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Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean

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Abstract

In the article, we prove that $\lambda_1 = 1/2 + \sqrt{[(\sqrt{2} + \log(1+\sqrt{2}))/2]^{1/\nu} - 1}/2$, $\mu_1 = 1/2 + \sqrt{6\nu}/(12\nu)$, $\lambda_2 = 1/2 + \sqrt{[(\pi+2)/4]^{1/\nu} - 1}/2$ and $\mu_2 = 1/2 + \sqrt{3\nu}/(6\nu)$ are the best possible parameters on the interval $[1/2, 1]$ such that the double inequalities

$$\begin{aligned} &C^{\nu}\big[\lambda_1 x + (1-\lambda_1)y, \lambda_1 y + (1-\lambda_1)x\big]A^{1-\nu}(x,y) \\ &\quad < \mathcal{R}_{QA}(x,y) < C^{\nu}\big[\mu_1 x + (1-\mu_1)y, \mu_1 y + (1-\mu_1)x\big]A^{1-\nu}(x,y), \\ &C^{\nu}\big[\lambda_2 x + (1-\lambda_2)y, \lambda_2 y + (1-\lambda_2)x\big]A^{1-\nu}(x,y) \\ &\quad < \mathcal{R}_{AQ}(x,y) < C^{\nu}\big[\mu_2 x + (1-\mu_2)y, \mu_2 y + (1-\mu_2)x\big]A^{1-\nu}(x,y) \end{aligned}$$

hold for all $x, y > 0$ with $x \neq y$ and $\nu \in [1/2, \infty)$, where $A(x,y)$ is the arithmetic mean, $C(x,y)$ is the contraharmonic mean, and $\mathcal{R}_{QA}(x,y)$ and $\mathcal{R}_{AQ}(x,y)$ are two Neuman means.

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1 Introduction

Let $x, y > 0$. Then the arithmetic mean $A(x,y)$, quadratic mean $Q(x,y)$ [1], contraharmonic mean $C(x,y)$ [2, 3], and Schwab–Borchardt mean $\mathrm{SB}(x,y)$ [4] are given by

$$A(x,y) = \frac{x+y}{2}, \qquad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}, \qquad C(x,y) = \frac{x^2+y^2}{x+y},$$
$$\mathrm{SB}(x,y) = \begin{cases} \frac{\sqrt{y^2-x^2}}{\arccos(x/y)}, & x < y, \\ x, & x = y, \\ \frac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)}, & x > y, \end{cases} \tag{1.1}$$

respectively, where $\cosh^{-1}(\sigma) = \log(\sigma + \sqrt{\sigma^2 - 1})$ is the inverse hyperbolic cosine function.

The Gaussian arithmetic–geometric mean $\mathrm{AGM}(x,y)$ [5–7] of two positive real numbers x and y is defined by the common limit of the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, which are given by

$$x_0 = x, \qquad y_0 = y, \qquad x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_n y_n}.$$

It is well known that the bivariate means have wide applications in mathematics, physics, engineering, and other natural sciences [8–55], many special functions can be expressed using bivariate means, for example, the complete elliptic integral

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2(t)}} \quad (0 < r < 1)$$

of the first kind [56–61] and the modulus $\mu(r)$ of the plane Grötzsch ring [62, 63] can be expressed by the Gaussian arithmetic–geometric mean $\mathrm{AGM}(x,y)$, the formula of the perimeter of an ellipse and the complete elliptic integral

$$\mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2(t)}\, dt$$

of the second kind [64–70] can be given in terms of the Toader mean [71–74]

$$T(a,b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2(t) + b^2 \sin^2(t)}\, dt.$$

Indeed, we have

$$\mathcal{K}(r) = \frac{\pi}{2} \frac{1}{\mathrm{AGM}(1, \sqrt{1-r^2})}, \qquad \mu(r) = \frac{\pi}{2} \frac{\mathrm{AGM}(1, \sqrt{1-r^2})}{\mathrm{AGM}(1,r)},$$
$$L(x,y) = 2\pi T(x,y), \qquad \mathcal{E}(r) = \frac{\pi}{2} T\bigl(1, \sqrt{1-r^2}\bigr).$$

