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Transpose of Nörlund matrices on the domain of summability matrices

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The paper is dedicated to Imam Khomeini, the late leader of the Islamic Revolution of Iran, on the occasion of the 30th anniversary of his demise.

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Abstract

Let $E = (E_{n,k})_{n,k\geq 0}$ be an invertible summability matrix with bounded absolute row sums and column sums, and let E_p denote the domain of E in the sequence space ℓ_p $(1 \leq p < \infty)$. In this paper, we consider the transpose of Nörlund matrix associated with a nonnegative and nonincreasing sequence as an operator mapping ℓ_p into the sequence space E_p and establish a general upper estimate for its operator norm, which depends on the ℓ_1-norm of the rows and columns of the matrix E. In particular, we apply our result to domains of some summability matrices such as Fibonacci, Karamata, Euler, and Taylor matrices. Our result is an extension of those given by G. Talebi and M.A. Dehghan (Linear Multilinear Algebra 64(2):196–207, 2016). It also provides some analogues of the results by G. Talebi (Indag. Math. 28(3):629–636, 2017).

MSC: 40G05; 11B39; 47A30

Keywords: Summability matrices; Fibonacci numbers; Nörlund matrices

1 Introduction

Let ω be the space of all real- or complex-valued sequences. We denote by ℓ_1 and ℓ_p the spaces of all absolutely and p-absolutely convergent series, respectively, where $1 < p < \infty$.

Let $E = (E_{n,k})_{n,k\geq 0}$ be an arbitrary invertible summability matrix with bounded absolute row and column sums, and let E_p denote the domain of E in the sequence space ℓ_p, that is,

$$E_p := (\ell_p)_E = \{x = (x_n) \in \omega, Ax \in \ell_p\}.$$

We summarize some important properties of the sequence space E_p and refer the reader to [13] for more details. The set E_p is a linear space with the coordinatewise addition and scalar multiplication, which is a normed space with the norm $\|x\|_{E_p} := \|Ex\|_{\ell_p}$; in particular, E_p is a BK space if the matrix E is lower triangular. Moreover, the inclusion $\ell_p \subseteq E_p$ holds for $1 \leq p < \infty$ whenever the absolute row sums and column sums of the matrix E are bounded. In addition, the inclusion $E_q \subseteq E_p$ holds for $1 \leq q \leq p$. Further, if the map $E : E_p \to \ell_p$ is onto, then the space E_p is linearly isomorphic to ℓ_p, and in such a case the columns of the matrix E^{-1} form a Schauder basis for E_p. We also refer the reader to [14], where the α-, β-, and γ-duals of the space E_p for $1 \leq p \leq \infty$ are established.

Let $W = (w_n)_{n=0}^{\infty}$ be a sequence of nonnegative real numbers with $w_0 > 0$. Set $W_n := \sum_{k=0}^{n} w_k$, $n \geq 0$, and define the Nörlund matrix $A_W^{\mathrm{NM}} := A(w_n) = (a_{n,k})_{n,k\geq 0}$, associated with the sequence W by

$$a_{n,k} = \begin{cases} \frac{w_{n-k}}{W_n} & (0 \leq k \leq n), \\ 0 & \text{otherwise.} \end{cases}$$

Since $A(w_n) = A(cw_n)$ for any $c > 0$, we may as well assume that $w_0 = 1$.

Recently, in [13], Theorem 4.2, the author considered the Nörlund matrices as operators from ℓ_p into E_p and established a general upper estimate for their operator norms. In this paper, we consider the same problem for the transpose of Nörlund matrices associated with the nonnegative and nonincreasing sequences as operators from ℓ_p into E_p. We obtain again a general upper estimate for their operator norms, which depend on the ℓ_1-norm of the columns and the rows of the summability matrix E. In particular, we apply our results to domains of some summability matrices such as Fibonacci, Karamata, Euler, and Taylor matrices. Our result is an extension of Theorem 3.8 in [15] and provides some analogue of those given in [13].

