce the Sobolev spaces associated with $X = \{X_1, \ldots, X_m\}$. Given $1 \leq p < \infty$, we define the Sobolev space by

$$S_X^{1,p}\left(\Omega, \mathbb{R}^N\right) = \left\{u \in L^p\left(\Omega, \mathbb{R}^N\right) : X_\alpha u \in L^p\left(\Omega, \mathbb{R}^N\right), \alpha = 1, 2, \ldots, m\right\}$$

with the norm

$$\|u\|_{S_X^{1,p}(\Omega,\mathbb{R}^N)} = \|u\|_{L^p(\Omega,\mathbb{R}^N)} + \sum_{\alpha=1}^{m} \|X_\alpha u\|_{L^p(\Omega,\mathbb{R}^N)}.$$

Here, $X_\alpha u$ is the distributional derivative of $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ defined by the identity

$$\int_\Omega X_\alpha u \cdot \phi \, dx = \int_\Omega u \cdot X_\alpha^* \phi \, dx, \quad \forall \phi \in C_0^\infty\left(\Omega, \mathbb{R}^N\right),$$

where $X_\alpha^* = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{\alpha k} \cdot)$ is the formal adjoint of X_α, not necessarily a vector field in general. The space $S_{X,0}^{1,p}(\Omega, \mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\Omega, \mathbb{R}^N)$ under the norm $\|\cdot\|_{S_X^{1,p}(\Omega,\mathbb{R}^N)}$. In particular, we denote $S_X^{1,2}(\Omega, \mathbb{R}^N)$ and $S_{X,0}^{1,2}(\Omega, \mathbb{R}^N)$ by $S_X^1(\Omega, \mathbb{R}^N)$ and $S_{X,0}^1(\Omega, \mathbb{R}^N)$, respectively.

The following Sobolev inequalities for vector fields can be found in [33] and [32].

Theorem 2.2 *For every compact set $K \subset \Omega$, there exist constants $C > 0$ and $\bar{R} > 0$ such that, for any metric ball $B = B(x_0, R)$ with $x_0 \in K$ and $0 < R \leq \bar{R}$, it holds that, for any $f \in S_X^{1,p}(B_R)$,*

$$\left(\fint_{B_R} |f - f_R|^{\kappa p} \, dx\right)^{\frac{1}{\kappa p}} \leq CR\left(\fint_{B_R} |Xf|^p \, dx\right)^{\frac{1}{p}},$$

where $f_R = \fint_{B_R} f \, dx$ is the integral average of f on B_R, and $1 \leq \kappa \leq Q/(Q-p)$, if $1 \leq p < Q$; $1 \leq \kappa < \infty$, if $p \geq Q$. Moreover,

$$\left(\fint_{B_R} |f|^{\kappa p} \, dx\right)^{\frac{1}{\kappa p}} \leq CR\left(\fint_{B_R} |Xf|^p \, dx\right)^{\frac{1}{p}},$$

whenever $f \in S_{X,0}^{1,p}(B_R)$.

Now we define the Morrey spaces, the Campanato spaces, VMO, and the Hölder spaces with respect to the Carnot–Caratheodory metric [16, 34]. To simplify our description, we introduce the following notations:

$$\Omega(x,R) = \Omega \cap B(x,R), \qquad f_{x,R} = \frac{1}{|\Omega(x,R)|}\int_{\Omega(x,R)} f(y)\,dy$$

and

$$d_0 = \min\{\operatorname{diam}\Omega, R_d\}.$$

Definition 2.3 For $1 < p < \infty$ and $\lambda \le Q$, we say that $f \in L^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ belongs to the Morrey space $L^{p,\lambda}_X(\Omega,\mathbb{R}^N)$ if

$$\|f\|_{L^{p,\lambda}_X(\Omega,\mathbb{R}^N)} = \sup_{x\in\Omega,0<\rho<d_0}\left(\frac{\rho^\lambda}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)}\bigl|f(y)\bigr|^p\,dy\right)^{\frac{1}{p}} < \infty;$$

$f \in L^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}_X(\Omega,\mathbb{R}^N)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}_X(\Omega,\mathbb{R}^N)} = \sup_{x\in\Omega,0<\rho<d_0}\left(\frac{\rho^\lambda}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)}\bigl|f(y)-f_{x,\rho}\bigr|^p\,dy\right)^{\frac{1}{p}} < \infty.$$

Definition 2.4 For $\alpha \in (0,1)$, the Hölder space $C^{0,\alpha}_X(\bar{\Omega},\mathbb{R}^N)$ is the collection of functions $f : \bar{\Omega} \to \mathbb{R}^N$ satisfying

$$\|f\|_{C^{0,\alpha}_X(\bar{\Omega},\mathbb{R}^N)} = \sup_{\Omega}|f| + \sup_{\bar{\Omega}}\frac{|f(x)-f(y)|}{d(x,y)^\alpha} < \infty.$$

We say that $f \in C^{0,\alpha}_X(\Omega,\mathbb{R}^N)$ if $f \in C^{0,\alpha}_X(K,\mathbb{R}^N)$ for every compact set $K \subset \Omega$.

