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Multiple stability and instability of Cohen–Grossberg neural network with unbounded time-varying delays

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Abstract

In this paper, multiple stability and instability of Cohen–Grossberg neural network with unbounded time-varying delays are studied. Based on the geometrical configuration of activation functions and some rigorous mathematical analysis, some algebraic criteria are proposed to guarantee coexistence of multiple stable equilibrium points and multiple unstable equilibrium points in the model. Moreover, using the partition space method, we prove that the discussed model has at least 3^n equilibrium points, 2^n of them are locally μ-stable and others are unstable. Finally, the numerical example and its simulation show the effectiveness of the proposed results.

Keywords: Cohen–Grossberg neural network; Multiple stability and instability; Unbounded time-varying delays

1 Introduction

The Cohen–Grossberg neural network model, proposed by Cohen and Grossberg in 1983 [1], has been attracting much attention because of its wide application in various engineering fields and because of it being highly inclusive of other neural networks such as Hopfield neural network, cellular neural network, recurrent neural network, and so on. Hence many scholars devoted themselves to the research of this aspect (see [2–18]). In some practical applications and hardware implementations of artificial neural networks, time delays are inevitable due to the finite switching speed of the amplifiers and the interneurons conduction distances, and they are even time-varying and unbounded in some cases such as the memory activation function of the human brain neural network model. Therefore, it is more suitable to introduce unbounded time-varying delays to the neural network, especially to Cohen–Grossberg models, and some results have been reported recently, for example, [19–28].

In the applications of pattern recognition, the addressable memories of patterns are stored as stable equilibrium points. Thus it is necessary that there exist multiple stable equilibrium points for neural networks. The coexistence of multiple equilibrium points and their local stability, which is usually referred to as the multistability of neural network models, has been reported in depth in the last years (see [29–43] and the references therein). Wang et al. in [35] studied a class of neural networks with r-level piecewise linear nondecreasing activation functions and showed that the n-neuron dynamical system had exact $(2r+1)^n$ equilibrium points, of which $(r+1)^n$ were locally exponentially stable and the others were unstable. By using the partition space method, [41] proved that neural networks with unbounded time-varying delays could exhibit at least 3^n equilibrium points, 2^n of them are locally μ-stable and others are unstable. In [43], based on the geometrical configuration of activation functions and mathematic tools, some novel algebraic criteria were proposed to guarantee the coexistence of 25^n equilibrium points, in which 9^n equilibrium points are locally μ-stable, for the memristor-based complex-valued neural networks with non-monotonic piecewise nonlinear activation functions and unbounded time-varying delays. From the references mentioned above, we find that the multistability of Cohen–Grossberg neural networks with unbounded time-varying delays is a challenging problem.

Motivated by the challenging problem, we investigate the multistability of a Cohen–Grossberg neural network with unbounded time-varying delay and nondecreasing activation functions in this paper and prove that the considered model has 3^n equilibrium points, and 2^n of them are locally μ-stable, the remaining ones are unstable. Compared with the literature [41], the results are more general. The rest of this paper is organized as follows. In Sect. 2, the Cohen–Grossberg model and some preliminaries are given. The main results are presented and proved in Sect. 3. The corollaries and comparison with the results of existing literature are presented in Sect. 4. A numerical example with its simulation is showed in Sect. 5 to illustrate the effectiveness the proposed results. Finally, conclusions are drawn in Sect. 6.

2 Preliminaries

In this paper, the following Cohen–Grossberg neural network is considered.

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i\bigl(x_i(t)\bigr)\Biggl[b_i\bigl(x_i(t)\bigr) - \sum_{j=1}^{n} c_{ij}g_j\bigl(x_j(t)\bigr) - \sum_{j=1}^{n} d_{ij}f_j\bigl(x_j\bigl(t-\tau(t)\bigr)\bigr) + I_i\Biggr], \quad t \geq 0, \quad (1)$$

where $i = 1,2,\dots,n$, $x_i(t)$ denotes the state variable associated with the ith neuron at time t; $a_i(x_i(t))$ represents an amplification function at time t; $b_i(x_i(t))$ is an appropriate inhibition behavior function at time t such that the solutions of model (1) remain bounded; $g_j(x_j(t))$ and $f_j(x_j(t-\tau(t)))$ denote the activation functions of the jth neuron unit at time t without and with time delays, respectively, and $C = (c_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ are the corresponding connection weights matrices; $\tau(t)$ corresponds to the transmission delay and satisfies $\tau(t) \geq 0$; I_i is the constant external input of the network on the ith neuron.

The initial conditions of model (1) are assumed to be $x_i(s) = \varphi_i(s)$, $s \leq 0$, $i = 1,2,\dots,n$, where $\varphi_i(s)$ is a real-valued continuous function bounded on $(-\infty,0]$, except that finite points existing at the left and right limits are continuous to the right. Throughout this paper, we make the following assumptions.

(H1) For each $i \in 1,2,\dots,n$, the amplification function $a_i(u)$ is nonnegative continuous and satisfies

$$0 < \underline{a}_i \leq a_i(u) \leq \bar{a}_i < \infty, \quad u \in R, i = 1,2,\dots,n.$$

And let two n-dimensional positive diagonal matrices $\hat{A} = \operatorname{diag}\{\bar{a}_1,\bar{a}_2,\dots,\bar{a}_n\}$ and $\check{A} = \operatorname{diag}\{\underline{a}_1,\underline{a}_2,\dots,\underline{a}_n\}$.

(H2) $b_i(u)$ is an odd function and monotone increasing, and there exists an n-dimensional positive diagonal matrix $B = \operatorname{diag}\{b_1, b_2, \dots, b_n\}$ such that

$$\frac{b_i(u) - b_i(v)}{u - v} \geq b_i, \quad u, v \in R, u \neq v, i = 1, 2, \dots, n.$$

