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Some results on quantum Hahn integral inequalities

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Abstract

In this paper the quantum Hahn difference operator and the quantum Hahn integral operator are defined via the quantum shift operator ${}_\theta\Phi_q(t) = qt + (1-q)\theta$, $t \in [a,b]$, $\theta = \omega/(1-q) + a$, $0 < q < 1$, $\omega \geq 0$. Some new fractional integral inequalities are established by using the quantum Hahn integral for one and two functions bounded by quantum integrable functions. The Hermite–Hadamard type of ordinary and fractional quantum Hahn integral inequalities as well as the Pólya–Szegö type fractional Hahn integral inequalities and the Grüss–Čebyšev type fractional Hahn integral inequality are also presented.

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1 Introduction and preliminaries

Let be f defined on an interval $I \subseteq \mathbb{R}$ containing $\omega_0 := \frac{\omega}{1-q}$. The Hahn difference operator $D_{q,\omega}$, introduced in [1], is defined as

$$D_{q,\omega}f(t) = \begin{cases} \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, & t \neq \omega_0, \\ f'(\omega_0), & t = \omega_0, \end{cases} \tag{1}$$

provided that f is differentiable at ω_0, where $q \in (0,1)$ and $\omega \geq 0$ are fixed.

The Hahn difference operator unifies (in the limit) the two most well-known and used quantum difference operators: the Jackson q-difference derivative D_q [2], where $q \in (0,1)$, defined by

$$D_qf(t) = \begin{cases} \frac{f(t)-f(qt)}{t(1-q)}, & t \neq 0, \\ f'(0), & t = 0, \end{cases} \tag{2}$$

provided that $f'(0)$ exists, for $\omega = 0$, and the forward difference D_ω for $q \to 1$, defined by

$$D_\omega f(t) = \frac{f(t+\omega)-f(t)}{\omega}, \tag{3}$$

where $\omega > 0$ is fixed. The Hahn difference operator is a successful tool for constructing families of orthogonal polynomials and investigating some approximation problems (cf. [3–6]). For some recent results to boundary value problems for Hahn difference operators, we refer to papers [7–10] and the references cited therein.

Let $[a,b] \subseteq \mathbb{R}$ be an interval. The two quantum numbers $0 < q < 1$, $\omega \geq 0$ can generate a point θ of Hahn calculus on an interval $[a,b]$ by

$$\theta = \frac{\omega}{1-q} + a, \tag{4}$$

which means that $\theta \in [a,b]$ for all results of our analysis. The quantum Hahn shifting operator is defined by

$${}_\theta\Phi_q(t) = qt + (1-q)\theta, \quad t \in [a,b]. \tag{5}$$

It is easy to see that the iterated k-times quantum shifting is presented by

$${}_\theta\Phi_q^k(t) = {}_\theta\Phi_q^{k-1}\big({}_\theta\Phi_q(t)\big) = q^k t + \big(1-q^k\big)\theta,$$

with ${}_\theta\Phi_q^0(t) = t$ for $t \in [a,b]$. Let us state the definitions of quantum Hahn calculus on an interval $[a,b]$ which are the results in [11] modified according to notation (5).

Definition 1 Let f be a function defined on $[a,b]$. The quantum Hahn difference operator is defined by

$${}_aD_{q,\omega}f(t) = \begin{cases} \frac{f(t)-f({}_\theta\Phi_q(t))}{t-{}_\theta\Phi_q(t)}, & t \neq \theta, \\ f'(\theta), & t = \theta, \end{cases} \tag{6}$$

provided that f is differentiable at θ.

Definition 2 Assume $f : [a,b] \to \mathbb{R}$ is a given function and two points $c,d \in [a,b]$. The q,ω-quantum Hahn integral of f from c to d is defined by

$$\int_c^d f(s)_a\, d_{q,\omega}s := \int_\theta^d f(s)_a\, d_{q,\omega}s - \int_\theta^c f(s)_a\, d_{q,\omega}s, \tag{7}$$

where

$$\int_\theta^t f(s)_a\, d_{q,\omega}s = \big[t - {}_\theta\Phi_q(t)\big] \sum_{i=0}^{\infty} q^i f\big({}_\theta\Phi_q^i(t)\big) \tag{8}$$

for $t \in [a,b]$, provided that the series converge at $t = c$ and $t = d$. The function f is called q, ω-integrable on $[a,b]$ if (8) exists for all $t \in [a,b]$.

Before going to state the definitions of fractional quantum Hahn calculus on an interval $[a,b]$, we should introduce the θ-power function which is defined by

$$(n-m)_\theta^{(0)} = 1, \qquad (n-m)_\theta^{(k)} = \prod_{i=0}^{k-1}\big(n - {}_\theta\Phi_q^i(m)\big), \quad k \in \mathbb{N} \cup \{\infty\}. \tag{9}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(n-m)_{\theta}^{(\gamma)} = \prod_{i=0}^{\infty} \frac{(n - {}_{\theta}\Phi_q^i(m))}{(n - {}_{\theta}\Phi_q^{\gamma+i}(m))}, \tag{10}$$

with ${}_{\theta}\Phi_q^{\sigma}(m) = q^{\sigma} m + (1-q^{\sigma})\theta$, $\sigma \in \mathbb{R}$.

The q-gamma function is defined by

$$\Gamma_q(\gamma) = \frac{(1-q)_0^{(\gamma-1)}}{(1-q)^{\gamma-1}}, \quad \gamma \in \mathbb{R} \setminus \{0,-1,-2,\dots\}. \tag{11}$$

Obviously, $\Gamma_q(\gamma+1) = [\gamma]_q \Gamma_q(\gamma)$, where $[x]_q = (1-q^x)/(1-q)$, $x \in \mathbb{R}$, is the quantum number or q-number.

Now the definitions of Riemann–Liouville type of fractional derivative and integral of quantum Hahn calculus on interval $[a,b]$ are presented in the following definitions. See [12].

Definition 3 The fractional quantum Hahn difference of Riemann–Liouville type of a function $f : [a,b] \to \mathbb{R}$ of order $\alpha \geq 0$ is defined by $({}_aD_{q,\omega}^0 f)(t) = f(t)$ and

$$\left({}_aD_{q,\omega}^{\alpha} f\right)(t) = \frac{1}{\Gamma_q(n-\alpha)} {}_aD_{q,\omega}^n \int_a^t \left(t - {}_{\theta}\Phi_q(s)\right)_{\theta}^{(n-\alpha-1)} f(s)_a \, d_{q,\omega}s, \quad \alpha > 0,$$

where n is the smallest integer greater than or equal to α.

Definition 4 Let $\alpha \geq 0$ and $f : [a,b] \to \mathbb{R}$ be a function. The fractional quantum Hahn integral of Riemann–Liouville type is defined by $({}_aI_{q,\omega}^0 f)(t) = f(t)$ and

$$\left({}_aI_{q,\omega}^{\alpha} f\right)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t \left(t - {}_{\theta}\Phi_q(s)\right)_{\theta}^{(\alpha-1)} f(s)_a \, d_{q,\omega}s, \quad \alpha > 0, t \in [a,b],$$

provided the right-hand side exists.

Theorem 1 [12] *Let* $\alpha, \beta \in \mathbb{R}^+$, $\lambda \in (-1,\infty)$, *and* $\theta \in [a,b]$. *The following formulas hold*:

(i) $({}_aI_{q,\omega}^{\alpha} (x-a)_{\theta}^{(\lambda)})(t) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (t-a)_{\theta}^{(\alpha+\lambda)}$;

(ii) $({}_aD_{q,\omega}^{\alpha} (x-a)_{\theta}^{(\lambda)})(t) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)} (t-a)_{\theta}^{(\lambda-\alpha)}$.