Definition 2.5 We say that $f \in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ belongs to $\mathrm{BMO}(\Omega,\mathbb{R}^N)$ if

$$\|f\|_* = \sup_{x\in\Omega,0<\rho<d_0}\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)}\bigl|f(y)-f_{x,\rho}\bigr|\,dy < \infty;$$

f belongs to $\mathrm{VMO}(\Omega,\mathbb{R}^N)$ if $f \in \mathrm{BMO}(\Omega,\mathbb{R}^N)$ and

$$\eta_r(f) = \sup_{x\in\Omega,0<\rho<r}\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)}\bigl|f(y)-f_{x,\rho}\bigr|\,dy \to 0, \quad r \to 0.$$

The integral characterization for a Hölder continuous function was shown in [34] and [35].

Lemma 2.6 *If* $-p < \lambda < 0$, *then* $\mathcal{L}^{p,\lambda}_X(\Omega,\mathbb{R}^N) \simeq C^{0,\alpha}_X(\Omega,\mathbb{R}^N)$, $\alpha = -\frac{\lambda}{p}$.

We end this section with a generalized iteration lemma, see [6, Proposition 2.1].

Lemma 2.7 *Let* H *be a nonnegative almost increasing function on the interval* $[0,T]$ *and* F *be a positive function on* $(0,T]$. *Suppose that*

(1) *for any* $0 < \rho \le R \le T$, *there exist* A, B, ε *and* $a > 0$ *such that*

$$H(\rho) \le \left(A\left(\frac{\rho}{R}\right)^a + \varepsilon\right)H(R) + BF(R);$$

(2) *there exists* $\tau \in (0,a)$ *such that* $\frac{\rho^\tau}{F(\rho)}$ *is almost increasing in* $(0,T]$. *Then there exist positive constants* ε_0 *and* C *such that, for any* $0 \le \varepsilon \le \varepsilon_0$,

$$H(\rho) \le C\frac{F(\rho)}{F(R)}H(R) + CB \cdot F(\rho), \quad 0 < \rho \le R \le T,$$

where ε_0 *depends only on* A, a *and* τ.

3 Obstacle problem for homogeneous systems

In this section, we deal with the $\mathfrak{K}_\psi^\theta$-obstacle problem related to the homogeneous nondiagonal quasilinear degenerate elliptic system

$$X_\alpha^*\left(A_{ij}^{\alpha\beta}(x,u)X_\beta u^j\right) = 0, \tag{3.1}$$

where $i,j = 1,2,\dots,N$, $\alpha,\beta = 1,2,\dots,m$, coefficients $A_{ij}^{\alpha\beta}(x,u)$ satisfy (H1). The main results are the higher integrability and a Campanato type estimate for the gradients of weak solutions. Let us recall that a function $u \in S^1_{X,\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is called a weak solution to (3.1) if

$$\int_\Omega A_{ij}^{\alpha\beta}(x,u)X_\beta u^j X_\alpha \phi^i\,dx = 0$$

for all $\phi \in C_0^\infty(\Omega,\mathbb{R}^N)$; a function $u \in \mathfrak{K}_\psi^\theta(\Omega,\mathbb{R}^N)$ is called a weak solution to the $\mathfrak{K}_\psi^\theta$-obstacle problem for (3.1) if

$$\int_\Omega A_{ij}^{\alpha\beta}(x,u)X_\beta u^j X_\alpha \phi^i\,dx \ge 0 \tag{3.2}$$

for all $\phi \in C_0^\infty(\Omega,\mathbb{R}^N)$ with $\phi + u \ge \psi$ a.e. Ω.

For the diagonal homogeneous linear degenerate elliptic system with constant coefficients, we have the following estimates (see [35, Theorem 3.2]).

Lemma 3.1 *Let* $u \in S^1_{X,\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *be a weak solution to the linear system*

$$X_\alpha^*\left(a^{\alpha\beta}X_\beta u^i\right) = 0, \quad i = 1,2,\dots,N,$$

with constant coefficients $a^{\alpha\beta} \in \mathbb{R}$ *for which* (1.2) *holds. Then, for any* $x_0 \in \Omega$, *there exist* $c > 0$ *and* $0 < R_0 < \min\{d_0, \mathrm{dist}(x_0,\partial\Omega)\}/2$ *such that, for any* ρ, R *with* $0 < \rho \le R \le R_0$, *it follows*

$$\int_{B_\rho(x_0)} |Xu|^2\,dx \le c\left(\frac{\rho}{R}\right)^Q \int_{B_R(x_0)} |Xu|^2\,dx. \tag{3.3}$$

In order to prove the higher integrability for gradients of weak solutions to the $\mathfrak{K}_\psi^\theta$-obstacle problem for (3.1), we need the Gehring lemma on the metric measure space (Y,d,μ), where d is a metric and μ is a doubling measure (see [36]).