Throughout the paper we suppose that $1 < p < \infty$ and $E = (e_{n,k})_{n,k\geq 0}$ is a summability matrix whose absolute row sums and column sums are bounded.

Theorem 1.1 *Suppose that $W = (w_n)_{n=0}^{\infty}$ is a nonnegative and nonincreasing sequence of real numbers with $w_0 = 1$. Then the transpose of the associated Nörlund matrix maps ℓ_p into E_p, and we have*

$$\|(A_W^{\mathrm{NM}})^t\|_{\ell_p,E_p} \leq p\Bigl(\sup_{k\in\mathbb{N}^0} \|\{e_{n,k}\}_{n\in\mathbb{N}^0}\|_{\ell_1}\Bigr)^{\frac{1}{p}} \Bigl(\sup_{n\in\mathbb{N}^0} \|\{e_{n,k}\}_{k\in\mathbb{N}^0}\|_{\ell_1}\Bigr)^{\frac{p-1}{p}}.$$

Proof Let us take any $x \in \ell_p$. Applying Hölder's inequality, we have

$$\begin{aligned}
\|(A_W^{\mathrm{NM}})^t x\|_{E_p}^p &= \sum_{k=0}^{\infty} \left| \sum_{n=0}^{\infty} e_{k,n} \sum_{j=n}^{\infty} \frac{w_{j-n}}{W_j} x_j \right|^p \\
&\leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |e_{k,n}| \left| \sum_{j=n}^{\infty} \frac{w_{j-n}}{W_j} x_j \right|^p \left| \sum_{n=0}^{\infty} e_{k,n} \right|^{p-1} \\
&\leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |e_{k,n}| \left| \sum_{j=n}^{\infty} \frac{w_{j-n}}{W_j} x_j \right|^p \left(\sum_{n=0}^{\infty} |e_{k,n}| \right)^{p-1} \\
&\leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |e_{k,n}| \left| \sum_{j=n}^{\infty} \frac{w_{j-n}}{W_j} x_j \right|^p \Bigl(\sup_{k\in\mathbb{N}^0} \|\{e_{k,n}\}_{n\in\mathbb{N}^0}\|_{\ell_1} \Bigr)^{p-1} \\
&= \Bigl(\sup_{k\in\mathbb{N}^0} \|\{e_{k,n}\}_{n\in\mathbb{N}^0}\|_{\ell_1} \Bigr)^{p-1} \sum_{n=0}^{\infty} \left| \sum_{j=n}^{\infty} \frac{w_{j-n}}{W_j} x_j \right|^p \sum_{k=0}^{\infty} |e_{k,n}| \\
&\leq \Bigl(\sup_{k\in\mathbb{N}^0} \|\{e_{k,n}\}_{n\in\mathbb{N}^0}\|_{\ell_1} \Bigr)^{p-1} \sum_{n=0}^{\infty} \left| \sum_{j=n}^{\infty} \frac{w_{j-n}}{W_j} x_j \right|^p \Bigl(\sup_{n\in\mathbb{N}^0} \|\{e_{k,n}\}_{k\in\mathbb{N}^0}\|_{\ell_1} \Bigr)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sup_{n\in\mathbb{N}^0} \left\|\{e_{k,n}\}_{k\in\mathbb{N}^0}\right\|_{\ell_1}\right)\left(\sup_{k\in\mathbb{N}^0} \left\|\{e_{k,n}\}_{n\in\mathbb{N}^0}\right\|_{\ell_1}\right)^{p-1} \sum_{n=0}^{\infty} \left|\sum_{j=n}^{\infty} \frac{w_{j-n}}{W_j} x_j\right|^p \\
&= \left(\sup_{n\in\mathbb{N}^0} \left\|\{e_{n,k}\}_{k\in\mathbb{N}^0}\right\|_{\ell_1}\right)\left(\sup_{k\in\mathbb{N}^0} \left\|\{e_{k,n}\}_{n\in\mathbb{N}^0}\right\|_{\ell_1}\right)^{p-1} \left\|\left(A_W^{\mathrm{NM}}\right)^t x\right\|_{\ell_p}^p \\
&\leq \left(\sup_{n\in\mathbb{N}^0} \left\|\{e_{n,k}\}_{k\in\mathbb{N}^0}\right\|_{\ell_1}\right)\left(\sup_{k\in\mathbb{N}^0} \left\|\{e_{k,n}\}_{n\in\mathbb{N}^0}\right\|_{\ell_1}\right)^{p-1} p^p \sum_{j=0}^{\infty} |x_j|^p,
\end{aligned}
$$