(H3) $g_j(\cdot)$ and $f_j(\cdot)$ are nondecreasing sigmoid continuous nonlinear function or nondecreasing piecewise continuous linear function, and there exist constants $p_j \leq q_j$, $m_j \leq M_j$, $m'_j \leq M'_j$, $m''_j \leq M''_j$, so that

$$\begin{aligned}
&m'_j = \lim_{x \to -\infty} g_j(x), \qquad M'_j = \lim_{x \to +\infty} g_j(x), \\
&m''_j = \lim_{x \to -\infty} f_j(x), \qquad M''_j = \lim_{x \to +\infty} f_j(x), \\
&0 \leq \underline{\sigma}_j^l \leq \frac{g_j(u) - g_j(v)}{u - v} \leq \bar{\sigma}_j^l, \qquad 0 \leq \underline{\delta}_j^l \leq \frac{f_j(u) - f_j(v)}{u - v} \leq \bar{\delta}_j^l, \quad \forall u, v \in (-\infty, p_j), \\
&0 \leq \underline{\sigma}_j^m \leq \frac{g_j(u) - g_j(v)}{u - v} \leq \bar{\sigma}_j^m, \qquad 0 \leq \underline{\delta}_j^m \leq \frac{f_j(u) - f_j(v)}{u - v} \leq \bar{\delta}_j^m, \quad \forall u, v \in [p_j, q_j], \\
&0 \leq \underline{\sigma}_i^r \leq \frac{g_j(u) - g_j(v)}{u - v} \leq \bar{\sigma}_j^r, \qquad 0 \leq \underline{\delta}_j^r \leq \frac{f_j(u) - f_j(v)}{u - v} \leq \bar{\delta}_j^r, \quad \forall u, v \in (p_j, +\infty),
\end{aligned}$$

where $m_j = \min\{m'_j, m''_j\}$, $M_j = \min\{M'_j, M''_j\}$, $\bar{\sigma}_j = \max\{\bar{\sigma}_j^l, \bar{\sigma}_j^m, \bar{\sigma}_j^r\}$, $\bar{\delta}_j = \max\{\bar{\delta}_j^l, \bar{\delta}_j^m, \bar{\delta}_j^r\}$, $j = 1, 2, \dots, n$, and define two n-dimensional positive diagonal matrices $\Sigma^g = \operatorname{diag}\{\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_n\}$ and $\Delta^f = \operatorname{diag}\{\bar{\delta}_1, \bar{\delta}_2, \dots, \bar{\delta}_n\}$. The superscripts "$l$", "$m$", "$r$" denote "left", "middle", and "right", respectively.

It is not hard to find such activation functions as the sigmoid continuous nonlinear function $f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, piecewise continuous linear function $g(x) = \frac{|x+1| - |x+1|}{2}$, which are different functions, but the properties of the functions can be discussed by common interval separation points. Based on the geometric structure of the activation function, we can define the interval of a one-dimensional real number space as follows:

$$(-\infty, +\infty) = (-\infty, p_i) \cup [p_i, q_i] \cup (q_i, +\infty), \quad i = 1, 2, \dots, n,$$

then the n-dimensional real number space R^n can be divided into 3^n non-intersection subregions. For convenience, let Φ denote the set of these subregions, and so

$$\Phi = \left\{ \prod_{i=1}^n w_i \mid w_i = (-\infty, p_i), [p_i, q_i] \text{ or } (q_i, +\infty) \right\}.$$

For each $\prod_{i=1}^n w_i \in \Phi$, we define the following index subsets with respect to different interval as $N_1 = \{i \mid w_i = (-\infty, p_i), i = 1, 2, \dots, n\}$, $N_2 = \{i \mid w_i = [p_i, q_i], i = 1, 2, \dots, n\}$, $N_3 = \{i \mid w_i = (q_i, +\infty), i = 1, 2, \dots, n\}$.

Furthermore, Φ can be separated into two parts: $\Phi_1 = \{\prod_{i=1}^n w_i \mid w_i = (-\infty, p_i) \text{ or } (q_i, +\infty)\}$, $\Phi_2 = \Phi - \Phi_1$. Obviously, Φ_1 is composed of 2^n subregions and Φ_2 contains $3^n - 2^n$ subregions.

To facilitate showing the existence of equilibrium points of model (1), we define new sets

$$\Omega = \left\{ \prod_{i=1}^{n} \nu_i \mid \nu_i = [-E_i, p_i], [p_i, q_i] \text{ or } [q_i, E_i] \right\},$$

$$\Omega_1 = \left\{ \prod_{i=1}^{n} \nu_i \mid \nu_i = [-E_i, p_i] \text{ or } [q_i, E_i] \right\},$$

where $E_i = 2b_i^{-1}[\sum_{j=1}^{n}(|c_{ij}| + |d_{ij}|)\max\{m_j, M_j\} + |I_i| + \max\{|b_i(p_i)|, |b_i(q_i|\}]$, $i = 1, 2, \ldots, n$.

For each $\prod_{i=1}^{n} \nu_i \in \Omega$, we can similarly define its index subsets: $N_1' = \{i \mid \nu_i = [-E_i, p_i], i = 1, 2, \ldots, n\}$, $N_2' = \{i \mid \nu_i = [p_i, q_i], i = 1, 2, \ldots, n\}$, $N_3' = \{i \mid \nu_i = [q_i, E_i], i = 1, 2, \ldots, n\}$.

3 Main results

Because an equilibrium point of system (1) is a constant satisfying the equation $b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}g_j(x_j(t)) - \sum_{j=1}^{n} d_{ij}f_j(x_j(t)) + I_i = 0$, it is obvious that of model (1) has the same equilibrium point with the following system:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i\bigl(x_i(t)\bigr)\left[b_i\bigl(x_i(t)\bigr) - \sum_{j=1}^{n} c_{ij}g_j\bigl(x_j(t)\bigr) - \sum_{j=1}^{n} d_{ij}f_j(x_j(t) + I_i\right], \quad t \geq 0, \tag{2}$$

for $i = 1, 2, \ldots, n$. Therefore, we can investigate the existence of multiple equilibrium points of model (2) instead of (1).

Theorem 1 *For any* $\prod_{i=1}^{n} w_i \in \Phi$, *if*

$$\begin{cases} -b_i(p_i) + c_{ii}g_i(p_i) + d_{ii}f_i(p_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \max\{(c_{ij} + d_{ij})m_j, (c_{ij} + d_{ij})M_j\} - I_i < 0, \\ \quad i \in N_1 \cup N_2, \\ -b_i(q_i) + c_{ii}g_i(q_i) + d_{ii}f_i(q_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \min\{(c_{ij} + d_{ij})m_j, (c_{ij} + d_{ij})M_j\} - I_i > 0, \\ \quad i \in N_2 \cup N_3, \end{cases} \tag{3}$$

where $N_i \cap N_j = \phi$, $i \neq j$, $i, j = 1, 2, 3$, $N_1 \cup N_2 \cup N_3 = \{1, 2, \ldots, n\}$, *then there is at least one equilibrium point of model* (1) *in* $\prod_{i=1}^{n} w_i$.