Lemma 3.2 *Let $q \in [\bar{q}, 2Q]$, where $\bar{q} > 1$ is fixed. Assume that functions F, G are nonnegative and $G \in L^q_{\mathrm{loc}}(Y, \mu)$, $F \in L^{r_0}_{\mathrm{loc}}(Y, \mu)$ for some $r_0 > q$. If there exists a constant $b > 1$ such that, for every ball $B \subset \sigma B \subset Y$, the following inequality holds:*

$$\fint_B G^q \, d\mu \le b\left[\left(\fint_{\sigma B} G \, d\mu\right)^q + \fint_{\sigma B} F^q \, d\mu\right],$$

then there exists a nonnegative constant $\varepsilon_0 = \varepsilon_0(b, \bar{q}, Q, C_d, \sigma)$ such that $G \in L^p_{\mathrm{loc}}(Y, \mu)$ for $p \in [q, q + \varepsilon_0)$. Moreover,

$$\left(\fint_B G^p \, d\mu\right)^{\frac{1}{p}} \le C\left[\left(\fint_{\sigma B} G^q \, d\mu\right)^{\frac{1}{q}} + \left(\fint_{\sigma B} F^p \, d\mu\right)^{\frac{1}{p}}\right]$$

for some positive constant $C = C(b, \bar{q}, Q, C_d, \sigma)$.

Theorem 3.3 (Higher integrability) *Let $u \in S^1_{X,\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the $\mathfrak{K}^\theta_\psi$-obstacle problem for (3.1) and $X\psi \in L^\sigma_X(\Omega, \mathbb{R}^{mN})(\sigma > 2)$. Then there exists $p > 2$ such that $u \in S^{1,p}_{X,\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Furthermore, for any $B_R \subset\subset \Omega$, we have*

$$\left(\fint_{B_{R/2}} |Xu|^p \, dx\right)^{\frac{1}{p}} \le c\left[\left(\fint_{B_R} |Xu|^2 \, dx\right)^{\frac{1}{2}} + \left(\fint_{B_R} |X\psi|^p \, dx\right)^{\frac{1}{p}}\right], \tag{3.4}$$

where the constant $c > 0$ does not depend on R.

Proof For the weak solution $u \in S^1_{X,\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and $B_R \subset\subset \Omega$, consider the function

$$\phi = -\eta^2\big(u - \psi - (u - \psi)_R\big),$$

where η is a cutoff function on B_R, i.e., $\eta \in C^\infty_0(B_R)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B_{R/2}$, and $|X\eta| \le c/R$. Since $u_R \ge \psi_R$, we have

$$\phi = \eta^2(\psi - u) + \eta^2(u_R - \psi_R) \ge \eta^2(\psi - u) + \eta^2(\psi_R - \psi_R) \ge \psi - u$$

a.e. in Ω and it is an admissible function for (3.2). Taking $\phi = -\eta^2(u - \psi - (u - \psi)_R)$ in (3.2), we immediately get

$$\begin{aligned}
&\int_{B_R} a^{\alpha\beta}(x)\delta_{ij}\eta^2 X_\beta u^j X_\alpha u^i \, dx \\
&\quad \le -\int_{B_R} B^{\alpha\beta}_{ij}(x,u)\eta^2 X_\beta u^j X_\alpha u^i \, dx \\
&\qquad + \int_{B_R} A^{\alpha\beta}_{ij}(x,u)\eta^2 X_\beta u^j X_\alpha \psi^i \, dx \\
&\qquad + 2\int_{B_R} A^{\alpha\beta}_{ij}(x,u)\eta(\psi - \psi_R)^i X_\beta u^j X_\alpha \eta \, dx \\
&\qquad - 2\int_{B_R} A^{\alpha\beta}_{ij}(x,u)\eta(u - u_R)^i X_\beta u^j X_\alpha \eta \, dx.
\end{aligned}$$

By means of assumption (H1), Young and Sobolev inequalities, we see that

$$\begin{aligned}\nu\int_{B_R}\eta^2|Xu|^2\,dx &\le (\varepsilon+\delta\nu)\int_{B_R}\eta^2|Xu|^2\,dx+\frac{c_\varepsilon|B_R|}{R^2}⨍_{B_R}|u-u_R|^2\,dx\\&\quad+\frac{c_\varepsilon}{R^2}\int_{B_R}|\psi-\psi_R|^2\,dx+c_\varepsilon\int_{B_R}|X\psi|^2\,dx\\&\le(\varepsilon+\delta\nu)\int_{B_R}\eta^2|Xu|^2\,dx+c_\varepsilon|B_R|\biggl(⨍_{B_R}|Xu|^{2q_0}\,dx\biggr)^{\frac{1}{q_0}}\\&\quad+c_\varepsilon\int_{B_R}|X\psi|^2\,dx.\end{aligned}$$

Choosing $\varepsilon=\nu/3$ and noting $\eta=1$ on $B_{R/2}$, it follows

$$\int_{B_{R/2}}|Xu|^2\,dx\le c|B_R|\biggl(⨍_{B_R}|Xu|^{2q_0}\,dx\biggr)^{\frac{1}{q_0}}+c\int_{B_R}|X\psi|^2\,dx.$$

Dividing by $|B_{R/2}|$ on both sides, we get

$$⨍_{B_{R/2}}|Xu|^2\,dx\le c\biggl(⨍_{B_R}|Xu|^{2q_0}\,dx\biggr)^{\frac{1}{q_0}}+c⨍_{B_R}|X\psi|^2\,dx.$$