Recently, the inequalities for bivariate means have attracted the attention of many mathematicians. Neuman [75] introduced the Neuman means

$$\mathcal{R}_{QA}(x,y) = \frac{1}{2}\left[Q(x,y) + \frac{A^2(x,y)}{\mathrm{SB}(Q(x,y), A(x,y))}\right],$$
$$\mathcal{R}_{AQ}(x,y) = \frac{1}{2}\left[A(x,y) + \frac{Q^2(x,y)}{\mathrm{SB}(A(x,y), Q(x,y))}\right]$$

and provided the formulas

$$\mathcal{R}_{QA}(x,y) = \frac{1}{2} A(x,y)\left[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\right], \tag{1.2}$$

$$\mathcal{R}_{AQ}(x,y) = \frac{1}{2} A(x,y)\left[1 + \frac{(1+u^2)\arctan(u)}{u}\right] \tag{1.3}$$

if $x > y > 0$, where $u = (x-y)/(x+y)$ and $\sinh^{-1}(\sigma) = \log(\sigma + \sqrt{\sigma^2+1})$ is the inverse hyperbolic sine function. Neuman [4] proved that the inequalities

$$A(x,y) < \mathcal{R}_{QA}(x,y) < \mathcal{R}_{AQ}(x,y) < Q(x,y) \tag{1.4}$$

hold for $x, y > 0$ with $x \neq y$.

Zhang et al. [76] proved that $\alpha_1 = 1/2 + \sqrt{2\sqrt{2}\log(1+\sqrt{2}) + \log^2(1+\sqrt{2}) - 2}/4 = 0.7817\ldots$, $\beta_1 = 1/2 + \sqrt{3}/6 = 0.7886\ldots$, $\alpha_2 = 1/2 + \sqrt{\pi^2 + 4\pi - 12}/8 = 0.9038\ldots$ and $\beta_2 = 1/2 + \sqrt{6}/6 = 0.9082\ldots$ are the best possible parameters on the interval $[1/2, 1]$ such that the double inequalities

$$\begin{aligned} &Q\bigl[\alpha_1 x + (1-\alpha_1)y, \alpha_1 y + (1-\alpha_1)x\bigr] \\ &\quad < \mathcal{R}_{QA}(x,y) < Q\bigl[\beta_1 x + (1-\beta_1)y, \beta_1 y + (1-\beta_1)x\bigr], \end{aligned} \tag{1.5}$$

$$\begin{aligned} &Q\bigl[\alpha_2 x + (1-\alpha_2)y, \alpha_2 y + (1-\alpha_2)x\bigr] \\ &\quad < \mathcal{R}_{AQ}(x,y) < Q\bigl[\beta_2 x + (1-\beta_2)y, \beta_2 y + (1-\beta_2)x\bigr] \end{aligned} \tag{1.6}$$

hold for $x, y > 0$ with $x \neq y$.

In [77], Yang et al. proved that the double inequalities

$$\begin{aligned} &\alpha\left[\frac{C(x,y)}{3} + \frac{2A(x,y)}{3}\right] + (1-\alpha)C^{1/3}(x,y)A^{2/3}(x,y) \\ &\quad < \mathcal{R}_{AQ}(x,y) < \beta\left[\frac{C(x,y)}{3} + \frac{2A(x,y)}{3}\right] + (1-\beta)C^{1/3}(x,y)A^{2/3}(x,y), \\ &\lambda\left[\frac{C(x,y)}{6} + \frac{5A(x,y)}{6}\right] + (1-\lambda)C^{1/6}(x,y)A^{5/6}(x,y) \\ &\quad < \mathcal{R}_{QA}(x,y) < \mu\left[\frac{C(x,y)}{6} + \frac{5A(x,y)}{6}\right] + (1-\mu)C^{1/6}(x,y)A^{5/6}(x,y) \end{aligned}$$

hold for for $x, y > 0$ with $x \neq y$ if and only if $\alpha \leq (3\pi + 6 - 12\sqrt[3]{2})/(16 - 12\sqrt[3]{2}) = 0.3470\ldots$, $\beta \geq 2/5$, $\lambda \leq [3\sqrt{2} + 3\log(1+\sqrt{2}) - 6\sqrt[6]{2}]/(7 - 6\sqrt[6]{2}) = 0.5730\ldots$ and $\mu \geq 16/25$.

