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Existence and Ulam–Hyers stability of coupled sequential fractional differential equations with integral boundary conditions

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Abstract

We study the existence and uniqueness of solutions for coupled sequential fractional differential equations involving Caputo fractional derivative of order $1 < \alpha \leq 2$ with integral boundary conditions. Moreover, we discuss Ulam–Hyers stability for the problem at hand.

MSC: 34A08; 34B

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1 Introduction

The theory of differential equations of fractional order, involving various types of boundary conditions, has been the subject of interest in the pure and applied sciences. In addition to the classical two-point boundary conditions, much attention is paid to non-local multipoint and integral boundary conditions. Non-local conditions are used to describe certain features of physical, chemical, or other processes occurring in the inner positions of a given region, while integral boundary conditions provide a plausible and practical approach to modeling blood flow problems. For more information and explanation, see, for example, [1, 2]. Some recent results on the boundary value problem of fractional order can be found in the series [3–21] and in the references cited there. Sequential differential equations with fractional derivatives also received considerable attention, for example, see [4–9].

The study of coupled systems involving fractional differential equations is also important because these systems occur in various problems of applied nature. Coupled systems of fractional differential equations have also been investigated by many authors. Such systems appear naturally in many real world situations. Some recent results on the topic can be found in [7, 17, 22–26].

We study the following nonlinear sequential fractional differential equation subject to nonseparated nonlocal integral fractional boundary conditions:

$$\begin{cases} (D^{\alpha} + \lambda D^{\alpha-1})x(t) = f_1(t, x(t), y(t)), & 1 < \alpha \leq 2, 0 \leq t \leq T, \\ (D^{\beta} + \lambda D^{\beta-1})y(t) = f_2(t, x(t), y(t)), & 1 < \beta \leq 2, 0 \leq t \leq T, \\ \nu_1 x(\eta) + \mu_1 x(T) = \int_0^T h_1(x(s))\,ds, & \nu_1 y(\eta) + \mu_1 y(T) = \int_0^T h_2(y(s))\,ds, \\ \nu_2 x'(\eta) + \mu_2 x'(T) = \int_0^T g_1(x(s),)\,ds, & \nu_2 y'(\eta) + \mu_2 y'(T) = \int_0^T g_2(y(s))\,ds, \end{cases} \tag{1.1}$$

where D^{α}, D^{β} denote the Caputo derivative, $0 < \eta < T$, $\lambda \in \mathbb{R}_+$, $\nu_1, \nu_2, \mu_1, \mu_2 \in \mathbb{R}$.

During the last few decades another part of research, which has been considered for fractional differential equations and got much attention from the researchers, is stability analysis. Numerous forms of stabilities have been studied in literature which are Mittag-Leffler stability, exponential stability, Lyapunov stability, etc. For historical background of Ulam–Hyers stability and recent results, we refer to works [27–36]. To the best of our knowledge, the Ulam–Hyers stability has been very rarely studied for coupled system of fractional differential equations. Therefore in this article we investigate existence and Ulam–Hyers stability to the considered problem. The rest of the paper is organized as follows. In Sect. 2, we recall some basic concepts of fractional calculus and obtain the integral solution for the linear variants of the given problems. Section 3 contains the existence results for problem (1.1) obtained by applying Leray–Schauder's nonlinear alternative, Banach's contraction mapping principle. In Sect. 4, Ulam–Hyers stability for problem (1.1) is studied. Finally, in Sect. 5, an example is provided to illustrate the theoretical results.

2 Preliminaries

We begin this section with some basic definitions of fractional calculus [2]. Later we prove an auxiliary lemma, which plays a key role in defining a fixed point problem associated with the given problem.

Definition 1 The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $f : [0, +\infty) \to R$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided that the right-hand side of the integral is pointwise defined on $(0, +\infty)$ and Γ is the gamma function.

Definition 2 The *Caputo* derivative of order $\alpha > 0$ for a function $f : [0, +\infty) \to R$ is written as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ is an integral part of α.

Lemma 1 ([2]) *Let $\alpha > 0$. Then the differential equation $D_{0+}^{\alpha} f(t) = 0$ has solutions*

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

and

$$I_{0+}^{\alpha}D_{0+}^{\alpha}f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \tag{2.1}$$

where $c_i \in \mathbb{R}$ *and* $i = 1, 2, \ldots, n = [\alpha] + 1$.

Let $C([0,T];\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ equipped with the sup-norm $\|x\|_\infty = \sup\{|x(t)| : 0 \le t \le T\}$. For computational convenience, in what follows we use the following notations:

$$\begin{aligned}
&a_{11} := \nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T}, \qquad a_{12} := \nu_1 \frac{1}{\lambda}\bigl(1 - e^{-\lambda\eta}\bigr) + \mu_1 \frac{1}{\lambda}\bigl(1 - e^{-\lambda T}\bigr),\\
&a_{21} := -\lambda\nu_2 e^{-\lambda\eta} - \lambda\mu_2 e^{-\lambda T}, \qquad a_{22} := \nu_2 e^{-\lambda\eta} + \mu_2 e^{-\lambda T},\\
&\Delta := a_{11}a_{22} - a_{12}a_{21}, \qquad \Delta \neq 0,\\
&\varphi_1(t) = \left(\frac{a_{12}}{\Delta}e^{-\lambda t} - \frac{a_{11}}{\Delta}\frac{1}{\lambda}\bigl(1 - e^{-\lambda t}\bigr)\right),\\
&\varphi_2(t) = \left(\frac{a_{21}}{\Delta}\frac{1}{\lambda}\bigl(1 - e^{-\lambda t}\bigr) - \frac{a_{22}}{\Delta}e^{-\lambda t}\right).
\end{aligned}$$