Now in Lemma 3.2 we set $G=|Xu|^{2q_0}$, $F=|X\psi|^{2q_0}$ and $q=1/q_0$. Then

$$|Xu|^{2q_0}\in L^r_{\mathrm{loc}}(\Omega),\quad \forall r\in[1/q_0,1/q_0+\varepsilon_0),$$

and

$$\biggl(⨍_{B_{R/2}}|Xu|^{2q_0r}\,dx\biggr)^{\frac{1}{r}}\le c\biggl(⨍_{B_R}|Xu|^2\,dx\biggr)^{q_0}+c\biggl(⨍_{B_R}|X\psi|^{2q_0r}\,dx\biggr)^{\frac{1}{r}}.$$

If we set $p=2q_0r$, then $p\in[2,2+2q_0\varepsilon_0)$, and

$$\biggl(⨍_{B_{R/2}}|Xu|^p\,dx\biggr)^{\frac{1}{p}}\le c\biggl[\biggl(⨍_{B_R}|Xu|^2\,dx\biggr)^{\frac{1}{2}}+\biggl(⨍_{B_R}|X\psi|^p\,dx\biggr)^{\frac{1}{p}}\biggr],$$

where c does not depend on R. The proof is complete. □

By virtue of the above result, we can establish a Campanato type estimate for the gradients of weak solutions to the $\mathfrak{K}^\theta_\psi$-obstacle problem for (3.1).

Theorem 3.4 *Let $u\in S^1_{X,\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a weak solution to the $\mathfrak{K}^\theta_\psi$-obstacle problem for (3.1) and $X\psi\in L^{\sigma,\lambda}_X(\Omega,\mathbb{R}^{mN})$, $\sigma>2$. Then, for any $x_0\in\Omega$, there exist $c>0$ and $0<R_0<\min\{d_0,\operatorname{dist}(x_0,\partial\Omega)\}/2$ such that, for any ρ, R with $0<\rho\le R\le R_0$, it follows*

$$\int_{B_\rho}|Xu|^2\,dx\le c\biggl[\biggl(\frac{\rho}{R}\biggr)^Q+\bigl(\eta_R\bigl(a^{\alpha\beta}\bigr)\bigr)^{\frac{p-2}{p}}+\delta\biggr]\int_{B_R}|Xu|^2\,dx+c\frac{|B_R|}{R^\lambda}\|X\psi\|^2_{L^{p,\lambda}},\qquad(3.5)$$

where $2<p<\sigma$.

Proof Let $B_R = B(x_0, R) \subset\subset \Omega$. In $B_{R/2}$ we split u as $u = U + w$, where $U \in S^1_X(B_{R/2}, \mathbb{R}^N)$ is the weak solution to the following boundary value problem for homogeneous system with constant coefficients:

$$\begin{cases} X^*_\alpha((a^{\alpha\beta}(x))_{R/2}\delta_{ij}X_\beta U^j) = 0, & \text{in } B_{R/2}, \\ U - u \in S^1_{X,0}(B_{R/2}, \mathbb{R}^N). \end{cases} \tag{3.6}$$

Denote $(a^{\alpha\beta})_{R/2} := (a^{\alpha\beta}(x))_{R/2}$. Therefore, by Lemma 3.1, there exist $c > 0$ and $0 < R_0 < \min\{d_0, \operatorname{dist}(x_0, \partial\Omega)\}/2$ such that, for all $0 < \rho < R/2$,

$$\int_{B_\rho} |XU|^2\,dx \le c\left(\frac{\rho}{R}\right)^Q \int_{B_{R/2}} |XU|^2\,dx. \tag{3.7}$$

Using (3.6), we know from (3.2) that, for all $\phi \in S^1_{X,0}(B_{R/2}, \mathbb{R}^N)$ with $\phi + u \ge \psi$ a.e. $B_{R/2}$,

$$\begin{aligned} \int_{B_{R/2}} (a^{\alpha\beta})_{R/2}\delta_{ij}X_\beta w^j X_\alpha \phi^i\,dx \ge{}& \int_{B_{R/2}} \bigl((a^{\alpha\beta})_{R/2} - a^{\alpha\beta}(x)\bigr)\delta_{ij}X_\beta u^j X_\alpha\phi^i\,dx \\ &- \int_{B_{R/2}} B^{\alpha\beta}_{ij}(x,u)X_\beta u^j X_\alpha \phi^i\,dx. \end{aligned} \tag{3.8}$$

Since $U - u \in S^1_{X,0}(B_{R/2}, \mathbb{R}^N)$, we can choose $\phi = U \vee \psi - u \in S^1_{X,0}(B_{R/2}, \mathbb{R}^N)$ as a test function in (3.8), where $U \vee \psi$ is a vector-valued function with components $U^i \vee \psi^i = \max\{U^i, \psi^i\}$. Then

$$\begin{aligned} &\int_{B_{R/2}} (a^{\alpha\beta})_{R/2}\delta_{ij}X_\beta w^j X_\alpha (u - U\vee\psi)^i\,dx \\ &\quad\le \int_{B_{R/2}} \bigl((a^{\alpha\beta})_{R/2} - a^{\alpha\beta}(x)\bigr)\delta_{ij}X_\beta u^j X_\alpha (u - U\vee\psi)^i\,dx \\ &\qquad - \int_{B_{R/2}} B^{\alpha\beta}_{ij}(x,u)X_\beta u^j X_\alpha (u - U\vee\psi)^i\,dx. \end{aligned} \tag{3.9}$$