Fractional differential equations attracted much attention in recent years due to their widespread application in many fields of science and engineering, such as fluid flow, signal and image processing, fractals theory, control theory, electromagnetic theory, fitting of experimental data, optics, potential theory, biology, chemistry, diffusion, and viscoelasticity, etc; see [13–15]. One of the aspects which are nowadays very much popular among the scientists for research is the integral inequalities with applications. The applications of inequalities are very much common for fixed point theorems and existence and uniqueness of solutions for differential equations.

In this paper we prove some new quantum Hahn fractional integral inequalities by using the quantum Hahn integral. The main results are included in Sect. 2, where inequalities are obtained for quantum integrable functions bounded by quantum integrable functions as well as quantum Hermite–Hadamard type Hahn inequalities. In Sect. 3, inequalities are produced for two quantum integrable functions bounded by quantum integrable functions as well as the Pólya–Szegö type fractional Hahn integral inequalities and the Grüss–Čebyšev type fractional Hahn integral inequality.

2 Some results on fractional integral inequalities for one unknown function

Let the points $\theta_i \in [a,b]$, $i = 1,2$, be defined by

$$\theta_i = \frac{\omega_i}{1-q_i} + a \tag{12}$$

for quantum numbers $0 < q_i < 1$ and $\omega_i \geq 0$, $i = 1,2$.

Theorem 2 *Let f be a q_i, ω_i-integrable function on $[a,b]$, $i = 1,2$. In addition, we assume that*:

(H_1) *There exist two q_i, ω_i-integrable functions φ_i, $i = 1,2$, on $[a,b]$ such that*:

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t) \quad \textit{for all } t \in [a,b].$$

Then, for $t \in [a,b]$, $\alpha, \beta > 0$, $0 < q_i < 1$, and $\omega_i \geq 0$, $i = 1,2$, we have

$$\begin{aligned} &\left({}_aI^{\beta}_{q_2,\omega_2}\varphi_2\right)(t)\left({}_aI^{\alpha}_{q_1,\omega_1}f\right)(t) + \left({}_aI^{\beta}_{q_2,\omega_2}f\right)(t)\left({}_aI^{\alpha}_{q_1,\omega_1}\varphi_1\right)(t) \\ &\quad \geq \left({}_aI^{\beta}_{q_2,\omega_2}\varphi_2\right)(t)\left({}_aI^{\alpha}_{q_1,\omega_1}\varphi_1\right)(t) + \left({}_aI^{\beta}_{q_2,\omega_2}f\right)(t)\left({}_aI^{\alpha}_{q_1,\omega_1}f\right)(t). \end{aligned} \tag{13}$$

Proof From condition (H_1), for all $\tau, \rho \in [a,b]$, we obtain

$$\left(\varphi_2(\tau) - f(\tau)\right)\left(f(\rho) - \varphi_1(\rho)\right) \geq 0,$$

which yields

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{14}$$

Multiplying both sides of (14) by $(t - {}_{\theta_1}\Phi_{q_1}(\rho))^{(\alpha-1)}_{\theta_1}/\Gamma_{q_1}(\alpha)$, $\rho \in [a,t)$ and taking the q_1, ω_1-integration with respect to ρ on $[a,t)$, we obtain

$$\begin{aligned} &\varphi_2(\tau)\int_a^t \frac{(t - {}_{\theta_1}\Phi_{q_1}(\rho))^{(\alpha-1)}_{\theta_1}}{\Gamma_{q_1}(\alpha)} f(\rho)\,{}_a d_{q_1,\omega_1}\rho + f(\tau)\int_a^t \frac{(t - {}_{\theta_1}\Phi_{q_1}(\rho))^{(\alpha-1)}_{\theta_1}}{\Gamma_{q_1}(\alpha)} \varphi_1(\rho)\,{}_a d_{q_1,\omega_1}\rho \\ &\quad \geq \varphi_2(\tau)\int_a^t \frac{(t - {}_{\theta_1}\Phi_{q_1}(\rho))^{(\alpha-1)}_{\theta_1}}{\Gamma_{q_1}(\alpha)} \varphi_1(\rho)\,{}_a d_{q_1,\omega_1}\rho \\ &\qquad + f(\tau)\int_a^t \frac{(t - {}_{\theta_1}\Phi_{q_1}(\rho))^{(\alpha-1)}_{\theta_1}}{\Gamma_{q_1}(\alpha)} f(\rho)\,{}_a d_{q_1,\omega_1}\rho, \end{aligned}$$

which leads to

$$\begin{aligned} &\varphi_2(\tau)\left({}_aI^{\alpha}_{q_1,\omega_1}f\right)(t) + f(\tau)\left({}_aI^{\alpha}_{q_1,\omega_1}\varphi_1\right)(t) \\ &\quad \geq \varphi_2(\tau)\left({}_aI^{\alpha}_{q_1,\omega_1}\varphi_1\right)(t) + f(\tau)\left({}_aI^{\alpha}_{q_1,\omega_1}f\right)(t). \end{aligned} \tag{15}$$

Multiplying both sides of (15) by $(t - {}_{\theta_2}\Phi_{q_2}(\tau))_{\theta_2}^{(\beta-1)}/\Gamma_{q_2}(\beta)$, $\tau \in [a,t)$ and taking the q_2, ω_2-integration with respect to τ on $[a,t)$, we get the desired result in (13). □

Corollary 1 *Let φ_1 and φ_2 be polynomial θ-power functions defined by*

$$\varphi_1(t) = \sum_{i=0}^{n} c_i (t-a)_{\theta_1}^{(i)}, \qquad \varphi_2(t) = \sum_{j=0}^{m} d_j (t-a)_{\theta_2}^{(j)},$$

satisfying (H_1). Then, for $t \in [a,b]$, $\alpha, \beta > 0$, $0 < q_i < 1$, and $\omega_i \geq 0$, $i = 1,2$, we have

$$\begin{aligned}
&\left({}_aI_{q_1,\omega_1}^{\alpha} f\right)(t) \sum_{j=0}^{m} d_j \frac{\Gamma_{q_2}(j+1)}{\Gamma_{q_2}(\beta+j+1)}(t-a)_{\theta_2}^{(\beta+j)} \\
&\quad + \left({}_aI_{q_2,\omega_2}^{\beta} f\right)(t) \sum_{i=0}^{n} c_i \frac{\Gamma_{q_1}(i+1)}{\Gamma_{q_1}(\alpha+i+1)}(t-a)_{\theta_1}^{(\alpha+i)} \\
&\geq \left(\sum_{j=0}^{m} d_j \frac{\Gamma_{q_2}(j+1)}{\Gamma_{q_2}(\beta+j+1)}(t-a)_{\theta_2}^{(\beta+j)}\right)\left(\sum_{i=0}^{n} c_i \frac{\Gamma_{q_1}(i+1)}{\Gamma_{q_1}(\alpha+i+1)}(t-a)_{\theta_1}^{(\alpha+i)}\right) \\
&\quad + \left({}_aI_{q_2,\omega_2}^{\beta} f\right)(t)\left({}_aI_{q_1,\omega_1}^{\alpha} f\right)(t).
\end{aligned}$$

Corollary 2 *Let f be a q_i, ω_i-integrable function on $[a,b]$, $i = 1,2$, satisfying*

(H_2) *$m \leq f(t) \leq M$ for all $t \in [a,b]$ and $m, M \in \mathbb{R}$.*

Then, for $\alpha, \beta > 0$, we have the following estimate:

$$M\left({}_aI_{q_1,\omega_1}^{\alpha} f\right)(t) + m\left({}_aI_{q_2,\omega_2}^{\beta} f\right)(t) \geq mM + \left({}_aI_{q_2,\omega_2}^{\beta} f\right)(t)\left({}_aI_{q_1,\omega_1}^{\alpha} f\right)(t).$$