The main purpose of the article is to generalize inequalities (1.5) and (1.6). To achieve this goal, we define the two-parameter contraharmonic and arithmetic mean $W_{\lambda,\nu}(x,y)$ as follows:

$$W_{\lambda,\nu}(x,y) = C^{\nu}\bigl[\lambda x + (1-\lambda)y, \lambda y + (1-\lambda)x\bigr]A^{1-\nu}(x,y), \tag{1.7}$$

where $\lambda \in [1/2, 1]$ and $\nu \in [1/2, \infty)$. We clearly see that the function $\lambda \to W_{\lambda,\nu}(x,y)$ is strictly increasing on $[1/2, 1]$ for $\nu \in [1/2, \infty)$ and $x, y > 0$ with $x \neq y$.

It follows from (1.1), (1.4) and (1.7) that

$$W_{\lambda,1/2}(x,y) = Q\bigl[\lambda x + (1-\lambda)y, \lambda y + (1-\lambda)x\bigr], \tag{1.8}$$

$$W_{\lambda,1}(x,y) = C\bigl[\lambda x + (1-\lambda)y, \lambda y + (1-\lambda)x\bigr], \tag{1.9}$$

$$W_{1/2,\nu}(x,y) = A(x,y),$$

$$W_{1,\nu}(x,y)=C^{\nu}(x,y)A^{1-\nu}(x,y)=A(x,y)\left[\frac{Q(x,y)}{A(x,y)}\right]^{2\nu}\geq Q(x,y),$$
$$W_{1/2,\nu}(x,y)<\mathcal{R}_{QA}(x,y)<\mathcal{R}_{AQ}(x,y)<W_{1,\nu}(x,y). \tag{1.10}$$

Inequalities (1.5), (1.6), and (1.10) give us the motivation to discuss the question: What are the best possible parameters $\lambda_1=\lambda_1(\nu)$, $\mu_1=\mu_1(\nu)$, $\lambda_2=\lambda_2(\nu)$ and $\mu_2=\mu_2(\nu)$ on the interval $[1/2,1]$ such that the double inequalities

$$W_{\lambda_1,\nu}(x,y)<\mathcal{R}_{QA}(x,y)<W_{\mu_1,\nu}(x,y),$$
$$W_{\lambda_2,\nu}(x,y)<\mathcal{R}_{AQ}(x,y)<W_{\mu_2,\nu}(x,y)$$

hold for all $x,y>0$ with $x\neq y$ and $\nu\in[1/2,\infty)$?

2 Lemmas

In order to prove our main results, we need to introduce and establish five lemmas which we present in this section.

Lemma 2.1 ([78, Theorem 1.25]) *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha<\beta$, $\Gamma,\Psi:[\alpha,\beta]\to\mathbb{R}$ be continuous on $[\alpha,\beta]$ and differentiable on (α,β) with $\Psi'(\tau)\neq 0$ on (α,β). Then the functions*

$$\frac{\Gamma(\tau)-\Gamma(\alpha)}{\Psi(\tau)-\Psi(\alpha)},\qquad\frac{\Gamma(\tau)-\Gamma(\beta)}{\Psi(\tau)-\Psi(\beta)}$$

are (strictly) increasing (decreasing) on (α,β) if $\Gamma'(\tau)/\Psi'(\tau)$ is (strictly) increasing (decreasing) on (α,β).

Lemma 2.2 *The function*

$$\phi(t)=\frac{\sqrt{1+t^2}\sinh^{-1}(t)}{t}$$

is strictly increasing from $(0,1)$ onto $(1,\sqrt{2}\log(1+\sqrt{2}))$.

Proof Differentiating $\phi(t)$ gives

$$\phi'(t)=\frac{\phi_1(t)}{t\sqrt{1+t^2}}, \tag{2.1}$$

where

$$\phi_1(t)=t\sqrt{1+t^2}-\sinh^{-1}(t). \tag{2.2}$$

It follows from (2.2) that

$$\phi_1(0^+)=0, \tag{2.3}$$
$$\phi_1'(t)=\frac{2t^2}{\sqrt{1+t^2}}>0 \tag{2.4}$$

for all $t\in(0,1)$.

Note that

$$\phi(0^+) = 1, \qquad \phi(1^-) = \sqrt{2}\log(1+\sqrt{2}). \tag{2.5}$$

Therefore, Lemma 2.2 follows from (2.1) and (2.3)–(2.5). □

Lemma 2.3 *The function*

$$\varphi(t) = \frac{t^3}{(1+t^2)\arctan(t) - t}$$

is strictly increasing from $(0,1)$ *onto* $(3/2, 2/(\pi-2))$.