Proof Let $(x_1, x_2, \ldots, x_n)$ be a point of $\prod_{i=1}^{n} \nu_i \in \Omega$. Then, for the ith component x_i, fixing other components $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, we can define a function as follows:

$$F_i(u) = -b_i(u) + c_{ii}g_i(u) + d_{ii}f_i(u) + \sum_{\substack{j=1\\ j\neq i}}^{n} c_{ij}g_j(x_j) + \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}f_j(x_j) - I_i,$$

where $i = 1, 2, \ldots, n$. Then, by (3), we can deduce the following:

(1) The function $F_i(u)$ is continuous in the interval $[-E_i, p_i]$, and $F_i(-E_i) > 0$, $F_i(p_i) < 0$. Then by the zero point theorem, there exists at least a point $\bar{x}_i \in (-E_i, p_i)$ such that $F_i(\bar{x}_i) = 0$. On account of $a_i(x_i) > 0$, by (H1), $\bar{x}_i$ is the equilibrium point of system (2) for the state component $x_i(t)$ of the ith neuron in the interval $(-E_i, p_i)$.

(2) If $u \in [p_i, q_i]$, then we have $F_i(p_i) > 0$, $F_i(q_i) < 0$. In view of $a_i(x_i) > 0$, there exists at least a point $\bar{x}_i \in (p_i, q_i)$, which is the equilibrium point of system (2) for the state component $x_i(t)$ of the ith neuron in the interval (p_i, q_i).

(3) There exists at least a point $\bar{x}_i \in (q_i, E_i)$, which is the equilibrium point of system (2) for the state component $x_i(t)$ of the ith neuron in the interval (q_i, E_i) because of $F_i(q_i) > 0$, $F_i(E_i) < 0$, and $a_i(x_i) > 0$.

Above all, the function $F_i(u)$ has at least one zero point in any subinterval. It follows that there is at least one equilibrium point of system (2) in $\prod_{i=1}^{n} v_i \in \Omega$.

In addition, we can define a mapping $\wp : \prod_{i=1}^{n} v_i \to \prod_{i=1}^{n} v_i$ for any given $x = (x_1, x_2, \ldots, x_n) \in \prod_{i=1}^{n} v_i$ such that $\wp(x_1, x_2, \ldots, x_n) = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, where $\bar{x}_i$ is a solution of equation $F_i(u) = 0$, $j = 1, 2, \ldots, n$. Since $f_j(\cdot), g_j(\cdot), j = 1, 2, \ldots, n$, are continuous by assumption (H3), the mapping $\wp$ is also continuous, then by Brouwer's fixed point theorem, there exists at least one fixed equilibrium point, denoted as $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)$, such that $\wp(x^\star) = x^\star$. Furthermore, components of the equilibrium point $x^\star$ satisfy $-E_i < x_i^\star < p_i$ for $i \in N_1'$, $p_i < x_i^\star < q_i$ for $i \in N_2'$, $q_i < x_i^\star < E_i$ for $i \in N_3'$. It means that all the equilibrium points in $\prod_{i=1}^{n} v_i$ are located in its interior. In view of the arbitrariness of the region $\prod_{i=1}^{n} v_i$ and the sameness of the equilibrium points between model (1) and model (2), model (1) has at least 3^n equilibrium points in Ω. From the definition of set Φ and Ω, we know that the corresponding region of $\prod_{i=1}^{n} v_i$ is $\prod_{i=1}^{n} w_i$ and satisfies $\prod_{i=1}^{n} v_i \subseteq \prod_{i=1}^{n} w_i$. Hence it is easy to see that $x^\star$ is also an equilibrium of model (1) in $\prod_{i=1}^{n} w_i$, and so there is at least one equilibrium point of model (1) in $\prod_{i=1}^{n} w_i$. □

For each equilibrium point $x^\star = (x_1^\star, \ldots, x_n^\star)$ of $\prod_{i=1}^{n} w_i \in \Phi_1$, we define its μ-stability in $\prod_{i=1}^{n} w_i$ (local μ-stability in Φ_1), and prove the μ-stability of all equilibrium points in Φ_1 in the following Definition 1 and Theorem 2, respectively.

Definition 1 Let $(x_1(t), x_2(t), \ldots, x_n(t))$ be an arbitrary solution of model (1) located in $\prod_{i=1}^{n} w_i \in \Phi_1$ with the initial state $x_i(s) = \varphi_i(s)$, $s \in (-\infty, 0]$, $i = 1, 2, \ldots, n$, and $\mu(t)$ be a nondecreasing function with $\mu(t) \to +\infty$ $(t \to +\infty)$. Then $x^\star$ is said to be μ-stable in $\prod_{i=1}^{n} w_i$ (locally μ-stable in Φ_1) if there is a positive constant M such that

$$\left|x_i(t) - x_i^\star\right| \le \frac{M}{\mu(t)}.$$

Theorem 2 *For any $\prod_{i=1}^{n} w_i \in \Phi_1$, given that*

$$\begin{cases} -b_i(p_i) + c_{ii}g_i(p_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \max\{c_{ij}m_j, c_{ij}M_j\} + \sum_{j=1}^{n} \max\{d_{ij}m_j, d_{ij}M_j\} - I_i < 0, \\ \quad i \in N_1, \\ -b_i(q_i) + c_{ii}g_i(q_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \min\{c_{ij}m_j, c_{ij}M_j\} + \sum_{j=1}^{n} \min\{d_{ij}m_j, d_{ij}M_j\} - I_i > 0, \\ \quad i \in N_3, \end{cases} \tag{4}$$

and the nondecreasing function $\mu(t) > 0$ with

$$\lim_{t\to+\infty} \mu(t) = +\infty, \qquad 0 \le \sup_{t\ge T^*} \frac{\dot{\mu}(t)}{\mu(t)} \le \alpha, \qquad \sup_{t\ge T^*} \frac{\mu(t)}{\mu(t-\tau(t))} \le 1 + \beta, \tag{5}$$

where α, β, T^* are nonnegative constants, $N_1 \cap N_3 = \phi$, $N_1 \cup N_3 = \{1, 2, \dots, n\}$. Then the equilibrium point $x^\star = (x_1^\star, \dots, x_n^\star)$ is μ-stable in $\prod_{i=1}^n w_i$ (locally μ-stable in Φ_1) if there are positive constants $\zeta_1, \zeta_2, \dots, \zeta_n$ such that

$$\begin{aligned}&(-\underline{a}_i\beta_i + \alpha)\zeta_i + \sum_{j\in N_1} \zeta_j \bar{a}_i \bar{\sigma}_j^l |c_{ij}| + \sum_{j\in N_3} \zeta_j \bar{a}_i \bar{\sigma}_j^r |c_{ij}| \\ &\quad + (1+\beta)\left(\sum_{j\in N_1} \zeta_j \bar{a}_i \bar{\delta}_j^l |d_{ij}| + \sum_{j\in N_3} \zeta_j \bar{a}_i \bar{\delta}_j^r |d_{ij}|\right) < 0,\end{aligned} \tag{6}$$

where $i = 1, 2, \dots, n$.