Lemma 2 *Let* $\rho, \gamma_1, \gamma_2 \in C([0,T];\mathbb{R})$. *Then the following boundary value problem*

$$\begin{cases}
(D^{\alpha} + \lambda D^{\alpha-1})z(t) = \rho(t), \quad 1 < \alpha \le 2, 0 \le t \le T,\\
\nu_1 z(\eta) + \mu_1 z(T) = \int_0^T \gamma_1(s)\,ds,\\
\nu_2 z'(\eta) + \mu_2 z'(T) = \int_0^T \gamma_2(s)\,ds,
\end{cases} \tag{2.2}$$

is equivalent to the fractional integral equation

$$\begin{aligned}
z(t) = {}&\int_0^t (t-s)^{\alpha-1}E_{1,\alpha}(-\lambda; t-s)\rho(s)\,ds\\
&+ \sum_{j=1}^{2}\nu_j\varphi_j(t)\int_0^{\eta}(\eta-s)^{\alpha-j}E_{1,\alpha+j-1}(-\lambda;\eta-s)\rho(s)\,ds\\
&+ \sum_{j=1}^{2}\mu_j\varphi_j(t)\int_0^{T}(T-s)^{\alpha-j}E_{1,\alpha+j-1}(-\lambda;T-s)\rho(s)\,ds\\
&- \sum_{j=1}^{2}\varphi_i(t)\int_0^{T}\gamma_i(s)\,ds.
\end{aligned} \tag{2.3}$$

Proof Applying $I^{\alpha-1}$ to both sides of (2.2) and using (2.1), we get

$$\begin{aligned}
&I^{\alpha-1}D^{\alpha-1}(D+\lambda)z(t) = I^{\alpha-1}\rho(t),\\
&(D+\lambda)z(t) = c_0 + I^{\alpha-1}\rho(t).
\end{aligned}$$

We solve the above linear ordinary differential equation:

$$\begin{aligned} z(t) &= c_1 e^{-\lambda t} + c_0 \frac{1}{\lambda} - c_0 \frac{1}{\lambda} e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} I^{\alpha-1} \rho(s)\,ds \\ &= c_1 e^{-\lambda t} + c_0 \frac{1}{\lambda}\bigl(1 - e^{-\lambda t}\bigr) + \int_0^t (t-r)^{\alpha-1} E_{1,\alpha}(-\lambda; t-r) \rho(r)\,dr. \end{aligned} \tag{2.4}$$

It is clear that

$$z'(t) = -\lambda c_1 e^{-\lambda t} + c_0 e^{-\lambda t} + \int_0^t (t-r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; t-r) \rho(r)\,dr. \tag{2.5}$$

The first boundary condition implies that

$$\begin{aligned} &\nu_1 z(\eta) + \mu_1 z(T) \\ &\quad= \nu_1 c_1 e^{-\lambda\eta} + \nu_1 c_0 \frac{1}{\lambda}\bigl(1 - e^{-\lambda\eta}\bigr) + \nu_1 \int_0^\eta (\eta-r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta-r) \rho(r)\,dr \\ &\qquad+ \mu_1 c_1 e^{-\lambda T} + \mu_1 c_0 \frac{1}{\lambda}\bigl(1 - e^{-\lambda T}\bigr) + \mu_1 \int_0^T (T-r)^{\alpha-1} E_{1,\alpha}(-\lambda; T-r) \rho(r)\,dr \\ &\quad= \int_0^T \gamma_1(s)\,ds. \end{aligned}$$

It follows that

$$\begin{aligned} &\bigl(\nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T}\bigr) c_1 + \left(\nu_1 \frac{1}{\lambda}\bigl(1 - e^{-\lambda\eta}\bigr) + \mu_1 \frac{1}{\lambda}\bigl(1 - e^{-\lambda T}\bigr)\right) c_0 \\ &\quad= \int_0^T \gamma_1(s)\,ds - \nu_1 \int_0^\eta (\eta-r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta-r) \rho(r)\,dr \\ &\qquad- \mu_1 \int_0^T (T-r)^{\alpha-1} E_{1,\alpha}(-\lambda; T-r) \rho(r)\,dr. \end{aligned}$$

The second boundary condition with (2.5) implies that

$$\begin{aligned} &\nu_2 z'(\eta) + \mu_2 z'(T) \\ &\quad= \nu_2\bigl(-\lambda c_1 e^{-\lambda\eta} + c_0 e^{-\lambda\eta}\bigr) + \nu_2 \int_0^\eta (\eta-r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta-r) \rho(r)\,dr \\ &\qquad+ \mu_2\bigl(-\lambda c_1 e^{-\lambda T} + c_0 e^{-\lambda T}\bigr) + \mu_2 \int_0^T (T-r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T-r) \rho(r)\,dr \\ &\quad= \int_0^T \gamma_2(s)\,ds. \end{aligned}$$

Thus

$$\begin{aligned} a_{11} c_1 + a_{12} c_0 = {}&\int_0^T \gamma_1(s)\,ds - \nu_1 \int_0^\eta (\eta-r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta-r) \rho(r)\,dr \\ &- \mu_1 \int_0^T (T-r)^{\alpha-1} E_{1,\alpha}(-\lambda; T-r) \rho(r)\,dr, \end{aligned}$$

$$\begin{aligned} a_{21}c_1 + a_{22}c_0 = {} & \int_0^T \gamma_2(s)\,ds - \nu_2 \int_0^\eta (\eta - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta - r)\rho(r)\,dr \\ & - \mu_2 \int_0^T (T - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T - r)\rho(r)\,dr. \end{aligned}$$