Proof Let $\varphi_1(t) = t^3$ and $\varphi_2(t) = (1+t^2)\arctan(t) - t$. Then we clearly see that

$$\varphi_1(0^+) = \varphi_2(0^+), \qquad \varphi(t) = \frac{\varphi_1(t)}{\varphi_2(t)}, \tag{2.6}$$

$$\frac{\varphi_1'(t)}{\varphi_2'(t)} = \frac{3t}{2\arctan(t)}. \tag{2.7}$$

It is not difficult to verify that the function $t \mapsto t/\arctan(t)$ is strictly increasing from $(0,1)$ onto $(1, 4/\pi)$. Then equation (2.7) leads to the conclusion that $\varphi_1'(t)/\varphi_2'(t)$ is strictly increasing on $(0,1)$.

Note that

$$\varphi(0^+) = \frac{3}{2}, \qquad \varphi(1^-) = \frac{2}{\pi-2}. \tag{2.8}$$

Therefore, Lemma 2.3 follows from Lemma 2.1, (2.6), (2.8), and the monotonicity of $\varphi_1'(t)/\varphi_2'(t)$. □

Lemma 2.4 *Let* $\theta \in [0,1]$, $\nu \in [1/2, \infty)$, $t \in (0,1)$ *and*

$$f_{\theta,\nu}(t) = \nu\log(1+\theta t^2) - \log\bigl[t\sqrt{1+t^2} + \sinh^{-1}(t)\bigr] + \log t + \log 2. \tag{2.9}$$

Then we have the following two conclusions:

(1) $f_{\theta,\nu}(t) > 0$ *for all* $t \in (0,1)$ *if and only if* $\theta \geq 1/(6\nu)$;

(2) $f_{\theta,\nu}(t) < 0$ *for all* $t \in (0,1)$ *if and only if* $\theta \leq [(\sqrt{2} + \log(1+\sqrt{2}))/2]^{1/\nu} - 1$.

Proof It follows from (2.9) that

$$f_{\theta,\nu}(0^+) = 0, \tag{2.10}$$

$$f_{\theta,\nu}(1^-) = \nu\log(1+\theta) - \log\bigl[\sqrt{2} + \log(1+\sqrt{2})\bigr] + \log 2, \tag{2.11}$$

$$f_{\theta,\nu}'(t) = \frac{t[(2\nu-1)(t\sqrt{1+t^2} - \sinh^{-1}(t)) + 4\nu\sinh^{-1}(t)]}{(1+\theta t^2)[t\sqrt{1+t^2} + \sinh^{-1}(t)]}\bigl[\theta - f_\nu(t)\bigr], \tag{2.12}$$

where

$$f_\nu(t) = \frac{t\sqrt{1+t^2} - \sinh^{-1}(t)}{(2\nu-1)t^2[t\sqrt{1+t^2} - \sinh^{-1}(t)] + 4\nu t^2\sinh^{-1}(t)}.$$

Let $\psi_1(t) = t\sqrt{1+t^2} - \sinh^{-1}(t)$ and $\psi_2(t) = (2\nu-1)t^2[t\sqrt{1+t^2} - \sinh^{-1}(t)] + 4\nu t^2 \sinh^{-1}(t)$. Then

$$\psi_1(0^+) = \psi_2(0^+) = 0, \qquad f_\nu(t) = \frac{\psi_1(t)}{\psi_2(t)}, \tag{2.13}$$

$$\frac{\psi_1'(t)}{\psi_2'(t)} = \frac{1}{(2\nu+1)\phi(t) + 2(2\nu-1)t^2 + 4\nu - 1}, \tag{2.14}$$

where $\phi(t)$ is defined in Lemma 2.2.

Equation (2.14) and Lemma 2.2 imply that $\psi_1'(t)/\psi_2'(t)$ is strictly decreasing on $(0,1)$. Therefore, the conclusion that $f_\nu(t)$ is strictly decreasing on $(0,1)$ follows from Lemma 2.1 and (2.13), together with the monotonicity of $\psi_1'(t)/\psi_2'(t)$ on the interval $(0,1)$. Moreover, making use of L'Hôpital's rule, we have that

$$f_\nu(0^+) = \frac{1}{6\nu}, \tag{2.15}$$

$$f_\nu(1^-) = \frac{\sqrt{2} - \log(1+\sqrt{2})}{(2\nu-1)\sqrt{2} + (2\nu+1)\log(1+\sqrt{2})} =: \theta_0. \tag{2.16}$$

We divide the proof into three cases.