Further, if $\alpha = \beta$, $q_1 = q_2$, and $\omega_1 = \omega_2$, then the following inequality holds:

$$(M+m)\left({}_aI_{q_1,\omega_1}^{\alpha} f\right)(t) \geq mM + \left({}_aI_{q_1,\omega_1}^{\alpha} f\right)^2(t).$$

Theorem 3 *Assume that:*

(H_3) *$0 < \varphi_1(t) \leq f(t) \leq \varphi_2(t)$ for all $t \in [a,b]$.*

Then the following inequalities

$$\begin{aligned}
&\frac{(t-a)_{\theta_2}^{(\beta)}}{\Gamma_{q_2}(\beta+1)}\left({}_aI_{q_1,\omega_1}^{\alpha} f^2\right)(t) + \frac{(t-a)_{\theta_1}^{(\alpha)}}{\Gamma_{q_1}(\alpha+1)}\left({}_aI_{q_2,\omega_2}^{\beta} \varphi_2^2\right)(t) \\
&\quad \geq 2\left({}_aI_{q_1,\omega_1}^{\alpha} f\right)(t)\left({}_aI_{q_2,\omega_2}^{\beta} \varphi_2\right)(t)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\frac{(t-a)_{\theta_2}^{(\beta)}}{\Gamma_{q_2}(\beta+1)}\left({}_aI_{q_1,\omega_1}^{\alpha} \varphi_1^2\right)(t) + \frac{(t-a)_{\theta_1}^{(\alpha)}}{\Gamma_{q_1}(\alpha+1)}\left({}_aI_{q_2,\omega_2}^{\beta} f^2\right)(t) \\
&\quad \geq 2\left({}_aI_{q_1,\omega_1}^{\alpha} \varphi_1\right)(t)\left({}_aI_{q_2,\omega_2}^{\beta} f\right)(t)
\end{aligned} \tag{17}$$

are fulfilled.

Proof From the well-known inequality

$$\frac{a}{b}+\frac{b}{a}\geq 2,\quad a,b\in\mathbb{R}^{+},$$

we can write

$$\frac{\varphi_2(\tau)}{f(\rho)}+\frac{f(\rho)}{\varphi_2(\tau)}\geq 2,$$

which leads to

$$\varphi_2^2(\tau)+f^2(\rho)\geq 2f(\rho)\varphi_2(\tau). \tag{18}$$

Multiplying both sides of (18) by $(t-{}_{\theta_1}\Phi_{q_1}(\rho))_{\theta_1}^{(\alpha-1)}/\Gamma_{q_1}(\alpha)$, $\rho\in[a,t)$ and applying q_1,ω_1-integration, we deduce the inequality

$$\frac{(t-a)_{\theta_1}^{(\alpha)}}{\Gamma_{q_1}(\alpha+1)}\varphi_2^2(\tau)+\big({}_aI_{q_1,\omega_1}^{\alpha}f^2\big)(t)\geq 2\varphi_2(\tau)\big({}_aI_{q_1,\omega_1}^{\alpha}f\big)(t).$$

By multiplying both sides of the last inequality with $(t-{}_{\theta_2}\Phi_{q_2}(\tau))_{\theta_2}^{(\beta-1)}/\Gamma_{q_2}(\beta)$, $\tau\in[a,t)$ and applying q_2,ω_2-integration, we obtain the result in (16).

To prove (17), we use the fact that

$$\frac{\varphi_1(\rho)}{f(\tau)}+\frac{f(\tau)}{\varphi_1(\rho)}\geq 2.$$

By the same method, we get inequality in (17). The proof is completed. □

Corollary 3 *Suppose that*:

(H_4) $0<m\leq f(t)\leq M$ *for all* $t\in[a,b]$.

Then the inequalities

$$\begin{aligned}&\frac{(t-a)_{\theta_2}^{(\beta)}}{\Gamma_{q_2}(\beta+1)}\big({}_aI_{q_1,\omega_1}^{\alpha}f^2\big)(t)+M^2\frac{(t-a)_{\theta_1}^{(\alpha)}(t-a)_{\theta_2}^{(\beta)}}{\Gamma_{q_1}(\alpha+1)\Gamma_{q_2}(\beta+1)}\\&\quad\geq 2M\frac{(t-a)_{\theta_2}^{(\beta)}}{\Gamma_{q_2}(\beta+1)}\big({}_aI_{q_1,\omega_1}^{\alpha}f\big)(t)\end{aligned} \tag{19}$$

and

$$\begin{aligned}&m^2\frac{(t-a)_{\theta_2}^{(\beta)}(t-a)_{\theta_1}^{(\alpha)}}{\Gamma_{q_2}(\beta+1)\Gamma_{q_1}(\alpha+1)}+\frac{(t-a)_{\theta_1}^{(\alpha)}}{\Gamma_{q_1}(\alpha+1)}\big({}_aI_{q_2,\omega_2}^{\beta}f^2\big)(t)\\&\quad\geq 2m\frac{(t-a)_{\theta_1}^{(\alpha)}}{\Gamma_{q_1}(\alpha+1)}\big({}_aI_{q_2,\omega_2}^{\beta}f\big)(t)\end{aligned} \tag{20}$$

are true.

Theorem 4 *Let $f(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ be (q,ω)-integrable functions on $[a,b]$ satisfying (H_3). Then, for $\alpha,\beta,\gamma>0$, we have the inequality*

$$\begin{aligned}
&\bigl({}_aI^{\alpha}_{q,\omega}\varphi_2^2\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}f^2\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_2^2\bigr)(t)+\bigl({}_aI^{\alpha}_{q,\omega}f^2\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}\varphi_1 f\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_1\varphi_2\bigr)(t)\\
&\qquad+\bigl({}_aI^{\alpha}_{q,\omega}\varphi_2 f\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}\varphi_1^2\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_1\varphi_2\bigr)(t)\\
&\qquad+\bigl({}_aI^{\alpha}_{q,\omega}\varphi_2 f\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}\varphi_1 f\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_1^2\bigr)(t)\\
&\quad\geq\bigl({}_aI^{\alpha}_{q,\omega}\varphi_2^2\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}\varphi_1 f\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_1\varphi_2\bigr)(t)\\
&\qquad+\bigl({}_aI^{\alpha}_{q,\omega}\varphi_2 f\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}f^2\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_1\varphi_2\bigr)(t)\\
&\qquad+\bigl({}_aI^{\alpha}_{q,\omega}\varphi_2 f\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}\varphi_1 f\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_2^2\bigr)(t)\\
&\qquad+\bigl({}_aI^{\alpha}_{q,\omega}f^2\bigr)(t)\bigl({}_aI^{\beta}_{q,\omega}\varphi_1^2\bigr)(t)\bigl({}_aI^{\gamma}_{q,\omega}\varphi_1^2\bigr)(t).
\end{aligned}\tag{21}$$

Proof From (H_3), we know that

$$\frac{\varphi_2(\rho)}{f(\rho)}\geq 1,\qquad \frac{f(\tau)}{\varphi_1(\tau)}\geq 1\quad\text{and}\quad\frac{\varphi_2(\eta)}{\varphi_1(\eta)}\geq 1.$$

Hence, we get the fact that

$$\left(\frac{\varphi_2(\rho)}{f(\rho)}-\frac{\varphi_1(\tau)}{f(\tau)}\right)\left(\frac{f(\tau)}{\varphi_1(\tau)}-\frac{\varphi_1(\eta)}{\varphi_2(\eta)}\right)\left(\frac{\varphi_2(\eta)}{\varphi_1(\eta)}-\frac{f(\rho)}{\varphi_2(\rho)}\right)\geq 0.$$