Proof From the comparison of (3) and (4), we know that there is at least one equilibrium point in $\prod_{i=1}^n w_i$ for model (1). The following proof will show that the equilibrium point is unique and μ-stable in $\prod_{i=1}^n w_i \in \Phi_1$.

Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ be an arbitrary solution of model (1) in $\prod_{i=1}^n v_i \in \Omega_1$ with the initial condition $x_i(s) = \varphi_i(s)$, $s \in (-\infty, 0]$. We claim that $x(t)$ would remain in $\prod_{i=1}^n v_i$ for all $t \geq 0$.

For $x_i(t)$, suppose that $i \in N_1'$. Then $i \in N_1$ on account of $[-E_i, p_i] \subset (-\infty, p_i]$. Further, by (4), one has $\varepsilon > 0$ small enough such that

$$-b_i(p_i - \varepsilon) + c_{ii}g_i(p_i - \varepsilon) + \sum_{\substack{j=1\\ j\neq i}}^n \max\{c_{ij}m_j, c_{ij}M_j\} + \sum_{j=1}^n \max\{d_{ij}m_j, d_{ij}M_j\} - I_i < 0. \tag{7}$$

And for ε above, we can find some $t^* \geq 0$ such that $p_i - \varepsilon \leq x_i(t^*) \leq p_i$. It follows that

$$\begin{aligned}\left.\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\right|_{t=t^*} &= -a_i\big(x_i(t^*)\big)\left[b_i\big(x_i(t^*)\big) - \sum_{j=1}^n c_{ij}g_j\big(x_j(t^*)\big) - \sum_{j=1}^n d_{ij}f_j\big(x_j\big(t^* - \tau(t^*)\big)\big) + I_i\right] \\ &\leq a_i\big(x_i(t^*)\big)\Bigg[-b_i(p_i - \varepsilon) \\ &\quad + c_{ii}g_i(p_i - \varepsilon) + \sum_{\substack{j=1\\ j\neq i}}^n \max\{c_{ij}m_j, c_{ij}M_j\} + \sum_{j=1}^n \max\{d_{ij}m_j, d_{ij}M_j\} - I_i\Bigg] \\ &< 0.\end{aligned} \tag{8}$$

On the other hand, we also can find some $t^* \geq 0$ such that $x_i(t^*) = -E_i$. By (H1), we get that

$$\begin{aligned}\left.\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\right|_{t=t^*} &\geq a_i(-E_i)\Bigg[b_i(E_i) + c_{ii}g_i(-E_i) + \sum_{\substack{j=1\\ j\neq i}}^n \min\{c_{ij}m_j, c_{ij}M_j\} \\ &\quad + \sum_{j=1}^n \min\{d_{ij}m_j, d_{ij}M_j\} - I_i\Bigg] \\ &> 0.\end{aligned} \tag{9}$$

Similarly, for $x_i(t), i \in N_3'$, there exists $\varepsilon > 0$ small enough such that

$$-b_i(q_i + \varepsilon) + c_{ii}g_i(q_i + \varepsilon) + \sum_{\substack{j=1\\ j\neq i}}^{n} \min\{c_{ij}m_j, c_{ij}M_j\} + \sum_{j=1}^{n} \min\{d_{ij}m_j, d_{ij}M_j\} - I_i > 0.$$

And for ε above, we can discover some $t^* \geq 0$ such that $q_i \leq x_i(t^*) \leq q_i + \varepsilon$. It follows that

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\bigg|_{t=t^*} &\geq a_i\bigl(x_i(t^*)\bigr)\Biggl[-b_i(q_i + \varepsilon) + c_{ii}g_i(q_i + \varepsilon) + \sum_{\substack{j=1\\ j\neq i}}^{n} \min\{c_{ij}m_j, c_{ij}M_j\} \\ &\quad + \sum_{j=1}^{n} \min\{d_{ij}m_j, d_{ij}M_j\} - I_i\Biggr] > 0. \end{aligned} \tag{10}$$

We also can find some $t^* \geq 0$ so that $x_i(t^*) = E_i$ and obtain the following inequality:

$$\begin{aligned} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\bigg|_{t=t^*} &\leq a_i(E_i)\Biggl[-b_i(E_i) + c_{ii}g_i(-E_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \max\{c_{ij}m_j, c_{ij}M_j\} \\ &\quad + \sum_{j=1}^{n} \max\{d_{ij}m_j, d_{ij}M_j\} - I_i\Biggr] \\ &< 0. \end{aligned} \tag{11}$$

From (8)–(11), we see that $x_i(t)$ would not escape from $[-E_i, p_i]$ when $i \in N_1'$, and $x_i(t)$ would remain in $[q_i, E_i]$ when $i \in N_3'$. Let i go through $1, 2, \ldots, n$, we get $x(t)$ is located in Ω_1. In view of $\Omega_1 \subseteq \Phi_1$, it implies that $\prod_{i=1}^{n} w_i \in \Phi_1$ is an invariant set of model (1) with the initial condition $x_i(s) = \varphi_i(s)$, $s \in (-\infty, 0]$.

Denote $u_i(t) = x_i(t) - x_i^\star$, $i = 1, 2, \ldots, n$, then we have

$$\begin{aligned} \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} &= -a_i\bigl(x_i(t)\bigr)\Biggl[\bigl(b_i\bigl(x_i(t)\bigr) - b_i\bigl(x_i^\star\bigr)\bigr) \\ &\quad - \sum_{j=1}^{n} c_{ij}\bigl(g_j\bigl(x_j(t)\bigr) - g_j\bigl(x_j^\star\bigr)\bigr) - \sum_{j=1}^{n} d_{ij}\bigl(f_j\bigl(x_j\bigl(t - \tau(t)\bigr)\bigr) - f_j\bigl(x_j^\star\bigr)\bigr)\Biggr]. \end{aligned} \tag{12}$$

Let $U_i(t) = \mu(t)u_i(t)$ and $U(t) = \sup_{s\leq t}(\max_{i=1,2,\ldots,n}(\zeta_i^{-1}|U_i(s)|))$, $t \geq T \geq T^*$. Then, for any t^*, $t^* \geq T \geq T^*$, we have $\max_{i=1,2,\ldots,n}(\zeta_i^{-1}|U_i(t^*)|) \leq U(t^*)$, which implies that $U(t)$ is bounded.