Case 1. $\theta \geq 1/(6\nu)$. Then (2.12) and (2.15), together with the monotonicity of $f_\nu(t)$ on the interval $(0,1)$, lead to the conclusion that $f_{\theta,\nu}(t)$ is strictly increasing on $(0,1)$. Therefore, $f_{\theta,\nu}(t) > 0$ for all $t \in (0,1)$ follows from (2.10) and the monotonicity of $f_{\theta,\nu}(t)$ on the interval $(0,1)$.

Case 2. $\theta \leq \theta_0$. Then from (2.12) and (2.16), together with the monotonicity of $f_\nu(t)$ on the interval $(0,1)$, we clearly see that $f_{\theta,\nu}(t)$ is strictly decreasing on $(0,1)$. Therefore, $f_{\theta,\nu}(t) < 0$ for all $t \in (0,1)$ follows from (2.10) and the monotonicity of $f_{\theta,\nu}(t)$ on the interval $(0,1)$.

Case 3. $\theta_0 < \theta < 1/(6\nu)$. Then from (2.12), (2.15), (2.16), and the monotonicity of $f_\nu(t)$ on the interval $(0,1)$, we clearly see that there exists $t_0 \in (0,1)$ such that $f_{\theta,\nu}(t)$ is strictly decreasing on $(0,t_0)$ and strictly increasing on $(t_0,1)$.

We divide the proof into two subcases.

Subcase 3.1. $[(\sqrt{2} + \log(1+\sqrt{2}))/2]^{1/\nu} - 1 < \theta < 1/(6\nu)$. Then (2.11) leads to

$$f_{\theta,\nu}(1^-) > 0. \tag{2.17}$$

Therefore, there exists $t^* \in (t_0,1)$ such that $f_{\theta,\nu}(t) < 0$ for $t \in (0,t^*)$ and $f_{\theta,\nu}(t) > 0$ for $t \in (t^*,1)$ follows from (2.10) and (2.17), together with the piecewise monotonicity of $f_{\theta,\nu}(t)$ on the interval $(0,1)$.

Subcase 3.2. $\theta_0 < \theta \leq [(\sqrt{2} + \log(1+\sqrt{2}))/2]^{1/\nu} - 1$. Then (2.11) leads to

$$f_{\theta,\nu}(1^-) \leq 0. \tag{2.18}$$

Therefore, $f_{\theta,\nu}(t) < 0$ for all $t \in (0,1)$ follows from (2.10) and (2.18), together with the piecewise monotonicity of $f_{\theta,\nu}(t)$ on the interval $(0,1)$. □

Lemma 2.5 *Let* $\vartheta \in [0,1]$, $\nu \in [1/2,\infty)$, $t \in (0,1)$ *and*

$$g_{\vartheta,\nu}(t) = \nu\log\bigl(1+\vartheta t^2\bigr) - \log\bigl[t + \bigl(1+t^2\bigr)\arctan(t)\bigr] + \log(t) + \log 2. \tag{2.19}$$

Then the following statements are true:

(1) $g_{\vartheta,\nu}(t) > 0$ *for all* $t \in (0,1)$ *if and only if* $\vartheta \geq 1/(3\nu)$;

(2) $g_{\vartheta,\nu}(t) < 0$ *for all* $t \in (0,1)$ *if and only if* $\vartheta \leq [(\pi+2)/4]^{1/\nu} - 1$.

Proof It follows from (2.19) that

$$g_{\vartheta,\nu}(0^+) = 0, \tag{2.20}$$

$$g_{\vartheta,\nu}(1^-) = \nu \log(1+\vartheta) - \log\left(\frac{\pi+2}{4}\right), \tag{2.21}$$

$$g'_{\vartheta,\nu}(t) = \frac{t[((2\nu-1)t^2+2\nu+1)\arctan(t)+(2\nu-1)t]}{(1+\vartheta t^2)[t+(1+t^2)\arctan(t)]}\left[\vartheta - g_\nu(t)\right], \tag{2.22}$$

where

$$g_\nu(t) = \frac{t-(1-t^2)\arctan(t)}{t^2[((2\nu-1)t^2+2\nu+1)\arctan(t)+(2\nu-1)t]}.$$

Let $\omega_1(t) = [t-(1-t^2)\arctan(t)]/t^2$ and $\omega_2(t) = [(2\nu-1)t^2+2\nu+1]\arctan(t)+(2\nu-1)t$. Then elaborate computations lead to

$$\omega_1(0^+) = \omega_2(0^+) = 0, \qquad g_\nu(t) = \frac{\omega_1(t)}{\omega_2(t)}, \tag{2.23}$$

$$\frac{\omega'_1(t)}{\omega'_2(t)} = \frac{1}{2[(2\nu-1)t^2+\nu]\varphi(t)+(2\nu-1)t^4}, \tag{2.24}$$

where $\varphi(t)$ is defined in Lemma 2.3.