It follows that

$$\begin{aligned}
&\varphi_2^2(\rho)f^2(\tau)\varphi_2^2(\eta)+f^2(\rho)\varphi_1(\tau)f(\tau)\varphi_1(\eta)\varphi_2(\eta)\\
&\qquad+\varphi_2(\rho)f(\rho)\varphi_1^2(\tau)\varphi_1(\eta)\varphi_2(\eta)+\varphi_2(\rho)f(\rho)\varphi_1(\tau)f(\tau)\varphi_1^2(\eta)\\
&\quad\geq\varphi_2^2(\rho)\varphi_1(\tau)f(\tau)\varphi_1(\eta)\varphi_2(\eta)+\varphi_2(\rho)f(\rho)f^2(\tau)\varphi_1(\eta)\varphi_2(\eta)\\
&\qquad+\varphi_2(\rho)f(\rho)\varphi_1(\tau)f(\tau)\varphi_2^2(\eta)+f^2(\rho)\varphi_1^2(\tau)\varphi_1^2(\eta).
\end{aligned}$$

Using the method to prove Theorem 2, we obtain the inequality in (21), which finishes the proof. □

Corollary 4 *Suppose that the (q,ω)-integrable function f satisfies (H_4). Then, for $t\in[a,b]$, we have*

$$\begin{aligned}
&\bigl(M^4-m^4\bigr)\bigl({}_aI^{\alpha}_{q,\omega}f^2\bigr)(t)\left(\frac{(t-a)_{\theta}^{(\alpha)}}{\Gamma_q(\alpha+1)}\right)^2\\
&\quad\geq m^2M^2(M-m)\bigl({}_aI^{\alpha}_{q,\omega}f\bigr)(t)\left(\frac{(t-a)_{\theta}^{(\alpha)}}{\Gamma_q(\alpha+1)}\right)^2\\
&\qquad+mM\bigl(M^2-m^2\bigr)\bigl({}_aI^{\alpha}_{q,\omega}f\bigr)^2(t)\frac{(t-a)_{\theta}^{(\alpha)}}{\Gamma_q(\alpha+1)}\\
&\qquad+mM(M-m)\bigl({}_aI^{\alpha}_{q,\omega}f^2\bigr)(t)\bigl({}_aI^{\alpha}_{q,\omega}f\bigr)(t)\frac{(t-a)_{\theta}^{(\alpha)}}{\Gamma_q(\alpha+1)}.
\end{aligned}\tag{22}$$

Next, we will prove the Hermite–Hadamard type of quantum Hahn integral inequality. Let $[a,b] \subset \mathbb{R}$ with $\theta \in [a,b]$ be a given interval containing a point θ. Now we claim that

$$\frac{(b+qa-\omega)}{1+q} \in [a,b]. \tag{23}$$

Since $\theta \in [a,b]$, we have $a+\omega/(1-q) \in [a,b]$, which yields $\omega \leq (1-q)(b-a)$. Hence we obtain

$$b - \left(\frac{b+qa-\omega}{1+q}\right) = \frac{q(b-a)+\omega}{1+q} \leq \frac{(b-a)}{1+q} \leq b-a,$$

which implies that (23) holds.

From Example 3.12, page 9 in [11], we get the following formula:

$$\begin{aligned} \int_a^b t\,{}_a\,d_{q,\omega}t &= \int_a^b \big((t-a)+a\big)\,{}_a\,d_{q,\omega}t \\ &= \frac{(b-a)^2-\omega(b-a)}{1+q} + a(b-a) \\ &= (b-a)\left(\frac{b+qa-\omega}{1+q}\right). \end{aligned} \tag{24}$$

Theorem 5 *Let $f:[a,b] \to \mathbb{R}$ be a convex differentiable function on (a,b), and quantum numbers $0<q<1$, $\omega \geq 0$ with $\theta \in [a,b]$. Then we have*

$$\begin{aligned} f\left(\frac{b+qa-\omega}{1+q}\right) &\leq \frac{1}{(b-a)}\int_a^b f(t)_a\,d_{q,\omega}t \\ &\leq \left[\frac{qf(a)+f(b)}{1+q} - \omega\left(\frac{f(b)-f(a)}{(b-a)(1+q)}\right)\right]. \end{aligned} \tag{25}$$

Proof From the fact that f is a differentiable function on (a,b) and (23), we have that $f'((b+qa-\omega)/(1+q))$ exists. Since f is convex, that is,

$$f\big((1-\lambda)a+\lambda b\big) \leq (1-\lambda)f(a)+\lambda f(b) \tag{26}$$

for all $\lambda \in [0,1]$, we set a tangent line under the curve of f by

$$h(t) = f\left(\frac{b+qa-\omega}{1+q}\right) + f'\left(\frac{b+qa-\omega}{1+q}\right)\left(t - \frac{b+qa-\omega}{1+q}\right),$$

which leads to $h(t) \leq f(t)$ for all $t \in [a,b]$. By taking the (q,ω)-integration and using (24), we have

$$\begin{aligned} \int_a^b h(t)_a\,d_{q,\omega}t &= \int_a^b \left[f\left(\frac{b+qa-\omega}{1+q}\right) + f'\left(\frac{b+qa-\omega}{1+q}\right)\left(t-\frac{b+qa-\omega}{1+q}\right)\right]_a d_{q,\omega}t \\ &= (b-a)f\left(\frac{b+qa-\omega}{1+q}\right) \\ &\quad + f'\left(\frac{b+qa-\omega}{1+q}\right)\left[\int_a^b t_a\,d_{q,\omega}t - (b-a)\left(\frac{b+qa-\omega}{1+q}\right)\right] \end{aligned}$$

$$= (b-a)f\left(\frac{b+qa-\omega}{1+q}\right)$$
$$+f'\left(\frac{b+qa-\omega}{1+q}\right)\left[(b-a)\left(\frac{b+qa-\omega}{1+q}\right)-(b-a)\left(\frac{b+qa-\omega}{1+q}\right)\right]$$
$$= (b-a)f\left(\frac{b+qa-\omega}{1+q}\right) \leq \int_a^b f(t)_a\, d_{q,\omega}t. \tag{27}$$

On the other hand, we set the line connected between the points $(a,f(a))$ and $(b,f(b))$ by

$$k(t) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(t-a), \tag{28}$$

which follows from (26) that $f(t) \leq k(t)$ for all $t \in [a,b]$. Applying the (q,ω)-integration on both sides of (28), one has

$$\int_a^b k(t)_a\, d_{q,\omega}t = \int_a^b \left[f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(t-a)\right]_a d_{q,\omega}t$$
$$= (b-a)f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{(b-a)^2-\omega(b-a)}{1+q}\right)$$
$$= (b-a)\left[\frac{qf(a)+f(b)}{1+q} - \omega\left(\frac{f(b)-f(a)}{(b-a)(1+q)}\right)\right]$$
$$\geq \int_a^b f(t)_a\, d_{q,\omega}t. \tag{29}$$

Combining inequalities (27)–(29), we have inequalities (25). This completes the proof. □

Remark 1 If $\omega = 0$, then (25) is reduced to

$$f\left(\frac{b+qa}{1+q}\right) \leq \frac{1}{(b-a)} \int_a^b f(t)_a\, d_{q,\omega}t \leq \left[\frac{qf(a)+f(b)}{1+q}\right], \tag{30}$$

which appeared in [16].