Solving the above system of equations for c_0 and c_1, we get

$$\begin{aligned} c_0 = {} & \frac{a_{11}}{\Delta}\biggl(\int_0^T \gamma_2(s)\,ds - \nu_2 \int_0^\eta (\eta - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta - r)\rho(r)\,dr \\ & - \mu_2 \int_0^T (T - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T - r)\rho(r)\,dr\biggr) \\ & - \frac{a_{21}}{\Delta}\biggl(\int_0^T \gamma_1(s)\,ds - \nu_1 \int_0^\eta (\eta - r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta - r)\rho(r)\,dr \\ & - \mu_1 \int_0^T (T - r)^{\alpha-1} E_{1,\alpha}(-\lambda; T - r)\rho(r)\,dr\biggr), \\ c_1 = {} & \frac{a_{22}}{\Delta}\biggl(\int_0^T \gamma_1(s)\,ds - \nu_1 \int_0^\eta (\eta - r)^{\alpha-1} E_{1,\alpha}(-\lambda; \eta - r)\rho(r)\,dr \\ & - \mu_1 \int_0^T (T - r)^{\alpha-1} E_{1,\alpha}(-\lambda; T - r)\rho(r)\,dr\biggr) \\ & - \frac{a_{12}}{\Delta}\biggl(\int_0^T \gamma_2(s)\,ds - \nu_2 \int_0^\eta (\eta - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; \eta - r)\rho(r)\,dr \\ & - \mu_2 \int_0^T (T - r)^{\alpha-2} E_{1,\alpha-1}(-\lambda; T - r)\rho(r)\,dr\biggr). \end{aligned}$$

Inserting c_0 and c_1 in (2.4), we obtain the desired formula (2.3).

Conversely, assume that u satisfies (2.3). By a direct computation, it follows that the solution given by (2.3) satisfies (2.2). □

Lemma 3 *For any* $g, h \in C([0,T];\mathbb{R})$, $\gamma > 0$, *we have*

$$\left|\int_0^t (t-s)^{\gamma-1} E_{1,\gamma}(-\lambda; t-s)\bigl(g(s) - h(s)\bigr)\,ds\right| \le t^\gamma E_{1,\gamma+1}(-\lambda; t)\|g - h\|_\infty.$$

Proof Indeed,

$$\begin{aligned} & \left|\int_0^t (t-s)^{\gamma-1} E_{1,\gamma}(-\lambda; t-s)\bigl(g(s) - h(s)\bigr)\,ds\right| \\ & \quad \le \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(k+\gamma)} \int_0^t (t-s)^{k+\gamma-1}\bigl|g(s) - h(s)\bigr|\,ds \\ & \quad \le \sum_{k=0}^\infty \frac{\lambda^k t^{k+\gamma}}{\Gamma(k+\gamma+1)} = t^\gamma E_{1,\gamma+1}\bigl(|\lambda|; t\bigr)\|g - h\|_\infty. \end{aligned}$$

□

3 Main results

By Lemma 2, we introduce a fixed point problem associated with the problem as follows:

$$\begin{aligned} (x,y) = (T_1, T_2)(x,y) = \mathfrak{T}(x,y) : {} & C\bigl([0,T];\mathbb{R}\bigr) \times C\bigl([0,T];\mathbb{R}\bigr) \\ & \to C\bigl([0,T];\mathbb{R}\bigr) \times C\bigl([0,T];\mathbb{R}\bigr), \end{aligned}$$

where

$$\begin{aligned} T_1(x,y)(t) &= \int_0^t (t-r)^{\alpha-1} E_{1,\alpha}(-\lambda; t-r) f_1\bigl(r,x(r),y(r)\bigr)\,dr \\ &\quad + \sum_{j=1}^{2} \nu_j \varphi_j(t) \int_0^{\eta} (\eta-r)^{\alpha-j} E_{1,\alpha+j-1}(-\lambda; \eta-r) f_1\bigl(r,x(r),y(r)\bigr)\,dr \\ &\quad + \sum_{j=1}^{2} \mu_j \varphi_j(t) \int_0^{T} (T-r)^{\alpha-j} E_{1,\alpha+j-1}(-\lambda; T-r) f_1\bigl(r,x(r),y(r)\bigr)\,dr \\ &\quad + \sum_{j=1}^{2} \varphi_i(t) \int_0^{T} h_i\bigl(r,x(r),y(r)\bigr)\,dr, \end{aligned} \tag{3.1}$$

$$\begin{aligned} T_2(x,y)(t) &= \int_0^t (t-r)^{\beta-1} E_{1,\beta}(-\lambda; t-r) f_2\bigl(r,x(r),y(r)\bigr)\,dr \\ &\quad + \sum_{j=1}^{2} \nu_j \varphi_j(t) \int_0^{\eta} (\eta-r)^{\beta-j} E_{1,\beta+j-1}(-\lambda; \eta-r) f_2\bigl(r,x(r),y(r)\bigr)\,dr \\ &\quad + \sum_{j=1}^{2} \mu_j \varphi_j(t) \int_0^{T} (T-r)^{\beta-j} E_{1,\beta+j-1}(-\lambda; T-r) f_2\bigl(r,x(r),y(r)\bigr)\,dr \\ &\quad + \sum_{j=1}^{2} \varphi_i(t) \int_0^{T} g_i\bigl(r,x(r),y(r)\bigr)\,dr. \end{aligned} \tag{3.2}$$

Evidently, the existence of fixed points of the operator $\mathfrak{T}$ is equivalent to the existence of solutions for problem (1.1).