Let $i_{t^*} = i_{t^*}(t^*)$ when $\max_{i=1,2,\ldots,n}(\zeta_i^{-1}|U_i(t^*)|) = U(t^*)$ holds. Differentiating $|U_{i_{t^*}}(t)|$ at time t^*, we can deduce that

$$\begin{aligned} \frac{\mathrm{d}|U_{i_{t^*}}(t)|}{\mathrm{d}t}\bigg|_{t=t^*} &= \operatorname{sign}\bigl(U_{i_{t^*}}(t^*)\bigr)\dot{\mu}(t^*)u_{i_{t^*}}(t^*) + \operatorname{sign}\bigl(U_{i_{t^*}}(t^*)\bigr)\mu(t^*) \\ &\quad \cdot \Biggl\{-a_{i_{t^*}}\bigl(x_{i_{t^*}}(t^*)\bigr)\Biggl[\bigl(b_{i_{t^*}}\bigl(x_{i_{t^*}}(t^*)\bigr) - b_{i_{t^*}}\bigl(x_{i_{t^*}}^\star\bigr)\bigr) \end{aligned}$$

$$\begin{aligned}
&-\sum_{j=1}^{n} c_{i_{t^*}j}\big(g_j\big(x_j(t^*)\big)-g_j\big(x_j^\star\big)\big)\\
&-\sum_{j=1}^{n} d_{i_{t^*}j}\big(f_j\big(x_j\big(t^*-\tau(t^*)\big)\big)-f_j\big(x_j^\star\big)\big)\Big]\Big\}.
\end{aligned}\tag{13}$$

By hypothesis (H2), there exists a positive constant β_i, so that

$$b_i(x_i)-b_i(y_i)=\beta_i(x_i-y_i),\tag{14}$$

where $\beta_i \geq b_i$, $i=1,2,\ldots,n$. Substituting (5), (6), (14) into (13) and applying (H1) and (H3), we can obtain the following inequality.

$$\begin{aligned}
\frac{\mathrm{d}|U_{i_{t^*}}(t)|}{\mathrm{d}t}\bigg|_{t=t^*} &= \operatorname{sign}\big(U_{i_{t^*}}(t^*)\big)\dot{\mu}(t^*)u_{i_{t^*}}(t^*)+\operatorname{sign}\big(U_{i_{t^*}}(t^*)\big)\mu(t^*)\\
&\quad\cdot\Big\{-a_{i_{t^*}}\big(x_{i_{t^*}}(t^*)\big)\Big[\big(b_{i_{t^*}}\big(x_{i_{t^*}}(t^*)\big)-b_{i_{t^*}}\big(x_{i_{t^*}}^\star\big)\big)\\
&\quad-\sum_{j=1}^{n} c_{i_{t^*}j}\big(g_j\big(x_j(t^*)\big)-g_j\big(x_j^\star\big)\big)\\
&\quad-\sum_{j=1}^{n} d_{i_{t^*}j}\big(f_j\big(x_j\big(t^*-\tau(t^*)\big)\big)-f_j\big(x_j^\star\big)\big)\Big]\Big\}\\
&\leq\left(-\underline{a}_{i_{t^*}}\beta_{i_{t^*}}+\frac{\dot{\mu}(t^*)}{\mu(t^*)}\right)\big|U_{i_{t^*}}(t^*)\big|+\sum_{j\in N_1}\bar{a}_{i_{t^*}}|c_{i_{t^*}j}|\bar{\sigma}_j^{l}\big|U_j(t^*)\big|\\
&\quad+\sum_{j\in N_3}\bar{a}_{i_{t^*}}|c_{i_{t^*}j}|\bar{\sigma}_j^{r}\big|U_j(t^*)\big|\\
&\quad+\sum_{j\in N_1}\bar{a}_{i_{t^*}}\frac{\mu(t^*)}{\mu(t^*-\tau(t^*))}|d_{i_{t^*}j}|\bar{\delta}_j^{l}\big|U_j\big(t^*-\tau(t^*)\big)\big|\\
&\quad+\sum_{j\in N_3}\bar{a}_{i_{t^*}}\frac{\mu(t^*)}{\mu(t^*-\tau(t^*))}|d_{i_{t^*}j}|\bar{\delta}_j^{r}\big|U_j\big(t^*-\tau(t^*)\big)\big|\\
&\leq\left\{\left(-\underline{a}_{i_{t^*}}\beta_{i_{t^*}}+\frac{\dot{\mu}(t^*)}{\mu(t^*)}\right)\zeta_{i_{t^*}}+\sum_{j\in N_1}\bar{a}_{i_{t^*}}|c_{i_{t^*}j}|\bar{\sigma}_j^{l}\zeta_j+\sum_{j\in N_3}\bar{a}_{i_{t^*}}|c_{i_{t^*}j}|\bar{\sigma}_j^{r}\zeta_j\right.\\
&\quad\left.+\frac{\mu(t^*)}{\mu(t^*-\tau(t^*))}\left(\sum_{j\in N_1}\bar{a}_{i_{t^*}}|d_{i_{t^*}j}|\bar{\delta}_j^{l}\zeta_j+\sum_{j\in N_3}\bar{a}_{i_{t^*}}|d_{i_{t^*}j}|\bar{\delta}_j^{r}\zeta_j\right)\right\}\cdot U(t^*)\\
&<0.
\end{aligned}\tag{15}$$

From (15), we can see that there exists a positive constant δ_1 such that $U(t)=U(t^*)$ for $t\in(t^*,t^*+\delta_1)$. Because of the arbitrariness of t^*, we can get that $U(t)=U(T)$ for all $t\geq T\geq T^*$, which implies $\zeta_i^{-1}|\mu(t)(x_i(t)-x_i^\star)|<\max_{i=1,2,\ldots,n}(\zeta_i^{-1}|U_i(t)|)<U(t)$. Therefore, by Definition 1, the equilibrium $x^\star$ is μ-stable in $\prod_{i=1}^{n} w_i\in\Phi_1$. □

Next, we show that there exists an unstable equilibrium point in Φ_2.