Example 1 Let $\alpha > 0$ be the order of quantum Hahn fractional integral. Then, by Theorem 1, we obtain

$${}_aI^{\alpha}_{q,\omega}(x)(b) = {}_aI^{\alpha}_{q,\omega}(x-a+a)(b)$$
$$= {}_aI^{\alpha}_{q,\omega}(x-a)(b) + a\,{}_aI^{\alpha}_{q,\omega}(1)(b)$$
$$= \frac{1}{\Gamma_q(\alpha+2)}(b-a)_{\theta}^{(\alpha+1)} + \frac{a}{\Gamma_q(\alpha+1)}(b-a)_{\theta}^{(\alpha)}.$$

Since

$$(b-a)_{\theta}^{(\alpha)} = \frac{(b-{}_{\theta}\Phi_q^0(a))(b-{}_{\theta}\Phi_q^1(a))(b-{}_{\theta}\Phi_q^2(a))(b-{}_{\theta}\Phi_q^3(a))\cdots}{(b-{}_{\theta}\Phi_q^{\alpha}(a))(b-{}_{\theta}\Phi_q^{\alpha+1}(a))(b-{}_{\theta}\Phi_q^{\alpha+2}(a))(b-{}_{\theta}\Phi_q^{\alpha+3}(a))\cdots},$$

we get the fact that $(b-a)_\theta^{(\alpha)}(b-{}_\theta\Phi_q^\alpha(a)) = (b-a)_\theta^{(\alpha+1)}$. Thus, we have

$$ {}_aI_{q,\omega}^\alpha(x)(b) = \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left(\frac{(b-{}_\theta\Phi_q^\alpha(a))}{[\alpha+1]_q}+a\right). $$

Now, we claim that the point

$$ \Delta := \frac{(b-{}_\theta\Phi_q^\alpha(a))}{[\alpha+1]_q}+a \in [a,b]. \tag{31} $$

Indeed, we will show that $b-\Delta \le b-a$. By direct computation with $\omega \le (1-q)(b-a)$, we have

$$ \begin{aligned} b-\Delta &= b-\left[\frac{(b-{}_\theta\Phi_q^\alpha(a))}{[\alpha+1]_q}+a\right] \\ &= \frac{(b-a)(1-q^{\alpha+1})-(b-a)(1-q)+(1-q^\alpha)\omega}{1-q^{\alpha+1}} \\ &\le \frac{(b-a)(1-q^{\alpha+1})-(b-a)(1-q)+(1-q^\alpha)(1-q)(b-a)}{1-q^{\alpha+1}} \\ &= \frac{(b-a)(1-q^\alpha)}{1-q^{\alpha+1}} \\ &\le b-a, \end{aligned} $$

which implies that (31) holds.

Now we are in a position to prove the Hermite–Hadamard type of quantum Hahn integral inequality.

Theorem 6 *Let the function f and the constants q,ω,θ be as in Theorem 5. Then, for $\alpha>0$, we have*

$$ \begin{aligned} f\left(\frac{(b-{}_\theta\Phi_q^\alpha(a))}{[\alpha+1]_q}+a\right) &\le \frac{\Gamma_q(\alpha+1)}{(b-a)_\theta^{(\alpha)}}\left({}_aI_{q,\omega}^\alpha f\right)(b) \\ &\le \left[\frac{q[\alpha]_qf(a)+f(b)}{[\alpha+1]_q}-\frac{\omega[\alpha]_q}{[\alpha+1]_q}\left(\frac{f(b)-f(a)}{b-a}\right)\right]. \end{aligned} \tag{32} $$

Proof To get the fractional quantum Hahn Hermite–Hadamard inequality, we will use a method similar to that in Theorem 5. Firstly, we define a function $h^*(t)$ by

$$ h^*(t) = f(\Delta)+f'(\Delta)(t-\Delta), $$

where $\Delta\in[a,b]$ is defined in (31). For $\alpha>0$, by taking the fractional (q,ω)-integral of order α with Example 1, we get

$$ \begin{aligned} \left({}_aI_{q,\omega}^\alpha h^*\right)(b) &= f(\Delta)\left({}_aI_{q,\omega}^\alpha(1)\right)(b)+f'(\Delta)\left[\left({}_aI_{q,\omega}^\alpha t\right)(b)-\Delta\left({}_aI_{q,\omega}^\alpha(1)\right)(b)\right] \\ &= \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}f(\Delta)+f'(\Delta)\left[\frac{(b-a)_\theta^{(\alpha+1)}}{\Gamma_q(\alpha+1)}\Delta-\Delta\frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\right] \end{aligned} $$

$$
\begin{aligned}
&= \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)} f(\Delta) \\
&\le \big({}_aI_{q,\omega}^\alpha f\big)(b).
\end{aligned} \tag{33}
$$

In the next step, we use the line $k(t)$ over the curve of f which is defined in (28). By the fractional (q,ω)-integral of order α and Theorem 1, one has

$$
\begin{aligned}
\big({}_aI_{q,\omega}^\alpha k\big)(b) &= f(a)\big({}_aI_{q,\omega}^\alpha(1)\big)(b) + \left(\frac{f(b)-f(a)}{b-a}\right)\big({}_aI_{q,\omega}^\alpha(t-a)\big)(b) \\
&= \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)} f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)\frac{(b-a)_\theta^{(\alpha+1)}}{\Gamma_q(\alpha+2)} \\
&= \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left[f(a) + \frac{(f(b)-f(a))(b-{}_\theta\Phi_q^\alpha(a))}{(b-a)[\alpha+1]_q}\right] \\
&= \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left[\frac{q[\alpha]_q f(a)+f(b)}{[\alpha+1]_q} - \frac{\omega[\alpha]_q}{[\alpha+1]_q}\left(\frac{f(b)-f(a)}{b-a}\right)\right] \\
&\ge \big({}_aI_{q,\omega}^\alpha f\big)(b).
\end{aligned} \tag{34}
$$

Combining both of inequalities (33)–(34), we obtain the desired result in (32). The proof is completed. □

Remark 2 If $\alpha = 1$, then inequality (32) is reduced to (25). If $\omega = 0$, then (32) is reduced to

$$
f\left(\frac{b+q[\alpha]_q a}{[\alpha+1]_q}\right) \le \frac{\Gamma_q(\alpha+1)}{(b-a)_a^{(\alpha)}}\big({}_aI_q^\alpha f\big)(b) \le \frac{q[\alpha]_q f(a)+f(b)}{[\alpha+1]_q}, \tag{35}
$$

which corrects the result in Theorem 3.3 [17], where $(b-a)_a^{(\alpha)}$ is defined by (10) with $\theta = a$. In addition, if $\alpha = 1$, then (35) is also reduced to (30).

3 Some results on fractional integral inequalities for two unknown functions

In this section, we prove some fractional integral inequalities for two unknown functions. Let g be a (q,ω)-integrable function on $[a,b]$. Assume that:

(H_5) There exist ψ_1 and ψ_2 (q,ω)-integrable functions on $[a,b]$ such that

$$
\psi_1(t) \le g(t) \le \psi_2(t) \quad \text{for all } t \in [a,b].
$$

Theorem 7 *Let* (H_1) *with* $q_i = q$, $\omega_i = \omega$, $i = 1,2$, *and* (H_5) *hold. Then, for* $\alpha,\beta > 0$, *the fractional Hahn integral inequalities are true on* $[a,b]$:

(i) $({}_aI_{q,\omega}^\alpha f)(t)({}_aI_{q,\omega}^\beta \psi_2)(t) + ({}_aI_{q,\omega}^\alpha \varphi_1)(t)({}_aI_{q,\omega}^\beta g)(t) \ge$
$({}_aI_{q,\omega}^\alpha f)(t)({}_aI_{q,\omega}^\beta g)(t) + ({}_aI_{q,\omega}^\alpha \varphi_1)(t)({}_aI_{q,\omega}^\beta \psi_2)(t)$.

(ii) $({}_aI_{q,\omega}^\alpha g)(t)({}_aI_{q,\omega}^\beta \varphi_2)(t) + ({}_aI_{q,\omega}^\alpha \psi_1)(t)({}_aI_{q,\omega}^\beta f)(t) \ge$
$({}_aI_{q,\omega}^\alpha g)(t)({}_aI_{q,\omega}^\beta f)(t) + ({}_aI_{q,\omega}^\alpha \psi_1)(t)({}_aI_{q,\omega}^\beta \varphi_2)(t)$.