For $\gamma = \alpha, \beta$, let

$$\begin{aligned} R^{\gamma} := \max\Biggl\{ \Biggl(& T^{\gamma} E_{1,\gamma+1}\bigl(|\lambda|; T\bigr) + \sum_{j=1}^{2} |\nu_j| \|\varphi_j\| \eta^{\gamma-j} E_{1,\gamma+j-1}\bigl(|\lambda|; \eta\bigr) \\ & + \sum_{j=1}^{2} |\mu_j| \|\varphi_j\| T^{\gamma-j} E_{1,\gamma+j-1}\bigl(|\lambda|; T\bigr) \Biggr), \bigl(\|\varphi_1\| + \|\varphi_2\|\bigr) T \Biggr\}. \end{aligned}$$

Here, we prove the existence and uniqueness of solutions for problem (1.1). We apply a fixed point theorem due to Banach.

Theorem 1 *Let $f_i, h_i, g_i : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that the following conditions hold*:

(A$_1$) *There exist $L_f, L_h, L_g > 0$ such that*

$$\begin{aligned} &\bigl|f_i(t,x_1,y_1) - f_i(t,x_2,y_2)\bigr| \le L_f\bigl(|x_1-x_2| + |y_1-y_2|\bigr), \\ &\bigl|h_i(t,x_1,y_1) - h_i(t,x_2,y_2)\bigr| \le L_h\bigl(|x_1-x_2| + |y_1-y_2|\bigr), \\ &\bigl|g_i(t,x_1,y_1) - g_i(t,x_2,y_2)\bigr| \le L_g\bigl(|x_1-x_2| + |y_1-y_2|\bigr), \\ &\forall (t,x_1,y_1), (t,x_2,y_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

(A$_2$) $1-2(L_f+L_h)R^{\alpha}>0$, $1-2(L_f+L_g)R^{\beta}>0$.
Then problem (1.1) *has a unique solution in* $C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$.

Proof Consider a ball

$$B_r:=\bigl\{u\in C\bigl([0,T],\mathbb{R}\bigr)\times C\bigl([0,T],\mathbb{R}\bigr):\|u\|_{\infty}\le r\bigr\}$$

with radius

$$r\ge \max\biggl\{\frac{(M_f+M_h)R^{\alpha}}{1-2(L_f+L_h)R^{\alpha}},\frac{(M_f+M_g)R^{\beta}}{1-2(L_f+L_g)R^{\beta}}\biggr\},$$

where

$$\begin{aligned}
&M_f:=\max\bigl(\bigl\|f_1(t,0,0)\bigr\|_{\infty},\bigl\|f_2(t,0,0)\bigr\|_{\infty}\bigr),\\
&M_h:=\max\bigl(\bigl\|h_1(t,0,0)\bigr\|_{\infty},\bigl\|h_2(t,0,0)\bigr\|_{\infty}\bigr),\\
&M_g:=\max\bigl(\bigl\|g_1(t,0,0)\bigr\|_{\infty},\bigl\|g_2(t,0,0)\bigr\|_{\infty}\bigr).
\end{aligned}$$

It is clear that for all $x,y\in\mathbb{R}$

$$\begin{aligned}
&\bigl|f_i(t,x,y)\bigr|\le L_f\bigl(|x|+|y|\bigr)+M_f,\\
&\bigl|h_i(t,x,y)\bigr|\le L_h\bigl(|x|+|y|\bigr)+M_h,\\
&\bigl|g_i(t,x,y)\bigr|\le L_g\bigl(|x|+|y|\bigr)+M_g.
\end{aligned}$$

Using this inequality and Lemma 3, from (3.1) it follows that

$$\begin{aligned}
&\bigl|T_1(x,y)(t)\bigr|\\
&\quad\le t^{\alpha}E_{1,\alpha+1}\bigl(|\lambda|;t\bigr)\bigl\|f_i\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\|+\sum_{j=1}^{2}|\nu_j|\bigl|\varphi_j(t)\bigr|\eta^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;\eta\bigr)\bigl\|f_i\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\|\\
&\qquad+\sum_{j=1}^{2}|\mu_j|\bigl|\varphi_j(t)\bigr|T^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;T\bigr)\bigl\|f_i\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\|_{\infty}\\
&\qquad+\sum_{j=1}^{2}\bigl|\varphi_i(t)\bigr|\int_0^T\bigl|h_i\bigl(r,x(r),y(r)\bigr)\bigr|\,dr\\
&\quad\le\Biggl(t^{\alpha}E_{1,\alpha+1}(-\lambda;t)+\sum_{j=1}^{2}|\nu_j|\bigl|\varphi_j(t)\bigr|\eta^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;\eta\bigr)\\
&\qquad+\sum_{j=1}^{2}|\mu_j|\bigl|\varphi_j(t)\bigr|T^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;T\bigr)\Biggr)\\
&\qquad\times\bigl(L_f\bigl(|x|+|y|\bigr)+M_f\bigr)+\bigl(\|\varphi_1\|+\|\varphi_2\|\bigr)T\bigl(L_h\bigl(|x|+|y|\bigr)+M_h\bigr)\\
&\quad\le\bigl((L_f+L_h)r+M_f+M_h\bigr)R^{\alpha}\le\frac{r}{2}.
\end{aligned}\tag{3.3}$$

In the like manner we have

$$\left|T_2(x,y)(t)\right| \leq \bigl((L_f+L_h)r+M_f+M_h\bigr)R^{\beta} \leq \frac{r}{2}. \tag{3.4}$$