From Lemma 2.3 and (2.24) we know that $\omega'_1(t)/\omega'_2(t)$ is strictly decreasing on $(0,1)$. Therefore, the conclusion that $g_\nu(t)$ is strictly decreasing on $(0,1)$ follows from Lemma 2.1 and (2.23), together with the monotonicity of $\omega'_1(t)/\omega'_2(t)$ on the interval $(0,1)$. Moreover, making use of L'Hôpital's rule, we have that

$$g_\nu(0^+) = \frac{1}{3\nu}, \tag{2.25}$$

$$g_\nu(1^-) = \frac{1}{(\pi+2)\nu-1}. \tag{2.26}$$

We divide the proof into three cases.

Case 1. $\vartheta \geq 1/(3\nu)$. Then (2.22) and (2.25), together with the monotonicity of $g_\nu(t)$ on the interval $(0,1)$, lead to the conclusion that $g_{\vartheta,\nu}(t)$ is strictly increasing on $(0,1)$. Therefore, $g_{\vartheta,\nu}(t) > 0$ for all $t \in (0,1)$ follows from (2.20) and the monotonicity of $g_{\vartheta,\nu}(t)$ on the interval $(0,1)$.

Case 2. $\vartheta \leq 1/[(\pi+2)\nu-1]$. Then from (2.22) and (2.26), together with the monotonicity of $g_\nu(t)$ on the interval $(0,1)$, we clearly see that $g_{\vartheta,\nu}(t)$ is strictly decreasing on $(0,1)$. Therefore, $g_{\vartheta,\nu}(t) < 0$ for all $t \in (0,1)$ follows from (2.20) and the monotonicity of $g_{\vartheta,\nu}(t)$ on the interval $(0,1)$.

Case 3. $1/[(\pi+2)\nu-1] < \vartheta < 1/(6\nu)$. Then it follows from (2.22), (2.25), (2.26), and the monotonicity of $g_\nu(t)$ on the interval $(0,1)$ that there exists $\rho_0 \in (0,1)$ such that $g_{\vartheta,\nu}(t)$ is strictly decreasing on $(0,\rho_0)$ and strictly increasing on $(\rho_0,1)$.

We divide the proof into two subcases.

Subcase 3.1. $[(\pi+2)/4]^{1/\nu}-1<\vartheta<1/(6\nu)$. Then (2.21) leads to

$$g_{\vartheta,\nu}(1^-)>0. \tag{2.27}$$

Therefore, there exists $\rho^*\in(\rho_0,1)$ such that $g_{\vartheta,\nu}(t)<0$ for $t\in(0,\rho^*)$ and $g_{\vartheta,\nu}(t)>0$ for $t\in(\rho^*,1)$ follows from (2.20) and (2.27), together with the piecewise of $g_{\vartheta,\nu}(t)$ on the interval $(0,1)$.

Subcase 3.2. $1/[(\pi+2)\nu-1]<\vartheta\le[(\pi+2)/4]^{1/\nu}-1$. Then (2.21) gives

$$g_{\vartheta,\nu}(1^-)\le 0. \tag{2.28}$$

Therefore, $g_{\vartheta,\nu}(t)<0$ for all $t\in(0,1)$ follows from (2.20) and (2.28), together with the piecewise of $g_{\vartheta,\nu}(t)$ on the interval $(0,1)$. □

3 Main results

Theorem 3.1 *Let* $\lambda_1,\mu_1\in[1/2,1]$ *and* $\nu\in[1/2,\infty)$. *Then the double inequality*

$$W_{\lambda_1,\nu}(x,y)<\mathcal{R}_{QA}(x,y)<W_{\mu_1,\nu}(x,y) \tag{3.1}$$

holds for all $x,y>0$ *with* $x\neq y$ *if and only if* $\lambda_1\le 1/2+\sqrt{[(\sqrt{2}+\log(1+\sqrt{2}))/2]^{1/\nu}-1}/2$ *and* $\mu_1\ge 1/2+\sqrt{6\nu}/(12\nu)$.