Theorem 3 *For any $\prod_{i=1}^{n} w_i \in \Phi_2$, given that* (3) *holds. If there are positive constants $\xi_1, \dots, \xi_n$ such that*

$$\min_{i\in N_2}\Big\{\big(-\beta_i + c_{ii}\sigma_i^{m^*}\big)\xi_i - \sum_{j\in N_1}\xi_j|c_{ij}|\bar{\sigma}_j^{l} - \sum_{j\in N_2}\xi_j|c_{ij}|\bar{\sigma}_j^{m} - \sum_{j\in N_3}\xi_j|c_{ij}|\bar{\sigma}_j^{r} - \sum_{j\in N_1}\xi_j|d_{ij}|\bar{\delta}_j^{l} - \sum_{j\in N_2}\xi_j|d_{ij}|\bar{\delta}_j^{m} - \sum_{j\in N_3}\xi_j|d_{ij}|\bar{\delta}_j^{r}\Big\} > \max\{\lambda, 0\}, \tag{16}$$

where

$$\lambda \triangleq \max_{i\in N_1\cup N_3}\Big\{\Big(-\beta_i\xi_i + \sum_{j\in N_1}\xi_j|c_{ij}|\bar{\sigma}_j^{l} + \sum_{j\in N_2}\xi_j|c_{ij}|\bar{\sigma}_j^{m} + \sum_{j\in N_3}\xi_j|c_{ij}|\bar{\sigma}_j^{r} + \sum_{j\in N_1}\xi_j|d_{ij}|\bar{\delta}_j^{l} + \sum_{j\in N_2}\xi_j|d_{ij}|\bar{\delta}_j^{m} + \sum_{j\in N_3}\xi_j|d_{ij}|\bar{\delta}_j^{r}\Big\}, \tag{17}$$

then the equilibrium point $x^\star$ of model (1) *in $\prod_{i=1}^{n} w_i \in \Phi_2$ is unstable.*

Proof Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ be an arbitrary solution of model (1) with the initial condition $x(s) = \varphi(s) \in \Phi_2$, $s \in (-\infty, 0]$, and let $u_i(t) = x_i(t) - x_i^\star$, $i = 1, 2, \dots, n$, $t \geq 0$, where $x^\star = (x_1^\star, \dots, x_n^\star)$ is one equilibrium point of model (1) in $\prod_{i=1}^{n} w_i \in \Phi_2$. Without loss of generality, suppose that $x(t)$ remains in $\prod_{i=1}^{n} w_i \in \Phi_2$. And we define $H(t) = \sup_{t-\tau(t)\leq s\leq t}\{\max_{i=1,2,\dots,n}\xi_i^{-1}|u_i(s)|\}$, $t \geq 0$. If $\max_{i\in N_2}\xi_i^{-1}|u_i(t)| = H(t)$ holds, denote a sequence item i_t of N_2 such that $\xi_{i_t}^{-1}|u_{i_t}(t)| = \max_{i\in N_2}\xi_i^{-1}|u_i(t)|$, and differentiate $|u_{i_t}(t)|$ at time t, then we can deduce

$$\frac{\mathrm{d}|u_{i_t}(t)|}{\mathrm{d}t}\bigg|_{(1)} = \operatorname{sign}\big(x_{i_t}(t) - x_{i_t}^\star\big)\cdot\Big\{-a_{i_t}\big(x_{i_t}(t)\big)\Big[\big(b_{i_t}\big(x_{i_t}(t)\big) - b_{i_t}\big(x_{i_t}^\star\big)\big) - \sum_{j=1}^{n}c_{i_tj}\big(g_j\big(x_j(t)\big) - g_j\big(x_j^\star\big)\big) - \sum_{j=1}^{n}d_{i_tj}\big(f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) - f_j\big(x_j^\star\big)\big)\Big]\Big\}.$$

On account of hypothesis (H3), that is, $0 \leq \underline{\sigma}_i^m \leq \frac{g_i(u)-g_i(v)}{u-v} \leq \bar{\sigma}_i^m$, $\forall u, v \in [p_i, q_i]$, we can find a positive constant $\sigma_i^{m^*}$ such that

$$\frac{g_i(u) - g_i(v)}{u - v} = \sigma_i^{m^*}. \tag{18}$$

Therefore by (14), (18), hypothesis (H1) and (H3), Eq. (17) can be further converted to

$$\begin{aligned}\frac{\mathrm{d}|u_{i_t}(t)|}{\mathrm{d}t}\bigg|_{(1)} = \Big\{&a_{i_t}\big(x_{i_t}(t)\big)\Big[-\operatorname{sign}\big(u_{i_t}(t)\big)\cdot\beta_{i_t}\cdot u_{i_t}(t) + \operatorname{sign}\big(u_{i_t}(t)\big)\cdot c_{i_ti_t}\big(\sigma_i^{m^*}\cdot u_{i_t}(t)\big)\\&+ \operatorname{sign}\big(u_{i_t}(t)\big)\cdot\sum_{\substack{j=1\\ j\neq i_t}}^{n}c_{i_tj}\big(g_j\big(u_j(t) + x_j^\star\big) - g_j\big(x_j^\star\big)\big)\\&+ \operatorname{sign}\big(u_{i_t}(t)\big)\cdot\sum_{j=1}^{n}d_{i_tj}\big(f_j\big(u_j\big(t-\tau_{ij}(t)\big) + x_j^\star\big) - f_j\big(x_j^\star\big)\big)\Big]\Big\}\end{aligned}$$