Noting $(u - U \vee \psi)^i = w^i + (U - U \vee \psi)^i$, it follows by using the Hölder inequality that

$$\begin{aligned} &\int_{B_{R/2}} (a^{\alpha\beta})_{R/2}\delta_{ij}X_\beta w^j X_\alpha w^i\,dx \\ &\quad\le \int_{B_{R/2}} (a^{\alpha\beta})_{R/2}\delta_{ij}X_\beta w^j X_\alpha (U\vee\psi - U)^i\,dx \\ &\qquad + \int_{B_{R/2}} \bigl((a^{\alpha\beta})_{R/2} - a^{\alpha\beta}(x)\bigr)\delta_{ij}X_\beta u^j X_\alpha w^i\,dx \\ &\qquad + \int_{B_{R/2}} \bigl((a^{\alpha\beta})_{R/2} - a^{\alpha\beta}(x)\bigr)\delta_{ij}X_\beta u^j X_\alpha (U - U\vee\psi)^i\,dx \\ &\qquad - \int_{B_{R/2}} B^{\alpha\beta}_{ij}X_\beta u^j X_\alpha w^i\,dx - \int_{B_{R/2}} B^{\alpha\beta}_{ij}X_\beta u^j X_\alpha (U - U\vee\psi)^i\,dx \\ &\quad\le \left(\varepsilon + \frac{\delta\nu}{2}\right)\int_{B_{R/2}} |Xw|^2\,dx + \left(c_\varepsilon + \frac{\delta\nu}{2}\right)\int_{B_{R/2}} \bigl|X(U - U\vee\psi)\bigr|^2\,dx \\ &\qquad + c_\varepsilon \int_{B_{R/2}} \bigl|(a^{\alpha\beta})_{R/2} - a^{\alpha\beta}(x)\bigr|^2 |Xu|^2\,dx + \delta\nu\int_{B_{R/2}} |Xu|^2\,dx. \end{aligned} \tag{3.10}$$

Recalling (1.2) and taking $\varepsilon = \frac{\nu}{3}$, we have

$$\begin{aligned}\int_{B_{R/2}} |Xw|^2\,dx &\le c\int_{B_{R/2}} \big|X(U - U\vee\psi)\big|^2\,dx \\ &\quad + c\int_{B_{R/2}} \big|\big(a^{\alpha\beta}\big)_{R/2} - a^{\alpha\beta}(x)\big|^2|Xu|^2\,dx + \delta\int_{B_{R/2}} |Xu|^2\,dx. \end{aligned} \tag{3.11}$$

Since $a^{\alpha\beta}(x) \in L^\infty \cap \mathrm{VMO}$ and invoking (3.4) in Theorem 3.3, we conclude that there exists $p > 2$ such that

$$\begin{aligned} &\int_{B_{R/2}} \big|\big(a^{\alpha\beta}\big)_{R/2} - a^{\alpha\beta}(x)\big|^2|Xu|^2\,dx \\ &\quad\le |B_{R/2}|\bigg(⨍_{B_{R/2}} \big|\big(a^{\alpha\beta}\big)_{R/2} - a^{\alpha\beta}(x)\big|^{\frac{2p}{p-2}}\,dx\bigg)^{\frac{p-2}{p}}\bigg(⨍_{B_{R/2}} |Xu|^p\,dx\bigg)^{\frac{2}{p}} \\ &\quad\le c\big(\eta_R\big(a^{\alpha\beta}\big)\big)^{\frac{p-2}{p}}|B_{R/2}|\bigg(⨍_{B_{R/2}} |Xu|^p\,dx\bigg)^{\frac{2}{p}} \\ &\quad\le c\big(\eta_R\big(a^{\alpha\beta}\big)\big)^{\frac{p-2}{p}}\int_{B_R} |Xu|^2\,dx + c|B_R|\bigg(⨍_{B_R} |X\psi|^p\,dx\bigg)^{\frac{2}{p}}. \end{aligned} \tag{3.12}$$

On the other hand, $U - U\vee\psi \in S^1_{X,0}(B_{R/2},\mathbb{R}^N)$ satisfies

$$\begin{aligned} &\int_{B_{R/2}} \big(a^{\alpha\beta}\big)_{R/2}\delta_{ij}X_\beta(U - U\vee\psi)^j X_\alpha\varphi^i\,dx \\ &\quad= -\int_{B_{R/2}} \big(a^{\alpha\beta}\big)_{R/2}\delta_{ij}X_\beta(U\vee\psi)^j X_\alpha\varphi^i\,dx \end{aligned} \tag{3.13}$$

for all $\varphi \in S^1_{X,0}(B_{R/2},\mathbb{R}^N)$. Thus choosing $\varphi = U - U\vee\psi$ in (3.13) and noting $(U\vee\psi)^i = \psi^i$ for $x \in \operatorname{supp}(U - U\vee\psi)^i$, we obtain from (1.2) that

$$\begin{aligned} \nu\int_{B_{R/2}} \big|X(U - U\vee\psi)\big|^2\,dx &\le \int_{B_{R/2}} \big(a^{\alpha\beta}\big)_{R/2}X_\beta(U - U\vee\psi)^i X_\alpha(U - U\vee\psi)^i\,dx \\ &= -\int_{B_{R/2}} \big(a^{\alpha\beta}\big)_{R/2}X_\beta(U\vee\psi)^i X_\alpha(U - U\vee\psi)^i\,dx \\ &= -\int_{\operatorname{supp}(U-U\vee\psi)^i} \big(a^{\alpha\beta}\big)_{R/2}X_\beta\psi^i X_\alpha(U - U\vee\psi)^i\,dx \\ &\le \varepsilon\int_{B_{R/2}} \big|X(U - U\vee\psi)\big|^2\,dx + c_\varepsilon\int_{B_{R/2}} |X\psi|^2\,dx. \end{aligned}$$