From (3.3) and (3.4) it follows that $\mathfrak{T}B_r \subset B_r$. Next, using condition (A_1), we obtain

$$\begin{aligned}\left|T_1(x_1,y_1)(t)-T_1(x_2,y_2)(t)\right| &\leq R^{\alpha}\bigl\|f_1\bigl(\cdot,x_1(\cdot),y_1(\cdot)\bigr)-f_1\bigl(\cdot,x_2(\cdot),y_2(\cdot)\bigr)\bigr\|_{\infty}\\ &\leq (L_f+L_h)R^{\alpha}\bigl\|(x_1,y_1)-(x_2,y_2)\bigr\|_{\infty}.\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}\left|T_2(x_1,y_1)(t)-T_2(x_2,y_2)(t)\right| &\leq R^{\beta}\bigl\|f_2\bigl(\cdot,x_1(\cdot),y_1(\cdot)\bigr)-f_2\bigl(\cdot,x_2(\cdot),y_2(\cdot)\bigr)\bigr\|_{\infty}\\ &\leq (L_f+L_g)R^{\beta}\bigl\|(x_1,y_1)-(x_2,y_2)\bigr\|_{\infty}.\end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\bigl\|T(x_1,y_1)-T(x_2,y_2)\bigr\| \leq \bigl[(L_f+L_h)R^{\alpha}+(L_f+L_g)R^{\beta}\bigr]\bigl\|(x_1,y_1)-(x_2,y_2)\bigr\|_{\infty}.$$

By (A_2) the operator $\mathfrak{T}$ is a contraction. Thus by the Banach fixed point theorem, $\mathfrak{T}$ has a unique fixed point in $C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$. This completes the proof. □

In the next result, we prove the existence of solutions for problem (1.1) by applying the Leray–Schauder alternative.

Theorem 2 *Let $f.[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following condition holds*:

(A_3) *There exist $\gamma_f,\gamma_h,\gamma_g \in C([0,T],\mathbb{R}_+)$ and a nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that*

$$\begin{aligned}&\bigl|f_i(t,x,y)\bigr| \leq \gamma_f(t)\psi_f\bigl(|x|+|y|\bigr), \quad \forall(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}.\\ &\bigl|h_i(t,x,y)\bigr| \leq \gamma_h(t)\psi_h\bigl(|x|+|y|\bigr),\\ &\bigl|g_i(t,x,y)\bigr| \leq \gamma_g(t)\psi_g\bigl(|x|+|y|\bigr), \quad i=1,2.\end{aligned}$$

(A_4) *There exists $M>0$ such that*

$$\frac{M}{(\|\gamma_f\|_{\infty}\psi_f(M)+\|\gamma_h\|_{\infty}\psi_h(M))R^{\alpha}+(\|\gamma_f\|_{\infty}\psi_f(M)+\|\gamma_g\|_{\infty}\psi_g(M))R^{\beta}} > 1.$$

Then BVP (1.1) *has at least one solution.*

Proof Step 1: Show that $\mathfrak{T}:C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ maps bounded sets into bounded sets and is continuous.

Let B_r be a ball in $C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$. Then

$$\bigl|f_1\bigl(t,x(t),y(t)\bigr)\bigr| \leq \|\gamma_f\|\psi_f(r), \qquad \bigl|f_2\bigl(t,x(t),y(t)\bigr)\bigr| \leq \|\gamma_f\|\psi_f(r),$$

and by Lemma 3

$$\begin{aligned}
&\bigl|T_1(x,y)(t)\bigr| \\
&\quad\le t^{\alpha}E_{1,\alpha+1}\bigl(|\lambda|;t\bigr)\bigl\|f_1\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\| + \sum_{j=1}^{2}|\nu_j|\bigl|\varphi_j(t)\bigr|\eta^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;\eta\bigr)\bigl\|f_i\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\| \\
&\qquad+\sum_{j=1}^{2}|\mu_j|\bigl|\varphi_j(t)\bigr|T^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;T\bigr)\bigl\|f_i\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\|_{\infty} \\
&\qquad+\sum_{j=1}^{2}\bigl|\varphi_i(t)\bigr|\int_0^T\bigl|h_i\bigl(r,x(r),y(r)\bigr)\bigr|\,dr \\
&\quad\le\Biggl(t^{\alpha}E_{1,\alpha+1}\bigl(|\lambda|;t\bigr)+\sum_{j=1}^{2}|\nu_j|\bigl|\varphi_j(t)\bigr|\eta^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;\eta\bigr) \\
&\qquad+\sum_{j=1}^{2}|\mu_j|\bigl|\varphi_j(t)\bigr|T^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;T\bigr)\Biggr) \\
&\qquad\times\bigl(\gamma_f(t)\psi_f\bigl(|x|+|y|\bigr)+M_f\bigr)+\bigl(\|\varphi_1\|+\|\varphi_2\|\bigr)T\bigl(L_h\bigl(|x|+|y|\bigr)+M_h\bigr) \\
&\quad\le\bigl(\|\gamma_f\|\psi_f\bigl(\bigl\|(x,y)\bigr\|\bigr)+\|\gamma_h\|\psi_h\bigl(\bigl\|(x,y)\bigr\|\bigr)\bigr)R^{\alpha}.
\end{aligned}$$

Similarly,

$$\bigl|T_2(x,y)(t)\bigr|\le\bigl(\|\gamma_f\|\psi_f\bigl(\bigl\|(x,y)\bigr\|\bigr)+\|\gamma_g\|\psi_g\bigl(\bigl\|(x,y)\bigr\|\bigr)\bigr)R^{\beta}.$$

It follows that $\mathfrak{T}(B_r)$ is bounded.