$$
\begin{aligned}
&\geq \Big\{a_{i_t}\big(x_{i_t}(t)\big)\Big[\big(-\beta_{i_t}+c_{i_t i_t}\sigma_i^{m^*}\big)\big|u_{i_t}(t)\big|-\sum_{j\in N_1}|c_{i_t j}|\bar{\sigma}_j^l\big|u_j(t)\big| \\
&\quad -\sum_{\substack{j\in N_2\\ j\neq i_t}}|c_{i_t j}|\bar{\sigma}_j^m\big|u_j(t)\big|-\sum_{j\in N_3}|c_{i_t j}|\bar{\sigma}_j^r\big|u_j(t)\big|-\sum_{j\in N_1}|d_{i_t j}|\bar{\delta}_j^l\big|u_j\big(t-\tau_{ij}(t)\big)\big| \\
&\quad -\sum_{j\in N_2}|d_{i_t j}|\bar{\delta}_j^m\big|u_j\big(t-\tau_{ij}(t)\big)\big|-\sum_{j\in N_3}|d_{i_t j}|\bar{\delta}_j^r\big|u_j\big(t-\tau_{ij}(t)\big)\big|\Big]\Big\} \\
&\geq a_{i_t}\big(x_{i_t}(t)\big)\Big\{\big(-\beta_{i_t}+c_{i_t i_t}\sigma_i^{m^*}\big)\xi_{i_t}-\Big[\sum_{\substack{j\in N_1\\ j\neq i_t}}\xi_j|c_{i_t j}|\bar{\sigma}_j^m+\sum_{j\in N_2}\xi_j|d_{i_t j}|\bar{\delta}_j^m\Big] \\
&\quad -\Big[\sum_{j\in N_1}\xi_j|c_{i_t j}|\bar{\sigma}_j^l+\sum_{j\in N_3}\xi_j|c_{i_t j}|\bar{\sigma}_j^r+\sum_{j\in N_1}\xi_j|d_{i_t j}|\bar{\delta}_j^l+\sum_{j\in N_3}\xi_j|d_{i_t j}|\bar{\delta}_j^r\Big]\Big\}H(t) \\
&> a_{i_t}\big(x_{i_t}(t)\big)\max\{\lambda,0\}H(t).
\end{aligned}
\tag{19}
$$

Inequality (19) implies that there exists a number $r>0$ such that $|u_{i_t}(s)|>|u_{i_t}(t)|$, $s\in(t,t+r)$.

Besides, suppose that there exists some time point t' such that

$$
\begin{aligned}
&\sup_{t'-\tau(t')\leq s\leq t'}\Big\{\max_{i\in N_1\cup N_3}\xi_i^{-1}\big|u_i(s)\big|\Big\}=\sup_{t'-\tau(t')\leq s\leq t'}\Big\{\max_{i\in N_2}\xi_i^{-1}\big|u_i(s)\big|\Big\}, \\
&\sup_{t'-\tau(t')\leq s\leq t'}\Big\{\max_{i\in N_1\cup N_3}\xi_i^{-1}\big|u_i(s)\big|\Big\}=\max_{i\in N_1\cup N_3}\xi_i^{-1}\big|u_i(t')\big|,
\end{aligned}
$$

and denote a sequence item $i'\in N_1\cup N_3$ such that $\xi_{i'}^{-1}|u_{i'}(t')|=\max_{i\in N_1\cup N_3}\xi_i^{-1}|u_i(t')|$. Hence we can get

$$
\begin{aligned}
\frac{\mathrm{d}|u_{i'}(t)|}{\mathrm{d}t}\bigg|_{t=t'}&=\Big\{a_{i'}\big(x_{i'}(t')\big)\Big[-\operatorname{sign}\big(u_{i'}(t)\big)\cdot\beta_{i'}\cdot u_{i'}(t') \\
&\quad+\operatorname{sign}\big(u_{i'}(t')\big)\cdot\sum_{j=1}^{n}c_{i'j}\big(g_j\big(u_j(t')+x_j^r\big)-g_j\big(x_j^r\big)\big) \\
&\quad+\operatorname{sign}\big(u_{i'}(t')\big)\cdot\sum_{j=1}^{n}d_{i'j}\big(f_j\big(u_j\big(t-\tau_{i'j}(t)\big)+x_j^r\big)-f_j\big(x_j^r\big)\big)\Big]\Big\} \\
&\leq\Big\{a_{i'}\big(x_{i'}(t')\big)\Big[-\beta_{i'}\big|u_{i'}(t')\big|+\sum_{j\in N_1}|c_{i'j}|\bar{\sigma}_j^l\big|u_j(t')\big|+\sum_{j\in N_2}|c_{i'j}|\bar{\sigma}_j^m\big|u_j(t')\big| \\
&\quad+\sum_{j\in N_3}|c_{i'j}|\bar{\sigma}_j^r\big|u_j(t')\big|+\sum_{j\in N_1}|d_{i'j}|\bar{\delta}_j^l\big|u_j\big(t'-\tau_{i'j}(t')\big)\big| \\
&\quad+\sum_{j\in N_2}|d_{i'j}|\bar{\delta}_j^m\big|u_j\big(t'-\tau_{i'j}(t')\big)\big|+\sum_{j\in N_3}|d_{i'j}|\bar{\delta}_j^r\big|u_j\big(t'-\tau_{ij}(t')\big)\big|\Big]\Big\} \\
&\leq\Big\{a_{i'}\big(x_{i'}(t')\big)\Big[(-\beta_{i'}\xi_{i'}+\sum_{j\in N_2}\xi_j|c_{i'j}|\bar{\sigma}_j^m+\sum_{j\in N_2}\xi_j|d_{i'j}|\bar{\delta}_j^m+\sum_{j\in N_1}\xi_j|c_{i'j}|\bar{\sigma}_j^l \\
&\quad+\sum_{j\in N_3}\xi_j|c_{i'j}|\bar{\sigma}_j^r+\sum_{j\in N_1}\xi_j|d_{i'j}|\bar{\delta}_j^l+\sum_{j\in N_3}\xi_j|d_{i'j}|\bar{\delta}_j^r\Big]\Big\}H(t') \\
&\leq a_{i'}\big(x_{i'}(t')\big)\lambda H(t').
\end{aligned}
\tag{20}
$$

In the meantime, we can obtain the derivative of $|u_{i'_{t'}}(t)|$ at time t' by (19) when $i' \in N_2$

$$\frac{\mathrm{d}|u_{i'_{t'}}(t)|}{\mathrm{d}t}\bigg|_{t=t'} > a_{i'_{t'}}\bigl(x_{i'_{t'}}\bigl(t'\bigr)\bigr)\max\{\lambda,0\}H\bigl(t'\bigr), \tag{21}$$

where $\xi_{i'_{t'}}^{-1}|u_{i'_{t'}}(t')| = \max_{i'\in N_2}\xi_{i'}^{-1}|u_{i'}(t')|$.