Letting $\varepsilon = \frac{\nu}{2}$, we get

$$\int_{B_{R/2}} \big|X(U - U\vee\psi)\big|^2\,dx \le c\int_{B_{R/2}} |X\psi|^2\,dx. \tag{3.14}$$

Inserting (3.12) and (3.14) into (3.11), we have

$$\int_{B_{R/2}} |Xw|^2\,dx \le c\Big[\big(\eta_R\big(a^{\alpha\beta}\big)\big)^{\frac{p-2}{p}} + \delta\Big]\int_{B_R} |Xu|^2\,dx + c|B_R|\bigg(⨍_{B_R} |X\psi|^p\,dx\bigg)^{\frac{2}{p}}. \tag{3.15}$$

Then it follows by using (3.7) and (3.15) that, for any $0 < \rho < R/2$,

$$\begin{aligned}\int_{B_\rho} |Xu|^2\,dx &\le 2\int_{B_\rho} |XU|^2\,dx + 2\int_{B_\rho} |Xw|^2\,dx \\ &\le c\left(\frac{\rho}{R}\right)^Q \int_{B_{R/2}} |XU|^2\,dx + 2\int_{B_{R/2}} |Xw|^2\,dx \\ &\le c\left(\frac{\rho}{R}\right)^Q \int_{B_{R/2}} |Xu|^2\,dx + c\int_{B_{R/2}} |Xw|^2\,dx \\ &\le c\left[\left(\frac{\rho}{R}\right)^Q + \left(\eta_R\left(a^{\alpha\beta}\right)\right)^{\frac{p-2}{p}} + \delta\right]\int_{B_R} |Xu|^2\,dx + c|B_R|\left(\fint_{B_R} |X\psi|^p\,dx\right)^{\frac{2}{p}} \\ &\le c\left[\left(\frac{\rho}{R}\right)^Q + \left(\eta_R\left(a^{\alpha\beta}\right)\right)^{\frac{p-2}{p}} + \delta\right]\int_{B_R} |Xu|^2\,dx + c\frac{|B_R|}{R^\lambda}\|X\psi\|^2_{L^{p,\lambda}}. \end{aligned} \tag{3.16}$$

It is obvious that the above inequality is valid for $R/2 \le \rho \le R$. Thus, for all $0 < \rho \le R$, we have

$$\int_{B_\rho} |Xu|^2\,dx \le c\left[\left(\frac{\rho}{R}\right)^Q + \left(\eta_R\left(a^{\alpha\beta}\right)\right)^{\frac{p-2}{p}} + \delta\right]\int_{B_R} |Xu|^2\,dx + c\frac{|B_R|}{R^\lambda}\|X\psi\|^2_{L^{p,\lambda}}. \qquad \square$$

4 Proofs of the main results

In this section, we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 For fixed $x_0 \in \Omega$, let $B_R = B(x_0, R) \subset\subset \Omega$. Assume that the function u is a weak solution to the $\mathfrak{K}^\theta_\psi$-obstacle problem for (1.1), i.e., for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^N)$ with $\phi + u \ge \psi$ a.e. Ω,

$$\int_\Omega A^{\alpha\beta}_{ij}(x,u)X_\beta u^j X_\alpha \phi^i\,dx \ge \int_\Omega B_i(x,u,Xu)\phi^i\,dx + \int_\Omega g^\alpha_i(x,u,Xu)X_\alpha\phi^i\,dx. \tag{4.1}$$

Let $U_0 \in \mathfrak{K}^u_\psi(B_R, \mathbb{R}^N)$ be a weak solution to the $\mathfrak{K}^u_\psi$-obstacle problem for (3.1), i.e.,

$$\int_{B_R} A^{\alpha\beta}_{ij}(x,u)X_\beta U_0^j X_\alpha \bar{\phi}^i\,dx \ge 0 \tag{4.2}$$

for any $\bar{\phi} \in C_0^\infty(B_R, \mathbb{R}^N)$ with $\bar{\phi} + U_0 \ge \psi$ a.e. B_R.