Step 2: Next we show that $\mathfrak{T}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $(x,y)\in B_r$. Then we obtain

$$\begin{aligned}
&\bigl|T_1(x,y)(t_1)-T_1(x,y)(t_2)\bigr| \\
&\quad\le\biggl|\int_0^{t_1}(t_1-r)^{\alpha-1}E_{1,\alpha}(-\lambda;t_1-r)f_1\bigl(r,x(r),y(r)\bigr)\,dr \\
&\qquad-\int_0^{t_2}(t_2-r)^{\alpha-1}E_{1,\alpha}(-\lambda;t_2-r)f_1\bigl(r,x(r),y(r)\bigr)\,dr\biggr| \\
&\qquad+\sum_{j=1}^{2}|\nu_j|\bigl|\varphi_j(t_1)-\varphi_j(t_2)\bigr|\eta^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;\eta\bigr)\bigl\|f_i\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\| \\
&\qquad+\sum_{j=1}^{2}|\mu_j|\bigl|\varphi_j(t_1)-\varphi_j(t_2)\bigr|T^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;T\bigr)\bigl\|f_i\bigl(\cdot,x(\cdot),y(\cdot)\bigr)\bigr\|_{\infty} \\
&\qquad+\sum_{j=1}^{2}\bigl|\varphi_j(t_1)-\varphi_j(t_2)\bigr|\int_0^T\bigl|h_i\bigl(r,x(r),y(r)\bigr)\bigr|\,dr.
\end{aligned}\tag{3.7}$$

It is clear that the last three terms approach to zero independently of $(x,y) \in B_r$ as $t_1 \to t_2$. Now, we estimate the first term of (3.7):

$$\begin{aligned}
&\biggl| \int_0^{t_1} (t_1 - r)^{\alpha-1} E_{1,\alpha}(-\lambda; t_1 - r) f_1\bigl(r, x(r), y(r)\bigr)\, dr \\
&\qquad - \int_0^{t_2} (t_2 - r)^{\alpha-1} E_{1,\alpha}(-\lambda; t_2 - r) f_1\bigl(r, x(r), y(r)\bigr)\, dr \biggr| \\
&\quad = \biggl| \int_0^{t_1} e^{-\lambda(t_1-s)} \bigl(I^{\alpha-1} f_1\bigl(\cdot, x(\cdot), y(\cdot)\bigr)\bigr)(s)\, ds - \int_0^{t_2} e^{-\lambda(t_2-s)} \bigl(I^{\alpha-1} f_1\bigl(\cdot, x(\cdot), y(\cdot)\bigr)\bigr)(s)\, ds \biggr| \\
&\quad \le \biggl| \int_{t_1}^{t_2} e^{-\lambda(t_1-s)} \bigl(I^{\alpha-1} f_1\bigl(\cdot, x(\cdot), y(\cdot)\bigr)\bigr)(s)\, ds \biggr| \\
&\qquad + \biggl| \int_0^{t_1} \bigl[e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\bigr] \bigl(I^{\alpha-1} f_1\bigl(\cdot, x(\cdot), y(\cdot)\bigr)\bigr)(s)\, ds \biggr|.
\end{aligned}$$

Obviously, the right-hand side of the above inequality tends to zero independently of $(x,y) \in B_r$ as $t_1 \to t_2$. A similar result is true for $T_1(x,y)$. As $\mathfrak{T}$ is uniformly bounded and equicontinuous, therefore it follows by the Arzelá–Ascoli theorem that $\mathfrak{T} : C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray–Schauder nonlinear alternative once we have proved the boundedness of the set of all solutions to equations $(x,y) = \theta\mathfrak{T}(x,y)$ for $0 \le \theta \le 1$.

Let (x,y) be a solution. Then, using the computations employed in proving that $\mathfrak{T}$ is bounded, we have

$$\begin{aligned}
\bigl|(x,y)(t)\bigr| &= \theta \bigl|\mathfrak{T}(x,y)(t)\bigr| \\
&\le \bigl(\|\gamma_f\| \psi_f\bigl(\bigl\|(x,y)\bigr\|\bigr) + \|\gamma_h\| \psi_h\bigl(\bigl\|(x,y)\bigr\|\bigr)\bigr) R^\alpha \\
&\quad + \bigl(\|\gamma_f\| \psi_f\bigl(\bigl\|(x,y)\bigr\|\bigr) + \|\gamma_g\| \psi_g\bigl(\bigl\|(x,y)\bigr\|\bigr)\bigr) R^\beta.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\bigl\|(x,y)\bigr\|_\infty \\
&\quad /\bigl(\bigl(\|\gamma_f\|_\infty \psi_f\bigl(\bigl\|(x,y)\bigr\|_\infty\bigr) + \|\gamma_h\|_\infty \psi_h\bigl(\bigl\|(x,y)\bigr\|_\infty\bigr)\bigr) R^\alpha \\
&\quad + \bigl(\|\gamma_f\|_\infty \psi_f\bigl(\bigl\|(x,y)\bigr\|_\infty\bigr) + \|\gamma_g\|_\infty \psi_g\bigl(\bigl\|(x,y)\bigr\|_\infty\bigr)\bigr) R^\beta\bigr) \le 1.
\end{aligned}$$

In view of (A_4), there exists M such that $\|(x,y)\|_\infty \ne M$. Let us set

$$\mathfrak{U} = \bigl\{(x,y) \in C\bigl([0,T],\mathbb{R}\bigr) \times C\bigl([0,T],\mathbb{R}\bigr) : \bigl\|(x,y)\bigr\|_\infty < M \bigr\}.$$