(iii) $({}_aI_{q,\omega}^\alpha \psi_1)(t)({}_aI_{q,\omega}^\beta \varphi_1)(t) + ({}_aI_{q,\omega}^\alpha g)(t)({}_aI_{q,\omega}^\beta f)(t) \ge$
$({}_aI_{q,\omega}^\alpha g)(t)({}_aI_{q,\omega}^\beta \varphi_1)(t) + ({}_aI_{q,\omega}^\alpha \psi_1)(t)({}_aI_{q,\omega}^\beta f)(t)$.

(iv) $({}_aI_{q,\omega}^\alpha g)(t)({}_aI_{q,\omega}^\beta f)(t) + ({}_aI_{q,\omega}^\alpha \psi_2)(t)({}_aI_{q,\omega}^\beta \varphi_2)(t) \ge$
$({}_aI_{q,\omega}^\alpha \psi_2)(t)({}_aI_{q,\omega}^\beta f)(t) + ({}_aI_{q,\omega}^\alpha g)(t)({}_aI_{q,\omega}^\beta \varphi_2)(t)$.

Proof To prove (i), from (H_1) and (H_5), we use the fact

$$\big(\psi_2(\tau) - g(\tau)\big)\big(f(\rho) - \varphi_1(\rho)\big) \geq 0,$$

which yields

$$\psi_2(\tau)f(\rho) + \varphi_1(\rho)g(\tau) \geq g(\tau)f(\rho) + \psi_2(\tau)\varphi_1(\rho).$$

By multiplying both sides of the above inequality with $(t - {}_{\theta}\Phi_q(\rho))_{\theta}^{(\alpha-1)}/\Gamma_q(\alpha)$, $\rho \in [a,t)$ and applying the Hahn (q,ω)-integration, we have

$$\psi_2(\tau)\big({}_aI_{q,\omega}^{\alpha}f\big)(t) + g(\tau)\big({}_aI_{q,\omega}^{\alpha}\varphi_1\big)(t) \geq g(\tau)\big({}_aI_{q,\omega}^{\alpha}f\big)(t) + \psi_2(\tau)\big({}_aI_{q,\omega}^{\alpha}\varphi_1\big)(t).$$

Multiplying both sides of the last inequality by $(t - {}_{\theta}\Phi_q(\tau))_{\theta}^{(\beta-1)}/\Gamma_q(\beta)$, $\tau \in [a,t)$ and taking the (q,ω)-integration with respect to τ on $[a,t)$, we get to inequality (i).

Finally, for proving (ii)–(iv), we use inequalities $(\varphi_2(\tau) - f(\tau))(g(\rho) - \psi_1(\rho)) \geq 0$, $(\varphi_1(\tau) - f(\tau))(\psi_1(\rho) - g(\rho)) \geq 0$, and $(f(\tau) - \varphi_2(\tau))(g(\rho) - \psi_2(\rho)) \geq 0$, $\tau, \rho \in [a,b]$, respectively. This completes the proof. □

Theorem 8 *Suppose that* (H_3) *holds and assume that*:

(H_6) $0 < \psi_1(t) \leq g(t) \leq \psi_2(t)$ *for all* $t \in [a,b]$.

Then the following inequalities:

(v) $({}_aI_{q,\omega}^{\alpha}\psi_1\psi_2 f^2)(t)({}_aI_{q,\omega}^{\alpha}\varphi_1\varphi_2 g^2)(t) \leq \frac{1}{4}(({}_aI_{q,\omega}^{\alpha}(\varphi_1\psi_1 + \varphi_2\psi_2)fg)(t))^2$,

(vi) $({}_aI_{q,\omega}^{\alpha}\varphi_1\varphi_2)(t)({}_aI_{q,\omega}^{\alpha}f^2)(t)({}_aI_{q,\omega}^{\beta}\psi_1\psi_2)(t)({}_aI_{q,\omega}^{\beta}g^2)(t) \leq$
$\frac{1}{4}(({}_aI_{q,\omega}^{\alpha}\varphi_1 f)(t)({}_aI_{q,\omega}^{\beta}\psi_1 g)(t) + ({}_aI_{q,\omega}^{\alpha}\varphi_2 f)(t)({}_aI_{q,\omega}^{\beta}\psi_2 g)(t))^2$,

are satisfied.

Proof For $\tau \in [a,t]$, $t > a$, we find that

$$A := \left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \geq 0 \tag{36}$$

and

$$B := \left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \geq 0. \tag{37}$$

Multiplying above two inequalities, that is,

$$AB = \left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \geq 0,$$

we obtain

$$\big(\varphi_1(\tau)\psi_1(\tau) + \varphi_2(\tau)\psi_2(\tau)\big)f(\tau)g(\tau) \geq \psi_1(\tau)\psi_2(\tau)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\tau). \tag{38}$$

Next multiplying both sides of (38) by $(t - {}_{\theta}\Phi_q(\tau))_{\theta}^{(\alpha-1)}/\Gamma_q(\alpha)$ and (q,ω)-integrating with respect to τ from 0 to t, we have

$$\big({}_aI_{q,\omega}^{\alpha}(\varphi_1\psi_1 + \varphi_2\psi_2)fg\big)(t) \geq \big({}_aI_{q,\omega}^{\alpha}\psi_1\psi_2 f^2\big)(t) + \big({}_aI_{q,\omega}^{\alpha}\varphi_1\varphi_2 g^2\big)(t).$$

By using the AM–GM inequality, that is, $x + y \geq 2\sqrt{xy}$, $x, y \in \mathbb{R}^+$, we obtain

$$\left({}_aI^{\alpha}_{q,\omega}(\varphi_1\psi_1 + \varphi_2\psi_2)fg\right)(t) \geq 2\sqrt{\left({}_aI^{\alpha}_{q,\omega}\psi_1\psi_2 f^2\right)(t)\left({}_aI^{\alpha}_{q,\omega}\varphi_1\varphi_2 g^2\right)(t)}.$$

To prove (vi), we use the fact that

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)} - \frac{f(\tau)}{g(\rho)}\right)\left(\frac{f(\tau)}{g(\rho)} - \frac{\varphi_1(\tau)}{\psi_2(\rho)}\right) \geq 0$$

for $\tau, \rho \in [a, b]$, which implies

$$\begin{aligned}&\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho) + \varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho)\\ &\quad \geq \psi_1(\rho)\psi_2(\rho)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\rho).\end{aligned}$$

By the method used in Theorem 7 with AM–GM inequality, we get the requested inequality in (vi). The proof is completed. □

Corollary 5 *Assume that* (H_4) *holds and also the following condition is satisfied*:

(H_7) $0 < n \leq g(t) \leq N$ *for all* $t \in [a, b]$.

Then we get the Pólya–Szegö type fractional Hahn integral inequalities:

(vii) $\frac{({}_aI^{\alpha}_{q,\omega}f^2)(t)({}_aI^{\alpha}_{q,\omega}g^2)(t)}{(({}_aI^{\alpha}_{q,\omega}fg)(t))^2} \leq \frac{1}{4}(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}})^2$.

(viii) $\frac{(t-a)_{\theta}^{(\alpha+\beta)}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} \frac{({}_aI^{\alpha}_{q,\omega}f^2)(t)({}_aI^{\beta}_{q,\omega}g^2)(t)}{(({}_aI^{\alpha}_{q,\omega}f)(t)({}_aI^{\beta}_{q,\omega}g)(t))^2} \leq \frac{1}{4}(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}})^2$.