Proof Since both $W_{\theta,\nu}(x,y)$ and $\mathcal{R}_{QA}(x,y)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $x>y>0$. Let $t=(x-y)/(x+y)\in(0,1)$ and $\theta\in[1/2,1]$. Then from (1.1), (1.2), and (1.7) we get

$$\frac{W_{\theta,\nu}(x,y))}{A(x,y)}=\left[1+(2\theta-1)^2t^2\right]^{\nu}, \tag{3.2}$$

$$\frac{\mathcal{R}_{QA}(x,y)}{A(x,y)}=\frac{1}{2}\left[\sqrt{1+t^2}+\frac{\sinh^{-1}(t)}{t}\right]. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}\log\left[\frac{W_{\theta,\nu}(x,y)}{\mathcal{R}_{QA}(x,y)}\right]&=\log\left[\frac{W_{\theta,\nu}(x,y)}{A(x,y)}\right]-\log\left[\frac{\mathcal{R}_{QA}(x,y)}{A(x,y)}\right]\\&=\nu\log\left[1+(2\theta-1)^2t^2\right]-\log\left[t\sqrt{1+t^2}+\sinh^{-1}(t)\right]\\&\quad+\log(t)+\log 2.\end{aligned} \tag{3.4}$$

Therefore, Theorem 3.1 follows easily from Lemma 2.4 and (3.4). □

Theorem 3.2 *Let* $\lambda_2,\mu_2\in[1/2,1]$ *and* $\nu\in[1/2,\infty)$. *Then the double inequality*

$$W_{\lambda_2,\nu}(x,y)<\mathcal{R}_{AQ}(x,y)<W_{\mu_2,\nu}(x,y) \tag{3.5}$$

holds for all $x,y>0$ *with* $x\neq y$ *if and only if* $\lambda_2\le 1/2+\sqrt{[(\pi+2)/4]^{1/\nu}-1}/2$ *and* $\mu_2\ge 1/2+\sqrt{3\nu}/(6\nu)$.

Proof Since both $W_{\vartheta,\nu}(x,y)$ and $\mathcal{R}_{AQ}(x,y)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $x > y > 0$. Let $t = (x-y)/(x+y) \in (0,1)$ and $\vartheta \in [1/2,1]$. Then it follows from (1.1), (1.3), and (1.7) that

$$\frac{W_{\vartheta,\nu}(x,y))}{A(x,y)} = \left[1 + (2\vartheta - 1)^2 t^2\right]^{\nu}, \tag{3.6}$$

$$\frac{\mathcal{R}_{AQ}(x,y)}{A(x,y)} = \frac{1}{2}\left[1 + \frac{(1+t^2)\arctan(t)}{t}\right]. \tag{3.7}$$

From (3.6) and (3.7) we have

$$\begin{aligned}\log\left[\frac{W_{\vartheta,\nu}(x,y))}{\mathcal{R}_{AQ}(x,y)}\right] &= \log\left[\frac{W_{\vartheta,\nu}(x,y)}{A(x,y)}\right] - \log\left[\frac{\mathcal{R}_{AQ}(x,y)}{A(x,y)}\right] \\ &= \nu\log\left[1 + (2\vartheta - 1)^2 t^2\right] - \log\left[t + \left(1+t^2\right)\arctan(t)\right] \\ &\quad + \log(t) + \log 2. \end{aligned} \tag{3.8}$$

Therefore, Theorem 3.2 follows easily from Lemma 2.5 and (3.8). □

Remark 3.3 Let $\nu = 1/2$. Then from (1.8) we clearly see that Theorems 3.1 and 3.2 become (1.5) and (1.6), respectively.

Let $\nu = 1$. Then from (1.9) and Theorems 3.1 and 3.2 we get Corollary 3.4 immediately.