Note that the operator $\mathfrak{T} : \overline{\mathfrak{U}} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathfrak{U}$, there is no $(x,y) \in \partial\mathfrak{U}$ such that $(x,y) = \theta\mathfrak{T}(x,y)$ for some $0 < \theta < 1$. Consequently, by the nonlinear alternative of Leray–Schauder type, we deduce that $\mathfrak{T}$ has a fixed point $(x,y) \in \overline{\mathfrak{U}}$ which is a solution of problem (1.1). This completes the proof. □

4 Ulam–Hyers stability

In this section, we discuss the Ulam–Hyers stability for problem (1.1) by means of integral representation of its solution given by

$$x(t) = T_1(x,y)(t), \qquad y(t) = T_2(x,y)(t),$$

where T_1 and T_2 are defined by (3.1) and (3.2).

Define the following nonlinear operators $Q_1, Q_2 : C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$:

$$Q_1(x,y)(t) := \bigl(D^{\alpha} + \lambda D^{\alpha-1}\bigr)x(t) - f_1\bigl(t,x(t),y(t)\bigr),$$
$$Q_2(x,y)(t) := \bigl(D^{\beta} + \lambda D^{\beta-1}\bigr)y(t) - f_2\bigl(t,x(t),y(t)\bigr).$$

For some $\varepsilon_1, \varepsilon_2 > 0$, we consider the following inequality:

$$\bigl\|Q_1(x,y)\bigr\| \le \varepsilon_1, \qquad \bigl\|Q_2(x,y)\bigr\| \le \varepsilon_2. \tag{4.1}$$

Definition 3 The coupled system (1.1) is said to be Ulam–Hyers stable if there exist $V_1, V_2 > 0$ such that, for every solution $(x^*,y^*) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of inequality (4.1), there exists a unique solution $(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ of problem (1.1) with

$$\bigl\|(x,y) - \bigl(x^*,y^*\bigr)\bigr\|_{\infty} \le V_1\varepsilon_1 + V_2\varepsilon_2. \tag{4.2}$$

Theorem 3 *Let the assumptions of Theorem 1 hold. Then problem (1.1) is Ulam–Hyers stable.*

Proof Let $(x,y) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ be the solution of problem (1.1) satisfying (3.1) and (3.2). Let (x^*,y^*) be any solution satisfying (4.1):

$$\bigl(D^{\alpha} + \lambda D^{\alpha-1}\bigr)x^*(t) = f_1\bigl(t,x^*(t),y^*(t)\bigr) + Q_1\bigl(x^*,y^*\bigr)(t),$$
$$\bigl(D^{\beta} + \lambda D^{\beta-1}\bigr)y^*(t) = f_2\bigl(t,x^*(t),y^*(t)\bigr) + Q_2\bigl(x^*,y^*\bigr)(t).$$

So

$$\begin{aligned} x^*(t) = T_1\bigl(x^*,y^*\bigr)(t) &+ \int_0^t (t-r)^{\alpha-1}E_{1,\alpha}(-\lambda;t-r)Q_1\bigl(x^*,y^*\bigr)(r)\,dr \\ &+ \sum_{j=1}^{2} \nu_j\varphi_j(t)\int_0^{\eta} (\eta-r)^{\alpha-j}E_{1,\alpha+j-1}(-\lambda;\eta-r)Q_1\bigl(x^*,y^*\bigr)(r)\,dr \\ &+ \sum_{j=1}^{2} \mu_j\varphi_j(t)\int_0^{T} (T-r)^{\alpha-j}E_{1,\alpha+j-1}(-\lambda;T-r)Q_1\bigl(x^*,y^*\bigr)(r)\,dr. \end{aligned}$$

It follows that

$$\begin{aligned} &\bigl|T_1\bigl(x^*,y^*\bigr)(t) - x^*(t)\bigr| \\ &\quad\le \int_0^t (t-r)^{\alpha-1}E_{1,\alpha}\bigl(|\lambda|;t-r\bigr)\,dr\varepsilon_1 + \sum_{j=1}^{2} |\nu_j|\|\varphi_j\| \int_0^{\eta} (\eta-r)^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;\eta-r\bigr)\,dr\varepsilon_1 \end{aligned}$$

$$\begin{aligned}&+\sum_{j=1}^{2}|\mu_j|\|\varphi_j\|\int_0^T (T-r)^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;T-r\bigr)\,dr\varepsilon_1\\
&\le\Biggl(T^{\alpha}E_{1,\alpha+1}\bigl(|\lambda|;T\bigr)+\sum_{j=1}^{2}|\nu_j|\|\varphi_j\|\eta^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;\eta\bigr)\\
&\quad+\sum_{j=1}^{2}|\mu_j|\|\varphi_j\|T^{\alpha-j}E_{1,\alpha+j-1}\bigl(|\lambda|;T\bigr)\Biggr)\varepsilon_1\\
&=:U^{\alpha}\varepsilon_1.\end{aligned}$$

Similarly,

$$\begin{aligned}&\bigl|T_2\bigl(x^*,y^*\bigr)(t)-y^*(t)\bigr|\\
&\le\Biggl(T^{\beta}E_{1,\beta+1}\bigl(|\lambda|;T\bigr)+\sum_{j=1}^{2}|\nu_j|\|\varphi_j\|\eta^{\beta-j}E_{1,\beta+j-1}\bigl(|\lambda|;\eta\bigr)\\
&\quad+\sum_{j=1}^{2}|\mu_j|\|\varphi_j\|T^{\beta-j}E_{1,\beta+j-1}\bigl(|\lambda|;T\bigr)\Biggr)\varepsilon_2\\
&=:U^{\beta}\varepsilon_1.\end{aligned}$$