Since $U_0 - u \in S^1_{X,0}(B_R, \mathbb{R}^N)$, we can take $\phi = U_0 - u := -w_0$ in (4.1) and hence

$$\begin{aligned}&\int_{B_R} A^{\alpha\beta}_{ij}(x,u)X_\beta(U_0 + w_0)^j X_\alpha w_0^i\,dx \\ &\quad\le \int_{B_R} B_i(x,u,Xu)w_0^i\,dx + \int_{B_R} g^\alpha_i(x,u,Xu)X_\alpha w_0^i\,dx.\end{aligned} \tag{4.3}$$

Noting that $w_0 + U_0 = u \ge \psi$ a.e. B_R, we obtain from (4.2) that

$$-\int_{B_R} A^{\alpha\beta}_{ij}(x,u)X_\beta U_0^j X_\alpha w_0^i\,dx \le 0. \tag{4.4}$$

Using (H1)–(H2), Hölder and Sobolev inequalities, (4.3) becomes

$$\begin{aligned}
&\nu \int_{B_R} |Xw_0|^2\,dx \\
&\quad\le \int_{B_R} a^{\alpha\beta}(x)\delta_{ij}X_\beta w_0^j X_\alpha w_0^i\,dx \\
&\quad\le -\int_{B_R} B_{ij}^{\alpha\beta}(x,u)X_\beta w_0^j X_\alpha w_0^i\,dx \\
&\qquad + \int_{B_R} B_i(x,u,Xu)w_0^i\,dx + \int_{B_R} g_i^\alpha(x,u,Xu)X_\alpha w_0^i\,dx \\
&\quad\le \nu\delta \int_{B_R} |Xw_0|^2\,dx + \int_{B_R} \bigl(|f| + L|Xu|^{\gamma_0}\bigr)\bigl|w_0^i\bigr|\,dx + \int_{B_R} \bigl(|\tilde{f}| + L|Xu|^{\gamma}\bigr)\bigl|X_\alpha w_0^i\bigr|\,dx \\
&\quad\le \nu\delta \int_{B_R} |Xw_0|^2\,dx + c\biggl(\int_{B_R} |w_0|^{2^*}\,dx\biggr)^{\frac{1}{2^*}} \biggl(\int_{B_R} \bigl(|f| + |Xu|^{\gamma_0}\bigr)^{2q_0}\,dx\biggr)^{\frac{1}{2q_0}} \\
&\qquad + c\biggl(\int_{B_R} |Xw_0|^2\,dx\biggr)^{\frac{1}{2}} \biggl(\int_{B_R} \bigl(|\tilde{f}| + |Xu|^{\gamma}\bigr)^2\,dx\biggr)^{\frac{1}{2}} \\
&\quad\le (\varepsilon + \nu\delta) \int_{B_R} |Xw_0|^2\,dx \\
&\qquad + c_\varepsilon\biggl[\biggl(\int_{B_R} \bigl(|f| + |Xu|^{\gamma_0}\bigr)^{2q_0}\,dx\biggr)^{\frac{1}{q_0}} + \int_{B_R} \bigl(|\tilde{f}| + |Xu|^{\gamma}\bigr)^2\,dx\biggr],
\end{aligned} \tag{4.5}$$

where $2^* = \frac{2Q}{Q-2}$. Choosing $\varepsilon > 0$ such that $\varepsilon + \nu\delta < \nu$, we have

$$\int_{B_R} |Xw_0|^2\,dx \le c\biggl(\int_{B_R} \bigl(|f| + |Xu|^{\gamma_0}\bigr)^{2q_0}\,dx\biggr)^{\frac{1}{q_0}} + c\int_{B_R} \bigl(|\tilde{f}| + |Xu|^{\gamma}\bigr)^2\,dx. \tag{4.6}$$

On the other hand, since $1 \le \gamma_0 < \frac{1}{q_0}$ and $0 \le \gamma < 1$, the Hölder inequality implies that

$$\biggl(\int_{B_R} |Xu|^{2\gamma_0 q_0}\,dx\biggr)^{\frac{1}{q_0}} \le |B_R|^{\frac{1-\gamma_0 q_0}{q_0}} \biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\gamma_0} \tag{4.7}$$

and

$$\int_{B_R} |Xu|^{2\gamma}\,dx \le |B_R|^{1-\gamma}\biggl(\int_{B_R} |Xu|^2\,dx\biggr)^{\gamma} \le \varepsilon\int_{B_R} |Xu|^2\,dx + c_\varepsilon|B_R|. \tag{4.8}$$

Putting (4.7) and (4.8) into (4.6), it follows

$$\begin{aligned}
\int_{B_R} |Xw_0|^2\,dx \le{}& c\bigl(\omega(R) + \varepsilon\bigr)\int_{B_R} |Xu|^2\,dx + c\biggl(\int_{B_R} |f|^{2q_0}\,dx\biggr)^{\frac{1}{q_0}} \\
&+ c\int_{B_R} |\tilde{f}|^2\,dx + c_\varepsilon|B_R|,
\end{aligned} \tag{4.9}$$

where $\omega(R) = |B_R|^{\frac{1-\gamma_0 q_0}{q_0}}(\int_{B_R} |Xu|^2\,dx)^{\gamma_0 - 1}$.