Corollary 3.4 *Let $\lambda_1, \mu_1, \lambda_2, \mu_2 \in [1/2,1]$. Then the double inequalities*

$$C\left[\lambda_1 x + (1-\lambda_1)y, \lambda_1 y + (1-\lambda_1)x\right] < \mathcal{R}_{QA}(x,y) < C\left[\mu_1 x + (1-\mu_1)y, \mu_1 y + (1-\mu_1)x\right],$$
$$C\left[\lambda_2 x + (1-\lambda_2)y, \lambda_2 y + (1-\lambda_2)x\right] < \mathcal{R}_{AQ}(x,y) < C\left[\mu_2 x + (1-\mu_2)y, \mu_2 y + (1-\mu_2)x\right]$$

hold for all $x, y > 0$ with $x \neq y$ if and only if $\lambda_1 \leq 1/2 + \sqrt{[(\sqrt{2} + \log(1+\sqrt{2}))/2] - 1}/2 = 0.6922\ldots$, $\mu_1 \geq 1/2 + \sqrt{6}/12 = 0.7041\ldots$, $\lambda_2 \leq 1/2 + \sqrt{[(\pi+2)/4] - 1}/2 = 0.7671\ldots$ and $\mu_2 \geq 1/2 + \sqrt{3}/6 = 0.7886\ldots$.

Let $u \in (0,1)$, $x = 1+u$, $y = 1-u$, $\lambda_1 = 1/2 + \sqrt{[(\sqrt{2} + \log(1+\sqrt{2}))/2]^{1/\nu} - 1}/2$, $\mu_1 = 1/2 + \sqrt{6\nu}/(12\nu)$, $\lambda_2 = 1/2 + \sqrt{[(\pi+2)/4]^{1/\nu} - 1}/2$ and $\mu_2 = 1/2 + \sqrt{3\nu}/(6\nu)$. Then (1.2), (1.3), and Theorems 3.1 and 3.2 lead to Corollary 3.5.

Corollary 3.5 *The double inequalities*

$$\begin{aligned}&2\left[\left(1-u^2\right) + \left(\frac{\sqrt{2} + \log(1+\sqrt{2})}{2}\right)^{1/\nu} u^2\right]^{\nu} - \sqrt{1+u^2} \\ &\quad < \frac{\sinh^{-1}(u)}{u} < 2\left(1 + \frac{u^2}{6\nu}\right)^{\nu} - \sqrt{1+u^2},\end{aligned}$$

$$\frac{2[(1-u^2) + (\frac{2+\pi}{4})^{1/\nu} u^2]^{\nu} - 1}{1+u^2} < \frac{\arctan(u)}{u} < \frac{2(1 + \frac{1}{3\nu}u^2)^{\nu} - 1}{1+u^2}$$

hold for all $u \in (0,1)$ and $\nu \in [1/2, \infty)$.

4 Results and discussion

In the article, we give the sharp bounds for the Neuman means

$$\mathcal{R}_{QA}(x,y)=\frac{1}{2}\left[Q(x,y)+\frac{A^2(x,y)}{\mathrm{SB}(Q(x,y),A(x,y))}\right]$$

and

$$\mathcal{R}_{AQ}(x,y)=\frac{1}{2}\left[A(x,y)+\frac{Q^2(x,y)}{\mathrm{SB}(A(x,y),Q(x,y))}\right]$$

in terms of the two-parameter contraharmonic and arithmetic mean

$$W_{\lambda,\nu}(x,y)=C^{\nu}\left[\lambda x+(1-\lambda)y,\lambda y+(1-\lambda)x\right]A^{1-\nu}(x,y),$$

and find new bounds for the functions $\sinh(u)/u$ and $\arctan(u)/u$ on the interval $(0,1)$.

5 Conclusion

In the article, we prove that the double inequalities

$$W_{\lambda_1,\nu}(x,y)<\mathcal{R}_{QA}(x,y)<W_{\mu_1,\nu}(x,y),\qquad W_{\lambda_2,\nu}(x,y)<\mathcal{R}_{AQ}(x,y)<W_{\mu_2,\nu}(x,y)$$

hold for all $x,y>0$ with $x\neq y$ if and only if $\lambda_1\leq 1/2+\sqrt{[(\sqrt{2}+\log(1+\sqrt{2}))/2]^{1/\nu}-1}/2$, $\mu_1\geq 1/2+\sqrt{6\nu}/(12\nu)$, $\lambda_2\leq 1/2+\sqrt{[(\pi+2)/4]^{1/\nu}-1}/2$ and $\mu_2\geq 1/2+\sqrt{3\nu}/(6\nu)$ if $\lambda_1,\mu_1,\lambda_2,\mu_2\in[1/2,1]$ and $\nu\in[1/2,\infty)$. Our results are a natural generalization of some previously known results, and our approach may lead to many follow-up studies.

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