Therefore, we deduce by the fixed point property of the operator $\mathfrak{T}$, given by (3.1) and (3.2), that

$$\begin{aligned}\bigl|x(t)-x^*(t)\bigr|&=\bigl|x(t)-T_1\bigl(x^*,y^*\bigr)(t)+T_1\bigl(x^*,y^*\bigr)(t)-x^*(t)\bigr|\\
&\le\bigl|T_1(x,y)(t)-T_1\bigl(x^*,y^*\bigr)(t)\bigr|+\bigl|T_1\bigl(x^*,y^*\bigr)(t)-x^*(t)\bigr|\\
&\le(L_f+L_h)U^{\alpha}\bigl\|(x,y)-\bigl(x^*,y^*\bigr)\bigr\|_{\infty}+U^{\alpha}\varepsilon_1,\end{aligned}\tag{4.3}$$

and similarly

$$\begin{aligned}\bigl|y(t)-y^*(t)\bigr|&=\bigl|y(t)-T_2\bigl(x^*,y^*\bigr)(t)+T_2\bigl(x^*,y^*\bigr)(t)-y^*(t)\bigr|\\
&\le(L_f+L_g)U^{\beta}\bigl\|(x,y)-\bigl(x^*,y^*\bigr)\bigr\|_{\infty}+U^{\beta}\varepsilon_2.\end{aligned}\tag{4.4}$$

From (4.3) and (4.4) it follows that

$$\bigl\|(x,y)-\bigl(x^*,y^*\bigr)\bigr\|_{\infty}\le\bigl((L_f+L_h)U^{\alpha}+(L_f+L_g)U^{\beta}\bigr)\bigl\|(x,y)-\bigl(x^*,y^*\bigr)\bigr\|_{\infty}+U^{\alpha}\varepsilon_1+U^{\beta}\varepsilon_2,$$

and

$$\begin{aligned}\bigl\|(x,y)-\bigl(x^*,y^*\bigr)\bigr\|_{\infty}&\le\frac{U^{\alpha}\varepsilon_1+U^{\beta}\varepsilon_2}{1-((L_f+L_h)U^{\alpha}+(L_f+L_g)U^{\beta})}\\
&=V_1\varepsilon_1+V_2\varepsilon_2,\end{aligned}$$

with

$$V_1=\frac{U^{\alpha}}{1-((L_f+L_h)U^{\alpha}+(L_f+L_g)U^{\beta})},$$

$$V_2 = \frac{U^\beta}{1-((L_f+L_h)U^\alpha + (L_f+L_g)U^\beta)}.$$

Thus, problem (1.1) is Ulam–Hyers stable. □

5 Application

We consider the following fractional order coupled system:

$$\begin{cases} (D^\alpha + \lambda D^{\alpha-1})x(t) = L_f \frac{|y(t)|}{1+|y(t)|}, \quad 1<\alpha\leq 2, 0\leq t\leq T, \\ (D^\beta + \lambda D^{\beta-1})y(t) = L_f(\sin x(t) + (\cos t)x(t)), \quad 1<\beta\leq 2, 0\leq t\leq T, \\ \nu_1 x(\eta) + \mu_1 x(T) = L_h \int_0^T \frac{|x(t)|}{11+|x(t)|}\,ds, \\ \nu_1 y(\eta) + \mu_1 y(T) = L_h \int_0^T (\sin y(t) + \cos y(t))\,ds, \\ \nu_2 x'(\eta) + \mu_2 x'(T) = L_g \int_0^T \frac{|x(t)|}{21+|x(t)|}\,ds, \\ \nu_2 y'(\eta) + \mu_2 y'(T) = L_g \int_0^T (\sin y(t) + \cos y(t))\,ds. \end{cases}$$

Here

$$f_1(t,x,y) = L_f\frac{|y|}{1+|y|}, \qquad f_2(t,x,y) = L_f\bigl(\sin x + (\cos t)x\bigr), \qquad h_1(x) = L_h\frac{|x|}{11+|x|},$$
$$h_2(y) = L_h(\sin y + \cos y), \qquad g_1(x) = L_g\frac{|x|}{21+|x|}, \qquad g_2(y) = L_g(\sin y + \cos y).$$

As

$$\bigl|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)\bigr| \leq L_f|y_1-y_2|, \qquad \bigl|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)\bigr| \leq L_f|x_1-x_2|,$$
$$\bigl|h_1(t,x_1) - h_1(t,x_2)\bigr| \leq L_h|x_1-x_2|, \qquad \bigl|h_2(t,y_1) - h_2(t,y_2)\bigr| \leq L_h|y_1-y_2|,$$
$$\bigl|g_1(t,x_1) - g_1(t,x_2)\bigr| \leq L_g|x_1-x_2|, \qquad \bigl|g_2(t,y_1) - g_2(t,y_2)\bigr| \leq L_g|y_1-y_2|,$$

therefore (A_1) is satisfied. It is obvious that $L_f, L_h, L_g > 0$ can be chosen so that condition (A_2) is satisfied. Therefore, coupled system (1.1) has a unique solution and is Ulam–Hyers stable.

6 Conclusion

Here we have studied the existence and uniqueness of the solutions as well as the Ulam–Hyers stability for a coupled sequential fractional system with integral boundary conditions. As a future work, one can generalize different concepts of stability and existence results to an impulsive fractional system, a neutral time-delay system/inclusion, and a time-delay system/inclusion with finite delay.

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