where the last inequality is based on [5], Theorem 1. This leads us to the desired inequality and completes the proof. □

Theorem 1.1 extends [15], Theorem 3.8, from the Fibonacci sequence space to the domains of invertible summability matrices in ℓ_p. To see this, consider the Fibonacci sequence space F_p defined in [7]:

$$
F_p = \left\{(x_n) \in \omega : \sum_{n=0}^{\infty} \left|\frac{1}{f_n f_{n+1}} \sum_{k=0}^{n} f_k^2 x_k\right|^p < \infty\right\},
$$

which is the domain of the Fibonacci matrix $F = (F_{n,k})_{n,k\geq 0}$ with the entries

$$
F_{n,k} = \begin{cases} \frac{f_k^2}{f_n f_{n+1}}, & 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}
$$

Here the Fibonacci numbers are the sequence of numbers $\{f_n\}_{n=0}^{\infty}$ defined by the linear recurrence equations

$$
f_n = f_{n-1} + f_{n-2}, \quad n \geq 1,
$$

where $f_{-1} = 0$ and $f_0 = 1$. For this matrix, the sums of all rows are 1, and by Lemma 2.4 of [7] its column sums are bounded. Hence, applying Theorem 1.1 to the Fibonacci sequence space F_p, we have the following result, which was previously obtained in Theorem 3.8 of [15].

Corollary 1.2 *Let $W = (w_n)_{n=0}^{\infty}$ be a nonnegative and nonincreasing sequence of real numbers with $w_0 = 1$. Then the transpose of the associated Nörlund matrix maps ℓ_p into the Fibonacci sequence space F_p, and we have*

$$
\left\|\left(A_W^{\mathrm{NM}}\right)^t\right\|_{\ell_p, F_p} \leq p\left(\sup_{k\in\mathbb{N}^0} \sum_{n=k}^{\infty} \frac{f_k^2}{f_n f_{n+1}}\right)^{\frac{1}{p}}.
$$

For further details on the normed spaces derived by the Fibonacci matrix, we refer the readers to the recent papers [4, 6, 8–10].

In the following, we present some additional particular cases of Theorem 1.1. First, consider the Karamata sequence space $\mathcal{K}_p^{\alpha,\beta}$ defined by [2]

$$
\mathcal{K}_p^{\alpha,\beta} = \left\{(x_n) \in \omega : \sum_{n=0}^{\infty} \left|\sum_{k=0}^{\infty} \sum_{\nu=0}^{k} \binom{n}{\nu} (1-\alpha-\beta)^{\nu} \alpha^{n-\nu} \binom{n+k-\nu-1}{k-\nu} \beta^{k-\nu} x_k\right|^p < \infty\right\},
$$

which is the domain of the Karamata matrix $K[\alpha,\beta] = (a_{n,k})_{n,k\geq 0}$ in the sequence space ℓ_p with entries

$$a_{n,k} = \sum_{\nu=0}^{k} \binom{n}{\nu} (1-\alpha-\beta)^{\nu}\alpha^{n-\nu} \binom{n+k-\nu-1}{k-\nu} \beta^{k-\nu},$$

where $\alpha,\beta \in (0,1)$. This matrix is a particular case of Sonnenschein matrices [3].

For the Karamata matrix, it is proved in [2], Theorem 1.2, that the sum of the first column is $\frac{1}{1-\alpha}$, the sums of all other columns are $\frac{1-\beta}{1-\alpha}$, and the sums of all rows are 1. Hence, applying Theorem 1.1 to the Karamata sequence space $\mathcal{K}_p^{\alpha,\beta}$, we have the following result.