Given the above, we can conclude that

$$H(t) = \sup_{t-\tau(t)\le s\le t}\Bigl\{\max_{i\in N_2}\xi_i^{-1}\bigl|u_i(s)\bigr|\Bigr\} \ge \sup_{-\tau(t)\le s\le 0}\Bigl\{\max_{i\in N_2}\xi_i^{-1}\bigl|u_i(s)\bigr|\Bigr\}$$

holds for all $t \ge 0$, and there exists an increasing time sequence $\{t_l\}_{l=1}^{\infty}$ with $\lim_{l\to\infty} t_l = +\infty$ such that $\sup_{t_l-\tau(t_l)\le s\le t_l}\{\max_{i\in N_2}\xi_i^{-1}|u_i(s)|\} = \max_{i\in N_2}\xi_i^{-1}|u_i(t_l)|$. Correspondingly, there exists an increasing time subsequence $\{t_{l_k}\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} t_{l_k} = +\infty$ such that $\xi_{i''}^{-1}|u_{i''}(t_{l_k})| = \sup_{t_{l_k}-\tau(t_{l_k})\le s\le t_{l_k}}\{\max_{i\in N_2}\xi_i^{-1}|u_i(s)|\} \ge \sup_{-\tau(t)\le s\le 0}\{\max_{i\in N_2}\xi_i^{-1}|u_i(s)|\} > 0$, $k = 1,2,\ldots$. Thus $u_{i''}(t)$ would not converge to 0 when $t \to +\infty$ means that the equilibrium point $x^\star$ is unstable in $\prod_{i=1}^{n} w_i \in \Phi_2$. □

4 Corollaries and comparison

According to the above theorems, we have the following two corollaries.

Corollary 1 *If conditions (3)–(6) and (16) hold, then model (1) has at least 3^n equilibrium points in R^n, 2^n of which are locally μ-stable in Φ_1, the remaining $3^n - 2^n$ points are unstable in Φ_2.*

Corollary 2 *When $a_i(x_i(t)) = 1$, model (1) becomes the Hopfield neural network model*

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -b_i\bigl(x_i(t)\bigr) + \sum_{j=1}^{n} c_{ij}g_j\bigl(x_j(t)\bigr) + \sum_{j=1}^{n} d_{ij}f_j\bigl(x_j\bigl(t-\tau(t)\bigr)\bigr) - I_i, \quad t \ge 0. \tag{22}$$

On the basis of conditions (3)–(6) and (16), model (22) has at least 3^n equilibrium points in R^n, 2^n of which are locally μ-stable in Φ_1, the remaining $3^n - 2^n$ points are unstable equilibrium points in Φ_2.

Remark 1 The present paper investigates the multistability of a Cohen–Grossberg neural network with unbounded time-varying delay and nondecreasing activation functions. Compared with [41], the obtained results are more general.

Remark 2 The net self-inhibition function $b_i(x_i(t))$ in this paper is an odd function and monotonically increasing, which includes the case of [41]. Thus model (22) is more general than [41].

Remark 3 The activation functions of [41] are identical whether with or without time delay, but the activation functions in this paper are different. Therefore the conclusion of this paper is closer to the practical application.

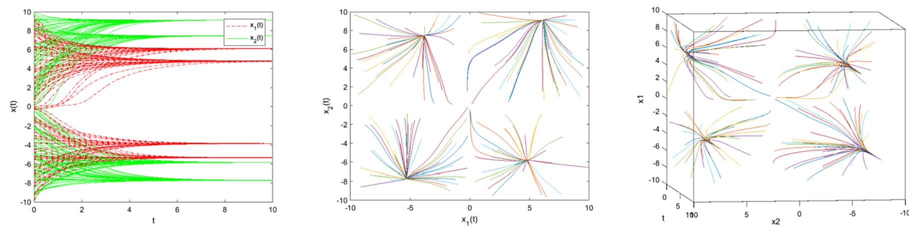

Figure 1 The state trajectories and phase plots of the neural network model (1). The above three graphs are the state trajectories and phrase plots, where the third graph is a 3*D* phase plot of the state variable (t, x_2, x_1)

5 Simulation example

Example We consider the following two-dimensional Cohen–Grossberg neural network model:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i\bigl(x_i(t)\bigr)\Biggl[b_i\bigl(x_i(t)\bigr) - \sum_{j=1}^{n} c_{ij}g_j\bigl(x_j(t)\bigr) - \sum_{j=1}^{n} d_{ij}f_j\bigl(x_j\bigl(t-\tau_{ij}(t)\bigr)\bigr) + I_i(t)\Biggr], \quad t \geq 0,$$

where $i = 1, 2$, $a(x) = 1 + 0.2\sin(x)$, $b_1(x_1(t)) = x_1(t)$, $b_2(x_2(t)) = -1.2x_2(t)$,

$$g(x) = \begin{cases} \tanh(0.2x) - \tanh(1) + \tanh(0.2), & x < -1, \\ \tanh(x), & -1 \leq x \leq 1, \\ \tanh(0.2x) + \tanh(1) - \tanh(0.2), & x > 1, \end{cases} \qquad f(x) = \frac{|x+1| - |x-1|}{2},$$

$$C = (c_{ij}) = \begin{pmatrix} 3.5 & 0.2 \\ 0.4 & 4.8 \end{pmatrix}, \qquad D = (d_{ij}) = \begin{pmatrix} 0.4 & 0.2 \\ 0.2 & 0.5 \end{pmatrix}, \qquad I = \begin{pmatrix} -0.3 \\ -0.6 \end{pmatrix},$$

$$\tau_{ij}(t) = \tau(t) = 0.2t.$$

After a simple calculation, we know that the above hypothesis satisfies the conditions of Theorems 1–3. Therefore there are at least nine equilibrium points in model (1) from Corollary 1, 4 of which are μ-stable equilibrium points and others are unstable points. The solution of model (1) is traced with 150 initial conditions, the simulation results are showed in the above three graphs of Fig. 1.

6 Conclusion

Stability of multiple unstable Cohen–Grossberg neural networks with unbounded time-varying delays is discussed analytically in this paper. Based on the geometric structure of two different activation functions and some rigorous mathematical analysis, the present paper proved that there exist multiple equilibrium points in the model, some of which are unstable, others are μ-stable. One numerical example and its simulation show the effectiveness of the conclusion. Here, we also need point out the following. On the one hand, the impulsive control is rarely used to deal with cases of unbounded time-varying delays, especially for multiple unstable Cohen–Grossberg neural networks with unbounded time-varying delays. Therefore, the stability under impulsive control of multiple unstable Cohen–Grossberg neural networks with unbounded time-varying delays is still a challenging problem. On the other hand, we use something like positivity-based method to study the stability of Cohen–Grossberg neural networks in this article. The positivity-based method is a valid approach for difference and delay differential systems (see [24–28, 44–48]). Therefore, the research on stability with positivity-based approach is an interesting and meaningful topic, and we will also consider the stability of other neural networks by employing the positivity-based approach in the near future.

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Authors' contributions
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