Consequently, we have by using (3.5) and (4.9) that, for any $0 < \rho \le R$,

$$\begin{aligned}
&\int_{B_\rho} |Xu|^2\,dx \\
&\le 2\int_{B_\rho} |XU_0|^2\,dx + 2\int_{B_\rho} |Xw_0|^2\,dx \\
&\le c\left[\left(\frac{\rho}{R}\right)^Q + \left(\eta_R\left(a^{\alpha\beta}\right)\right)^{\frac{p-2}{p}} + \delta\right]\int_{B_R} |XU_0|^2\,dx + c\frac{|B_R|}{R^\lambda}\|X\psi\|^2_{L^{p,\lambda}} + c\int_{B_\rho} |Xw_0|^2\,dx \\
&\le c\left[\left(\frac{\rho}{R}\right)^Q + \left(\eta_R\left(a^{\alpha\beta}\right)\right)^{\frac{p-2}{p}} + \delta\right]\int_{B_R} |Xu|^2\,dx + c\frac{|B_R|}{R^\lambda}\|X\psi\|^2_{L^{p,\lambda}} + c\int_{B_R} |Xw_0|^2\,dx \\
&\le c\left[\left(\frac{\rho}{R}\right)^Q + \left(\eta_R\left(a^{\alpha\beta}\right)\right)^{\frac{p-2}{p}} + \delta + \omega(R) + \varepsilon\right]\int_{B_R} |Xu|^2\,dx \\
&\quad + c\left(\int_{B_R} |f|^{2q_0}\,dx\right)^{\frac{1}{q_0}} + c\int_{B_R} |\tilde{f}|^2\,dx + c_\varepsilon|B_R| + c\frac{|B_R|}{R^\lambda}\|X\psi\|^2_{L^{p,\lambda}} \\
&\le c\left[\left(\frac{\rho}{R}\right)^Q + \left(\eta_R\left(a^{\alpha\beta}\right)\right)^{\frac{p-2}{p}} + \delta + \omega(R) + \varepsilon\right]\int_{B_R} |Xu|^2\,dx \\
&\quad + c\frac{|B_R|^{\frac{Q+2}{Q}}}{R^\lambda}\|f\|^2_{L^{2q_0,\lambda q_0}} + c\frac{|B_R|}{R^\lambda}\|\tilde{f}\|^2_{L^{2,\lambda}} + c\frac{|B_R|}{R^\lambda}\|X\psi\|^2_{L^{p,\lambda}} + c\frac{|B_R|}{R^\lambda} \\
&\le c\left[\left(\frac{\rho}{R}\right)^Q + \vartheta\right]\int_{B_R} |Xu|^2\,dx + c\frac{|B_R|}{R^\lambda}\left[\|f\|^2_{L^{2q_0,\lambda q_0}} + \|\tilde{f}\|^2_{L^{2,\lambda}} + \|X\psi\|^2_{L^{p,\lambda}} + 1\right], \qquad (4.10)
\end{aligned}$$

where $\vartheta = (\eta_R(a^{\alpha\beta}))^{\frac{p-2}{p}} + \delta + \omega(R) + \varepsilon$. By the absolute continuity of Lebesgue integral, we see that $\omega(R) \to 0$ as $R \to 0$. Making use of the VMO hypothesis on the coefficients $a^{\alpha\beta}(x)$, we know that there exists $0 < R_0 \le d_0$ such that ϑ is small enough for any $0 < R \le R_0$. Taking

$$F(\rho) = \frac{|B_\rho|}{\rho^\lambda} \quad \text{and} \quad Q - \lambda < \tau < Q,$$

we claim that $\frac{\rho^\tau}{F(\rho)}$ is almost increasing and it follows from (2.2) that

$$\frac{(t\rho)^\tau}{F(t\rho)}\Big/\frac{\rho^\tau}{F(\rho)} = \frac{t^{\tau+\lambda}|B_\rho|}{|B_{t\rho}|} \le C_d^2 \cdot t^{\tau+\lambda-Q} \le C_d^2, \quad \forall t \in (0,1).$$

Thus we obtain by Lemma 2.7 that, for any $0 < \rho \le R$,

$$\int_{B_\rho} |Xu|^2\,dx \le c\frac{|B_\rho|}{\rho^\lambda}\left[\frac{R^\lambda}{|B_R|}\int_{B_R} |Xu|^2\,dx + \|f\|^2_{L^{2q_0,\lambda q_0}} + \|\tilde{f}\|^2_{L^{2,\lambda}} + \|X\psi\|^2_{L^{p,\lambda}} + 1\right], \qquad (4.11)$$

which implies $Xu \in L^{2,\lambda}_{X,\mathrm{loc}}(\Omega, \mathbb{R}^{mN})$. The proof is finished. □

From Theorem 1.1 and Lemma 2.6, we can prove the Hölder continuity for the weak solutions.

Proof of Theorem 1.2 Let $u \in S^1_{X,\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the $\mathfrak{K}^\theta_\psi$-obstacle problem for (1.1) and $0 < \lambda < 2$. For any $B_\rho \subset B_R \subset\subset \Omega$, it follows by Theorem 2.2 and (4.11) that

$$\begin{aligned}\int_{B_\rho} |u - u_\rho|^2\,dx &\le c\rho^2 \int_{B_\rho} |Xu|^2\,dx \\ &\le c\frac{|B_\rho|}{\rho^{\lambda-2}}\left[\frac{R^\lambda}{|B_R|}\int_{B_R} |Xu|^2\,dx + \|f\|^2_{L^{2q_0,\lambda q_0}} + \|\tilde{f}\|^2_{L^{2,\lambda}} + \|X\psi\|^2_{L^{p,\lambda}} + 1\right],\end{aligned}$$

which implies $u \in \mathcal{L}^{2,\lambda-2}_{X,\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Noting $0 < \lambda < 2$, Theorem 1.2 follows immediately from Lemma 2.6. □

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