Corollary 1.3 *Let $W = (w_n)_{n=0}^{\infty}$ be a nonnegative and nonincreasing sequence of real numbers with $w_0 = 1$. Then the transpose of the associated Nörlund matrix maps ℓ_p into the Karamata sequence space $\mathcal{K}_p^{\alpha,\beta}$, and we have*

$$\big\|\big(A_W^{\mathrm{NM}}\big)^t\big\|_{\ell_p,\mathcal{K}_p^{\alpha,\beta}} \leq p\left(\frac{1}{1-\alpha}\right)^{\frac{1}{p}}.$$

Next, consider the Euler sequence space e_p^{α} defined by [1]

$$e_p^{\alpha} = \left\{(x_n) \in \omega : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \binom{n}{k} (1-\alpha)^{n-k}\alpha^k x_k \right|^p < \infty \right\},$$

where $\alpha \in (0,1)$ (see also [11]). Clearly, $e_p^{\alpha} = \mathcal{K}_p^{1-\alpha,0}$. Therefore we have the following estimation for the operator norm of the transpose of Nörlund matrix as operator mapping ℓ_p into e_p^{α}.

Corollary 1.4 *Let $W = (w_n)_{n=0}^{\infty}$ be a nonnegative and nonincreasing sequence of real numbers with $w_0 = 1$. Then the transpose of the associated Nörlund matrix maps ℓ_p into the Euler sequence space e_p^{α}, and we have*

$$\big\|\big(A_W^{\mathrm{NM}}\big)^t\big\|_{\ell_p,e_p^{\alpha}} \leq p\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}.$$

As the third particular case, we consider the Taylor sequence space [13]

$$t_p^{\theta} = \left\{(x_n) \in \omega : \sum_{n=0}^{\infty} \left| \sum_{k=n}^{\infty} \binom{n}{k} (1-\theta)^{n+1}\theta^{k-n} x_k \right|^p < \infty \right\}, \quad \theta \in (0,1),$$

which is the domain of the Taylor matrix $T^{\theta} = (t_{n,k}^{\theta})_{n,k\geq 0}$ in ℓ_p with entries [12]

$$t_{n,k}^{\theta} = \begin{cases} 0, & 0 \leq k < n, \\ \binom{n}{k}(1-\theta)^{n+1}\theta^{k-n}, & k \geq n. \end{cases}$$

For this matrix, we have $\sup_n \|\{t_{n,k}^{\theta}\}_{k\in\mathbb{N}^0}\|_{\ell_1} = 1$ and $\sup_k \|\{t_{n,k}^{\theta}\}_{n\in\mathbb{N}^0}\|_{\ell_1} = (1-\theta)$. Hence Theorem 1.1 enables us to obtain the following estimation for the operator norm of the transpose of Nörlund matrix as an operator mapping ℓ_p into t_p^{θ}.

Corollary 1.5 *Let $W = (w_n)_{n=0}^{\infty}$ be a nonnegative and nonincreasing sequence of real numbers with $w_0 = 1$. Then the transpose of the associated Nörlund matrix maps ℓ_p into the Taylor sequence space t_p^{θ}, and we have*

$$\left\| \left(A_W^{\mathrm{NM}}\right)^t \right\|_{\ell_p, t_p^{\theta}} \leq p(1-\theta)^{\frac{1}{p}}.$$

2 Conclusions

In this paper, we consider the transpose of Nörlund matrix associated with a nonnegative and nonincreasing sequence as an operator mapping ℓ_p into the sequence space E_p and establish a general upper estimate for its operator norm, which depends on the ℓ_1-norm of the rows and the columns of the matrix E, where $E = (E_{n,k})_{n,k\geq 0}$ is an invertible summability matrix with bounded absolute row sums and column sums, and E_p denotes the domain of E in ℓ_p.

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