Lemma 1 ([18]) *If A and B are nonnegative constants, then*

$$A^{\lambda} + (\lambda - 1)B^{\lambda} - \lambda AB^{\lambda-1} \geq 0 \quad \text{for } \lambda > 1, \tag{39}$$

and

$$A^{\lambda} + (\lambda - 1)B^{\lambda} - \lambda AB^{\lambda-1} \leq 0 \quad \text{for } 0 < \lambda < 1, \tag{40}$$

where the equality holds if and only if $A = B$.

Theorem 9 *Suppose that the assumptions of Theorem 8 hold. Then, for* $t \in [a, b]$, *we have*

$$\begin{aligned}&\left({}_aI^{\alpha}_{q,\omega}(\varphi_2\psi_2 g - \psi_1\psi_2 f)^{\lambda}\right)(t) + (\lambda - 1)\left({}_aI^{\alpha}_{q,\omega}(\psi_1\psi_2 f - \varphi_1\psi_1 g)^{\lambda}\right)(t)\\ &\quad \geq \lambda\left({}_aI^{\alpha}_{q,\omega}(\varphi_2\psi_2 g - \psi_1\psi_2 f)(\psi_1\psi_2 f - \varphi_1\psi_1 g)^{\lambda-1}\right)(t), \quad \lambda > 1,\end{aligned} \tag{41}$$

and

$$\begin{aligned}&\left({}_aI^{\alpha}_{q,\omega}(\varphi_2\psi_2 g - \psi_1\psi_2 f)^{\lambda}\right)(t) + (\lambda - 1)\left({}_aI^{\alpha}_{q,\omega}(\psi_1\psi_2 f - \varphi_1\psi_1 g)^{\lambda}\right)(t)\\ &\quad \leq \lambda\left({}_aI^{\alpha}_{q,\omega}(\varphi_2\psi_2 g - \psi_1\psi_2 f)(\psi_1\psi_2 f - \varphi_1\psi_1 g)^{\lambda-1}\right)(t), \quad 0 < \lambda < 1.\end{aligned} \tag{42}$$

Proof Set the constants A, B as in (36)–(37), respectively. For any $\lambda > 1$, we obtain from Lemma 1 that

$$\left(\frac{\varphi_2(\tau)g(\tau) - \psi_1(\tau)f(\tau)}{\psi_1(\tau)g(\tau)}\right)^{\lambda} + (\lambda - 1)\left(\frac{\psi_2(\tau)f(\tau) - \varphi_1(\tau)g(\tau)}{\psi_2(\tau)g(\tau)}\right)^{\lambda}$$
$$\geq \lambda\left(\frac{\varphi_2(\tau)g(\tau) - \psi_1(\tau)f(\tau)}{\psi_1(\tau)g(\tau)}\right)\left(\frac{\psi_2(\tau)f(\tau) - \varphi_1(\tau)g(\tau)}{\psi_2(\tau)g(\tau)}\right)^{\lambda-1}.$$

Multiplying both sides by $(\psi_1(\tau)\psi_2(\tau)g(\tau))^{\lambda}$, we have

$$\big(\varphi_2(\tau)\psi_2(\tau)g(\tau) - \psi_1(\tau)\psi_2(\tau)f(\tau)\big)^{\lambda} + (\lambda - 1)\big(\psi_1(\tau)\psi_2(\tau)f(\tau) - \varphi_1(\tau)\psi_1(\tau)g(\tau)\big)^{\lambda}$$
$$\geq \lambda\big(\varphi_2(\tau)\psi_2(\tau)g(\tau) - \psi_1(\tau)\psi_2(\tau)f(\tau)\big)\big(\psi_1(\tau)\psi_2(\tau)f(\tau) - \varphi_1(\tau)\psi_1(\tau)g(\tau)\big)^{\lambda-1},$$

which implies inequality (41) by multiplying by $(t - {}_{\theta}\Phi_q(\tau))_{\theta}^{(\alpha-1)}/\Gamma_q(\alpha)$ and applying Hahn (q,ω)-integration. In a similar way, the proof of inequality (42) can be obtained. The proof is completed. □

Corollary 6 *If* (H_4) *and* (H_7) *hold, then we get*

$$\big({}_aI_{q,\omega}^{\alpha}(MNg - nNf)^{\lambda}\big)(t) + (\lambda - 1)\big({}_aI_{q,\omega}^{\alpha}(nNf - mng)^{\lambda}\big)(t)$$
$$\geq \lambda\big({}_aI_{q,\omega}^{\alpha}(MNg - nNf)(nNf - mng)^{\lambda-1}\big)(t), \quad \lambda > 1,$$

and

$$\big({}_aI_{q,\omega}^{\alpha}(MNg - nNf)^{\lambda}\big)(t) + (\lambda - 1)\big({}_aI_{q,\omega}^{\alpha}(nNf - mng)^{\lambda}\big)(t)$$
$$\leq \lambda\big({}_aI_{q,\omega}^{\alpha}(MNg - nNf)(nNf - mng)^{\lambda-1}\big)(t), \quad 0 < \lambda < 1.$$

In particular case, if $M = N$, $m = n$, *and* $\lambda = 2$, *we have*

$$\big(M^2 + m^2\big)^2\big({}_aI_{q,\omega}^{\alpha}g^2\big)(t) + 4m^2M^2\big({}_aI_{q,\omega}^{\alpha}f^2\big)(t)$$
$$\geq 4\big(mM^3 + m^3M\big)\big({}_aI_{q,\omega}^{\alpha}fg\big)(t).$$

Next, we are going to prove the Grüss–Čebyšev type fractional Hahn (q,ω)-integral inequality on the interval $[a,b]$. Let us prove the fractional (q,ω)-Korkine equality on the interval $[a,b]$.

Lemma 2 *Let* $f, g : [a,b] \to \mathbb{R}$ *be* (q,ω)*-integrable functions. Then, for* $0 < q < 1$, $\omega \geq 0$, $\theta \in [a,b]$, *and* $\alpha > 0$, *we have*

$$\frac{1}{2}\big\{{}_aI_{q,\omega}^{2\alpha}\big(f(s) - f(r)\big)\big(g(s) - g(r)\big)\big\}(b)$$
$$= \frac{(b-a)_{\theta}^{(\alpha)}}{\Gamma_q(\alpha+1)}\big({}_aI_{q,\omega}^{\alpha}fg\big)(b) - \big({}_aI_{q,\omega}^{\alpha}f\big)(b)\big({}_aI_{q,\omega}^{\alpha}g\big)(b). \tag{43}$$

Proof From Definitions 2 and 4, we have

$$
\begin{aligned}
&\big\{{}_aI_{q,\omega}^{2\alpha}\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big\}(b)\\
&= \frac{1}{\Gamma_q^2(\alpha)}\int_a^b\int_a^b \big(b-{}_\theta\Phi_q(s)\big)_\theta^{(\alpha-1)}\big(b-{}_\theta\Phi_q(r)\big)_\theta^{(\alpha-1)}\\
&\quad\times\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)_a\,d_{q,\omega}s_a\,d_{q,\omega}r\\
&= \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}f\big({}_\theta\Phi_q^i(b)\big)g\big({}_\theta\Phi_q^i(b)\big)\right)\\
&\quad-\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}f\big({}_\theta\Phi_q^i(b)\big)\right)\\
&\quad\times\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}g\big({}_\theta\Phi_q^i(b)\big)\right)\\
&\quad-\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}g\big({}_\theta\Phi_q^i(b)\big)\right)\\
&\quad\times\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}f\big({}_\theta\Phi_q^i(b)\big)\right)\\
&\quad+\frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}f\big({}_\theta\Phi_q^i(b)\big)g\big({}_\theta\Phi_q^i(b)\big)\right)\\
&= \frac{2(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}f\big({}_\theta\Phi_q^i(b)\big)g\big({}_\theta\Phi_q^i(b)\big)\right)\\
&\quad-2\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}f\big({}_\theta\Phi_q^i(b)\big)\right)\\
&\quad\times\left(\frac{b-{}_\theta\Phi_q(b)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty}q^i\big(b-{}_\theta\Phi_q^{i+1}(b)\big)_\theta^{(\alpha-1)}g\big({}_\theta\Phi_q^i(b)\big)\right)\\
&= \frac{2(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\big({}_aI_{q,\omega}^{\alpha}fg\big)(b)-2\big({}_aI_{q,\omega}^{\alpha}f\big)(b)\big({}_aI_{q,\omega}^{\alpha}g\big)(b),
\end{aligned}
$$

which is (2), as desired. The proof is completed. □

The following example is needed to prove our next result.

Example 2 Let $f(t)=t^2$ and $g(t)=t$, $t\in[a,b]$. Then the fractional Hahn (q,ω)-integral of order $\alpha>0$ of $f(t)$ from a to b is

$$
\begin{aligned}
\big({}_aI_{q,\omega}^{\alpha}f\big)(b) &= \big({}_aI_{q,\omega}^{\alpha}(t-a+a)^2\big)(b)\\
&= \big({}_aI_{q,\omega}^{\alpha}(t-a)^2\big)(b)+2a\big({}_aI_{q,\omega}^{\alpha}(t-a)\big)(b)+a^2\big({}_aI_{q,\omega}^{\alpha}(1)\big)(b)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma_q(3)}{\Gamma_q(\alpha+3)}(b-a)_\theta^{(\alpha+2)} + \frac{2a}{\Gamma_q(\alpha+2)}(b-a)_\theta^{(\alpha+1)} + \frac{a^2}{\Gamma_q(\alpha+1)}(b-a)_\theta^{(\alpha)} \\
&= \frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left[\frac{\Gamma_q(3)(b-{}_\theta\Phi_q^{\alpha+1}(a))(b-{}_\theta\Phi_q^{\alpha}(a))}{[\alpha+2]_q[\alpha+1]_q} + \frac{2a(b-{}_\theta\Phi_q^{\alpha}(a))}{[\alpha+1]_q} + a^2\right].
\end{aligned}
$$

In addition, from Example 1, the square of $({}_aI_{q,\omega}^{\alpha}g)(b)$ is

$$
\begin{aligned}
\left[\left({}_aI_{q,\omega}^{\alpha}g\right)(b)\right]^2 &= \left[\frac{1}{\Gamma_q(\alpha+2)}(b-a)_\theta^{(\alpha+1)} + \frac{a}{\Gamma_q(\alpha+1)}(b-a)_\theta^{(\alpha)}\right]^2 \\
&= \left[\frac{(b-{}_\theta\Phi_q^{\alpha}(a))(b-a)_\theta^{(\alpha)}}{[\alpha+1]_q\Gamma_q(\alpha+1)} + \frac{a(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\right]^2 \\
&= \frac{(b-a)_\theta^{2(\alpha)}}{\Gamma_q^2(\alpha+1)}\left[\frac{(b-{}_\theta\Phi_q^{\alpha}(a))^2}{[\alpha+1]^2} + \frac{2a(b-{}_\theta\Phi_q^{\alpha}(a))}{[\alpha+1]_q} + a^2\right].
\end{aligned}
$$

Theorem 10 *Let $f,g:[a,b]\to\mathbb{R}$ be L_1, L_2-Lipschitzian (q,ω)-integrable functions, that is,*

$$
\begin{aligned}
\left|f(s)-f(r)\right| &\le L_1|s-r|, \\
\left|g(s)-g(r)\right| &\le L_2|s-r|
\end{aligned}
\tag{44}
$$

for all $s,r\in[a,b]$, $L_1,L_2>0$. Then, for $0<q<1$, $\omega\ge 0$, $\theta\in[a,b]$, and $\alpha>0$, the following inequality holds:

$$
\begin{aligned}
&\frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left({}_aI_{q,\omega}^{\alpha}fg\right)(b) - \left({}_aI_{q,\omega}^{\alpha}f\right)(b)\left({}_aI_{q,\omega}^{\alpha}g\right)(b) \\
&\quad\le L_1L_2\frac{q[\alpha]_q(b-a)_\theta^{2(\alpha)}(b-{}_\theta\Phi_q^{\alpha}(a))^2}{[\alpha+1]_q^2[\alpha+2]_q\Gamma_q^2(\alpha+1)}.
\end{aligned}
\tag{45}
$$

Proof From inequalities (44), we have

$$
\left|\left(f(s)-f(r)\right)\left(g(s)-g(r)\right)\right| \le L_1L_2(s-r)^2
$$

for all $s,r\in[a,b]$. Taking the double fractional Hahn (q,ω)-integration of order α with respect to $s,r\in[a,b]$, we get

$$
\begin{aligned}
&\left({}_aI_{q,\omega}^{2\alpha}\left|\left(f(s)-f(r)\right)\left(g(s)-g(r)\right)\right|\right)(b) \\
&\quad\le L_1L_2\left({}_aI_{q,\omega}^{2\alpha}(s-r)^2\right)(b) \\
&\quad= L_1L_2\left[\frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left({}_aI_{q,\omega}^{\alpha}s^2\right)(b) - 2\left({}_aI_{q,\omega}^{\alpha}s\right)(b)\left({}_aI_{q,\omega}^{\alpha}r\right)(b)\right. \\
&\qquad\left.+\frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left({}_aI_{q,\omega}^{\alpha}r^2\right)(b)\right] \\
&\quad= 2L_1L_2\left[\frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\left({}_aI_{q,\omega}^{\alpha}s^2\right)(b) - \left(\left({}_aI_{q,\omega}^{\alpha}s\right)(b)\right)^2\right].
\end{aligned}
\tag{46}
$$

Applying the results from Example 2, it follows that

$$\begin{aligned}
&\frac{(b-a)_\theta^{(\alpha)}}{\Gamma_q(\alpha+1)}\big({}_aI_{q,\omega}^{\alpha}s^2\big)(b)-\big(\big({}_aI_{q,\omega}^{\alpha}s\big)(b)\big)^2\\
&\quad=\frac{(b-a)_\theta^{2(\alpha)}(b-{}_\theta\Phi_q^{\alpha}(a))}{[\alpha+1]_q\Gamma_q^2(\alpha+1)}\biggl[\frac{\Gamma_q(3)(b-{}_\theta\Phi_q^{\alpha+1}(a))}{[\alpha+2]_q}-\frac{(b-{}_\theta\Phi_q^{\alpha}(a))}{[\alpha+1]_q}\biggr]\\
&\quad=\frac{(b-a)_\theta^{2(\alpha)}(b-{}_\theta\Phi_q^{\alpha}(a))}{[\alpha+1]_q\Gamma_q^2(\alpha+1)}\biggl[\frac{q(b-a)(1-q)(1-q^{\alpha})}{(1-q^{\alpha+1})(1-q^{\alpha+2})}+\frac{q\omega(-q^{2\alpha}+2q^{\alpha}-1)}{(1-q^{\alpha+1})(1-q^{\alpha+2})}\biggr]\\
&\quad=\frac{q[\alpha]_q(b-a)_\theta^{2(\alpha)}(b-{}_\theta\Phi_q^{\alpha}(a))^2}{[\alpha+1]_q^2[\alpha+2]_q\Gamma_q^2(\alpha+1)}.
\end{aligned}\tag{47}$$

Hence, (46) and (47) lead to

$$\big({}_aI_{q,\omega}^{2\alpha}\big|\big(f(s)-f(r)\big)\big(g(s)-g(r)\big)\big|\big)(b)\le 2L_1L_2\frac{q[\alpha]_q(b-a)_\theta^{2(\alpha)}(b-{}_\theta\Phi_q^{\alpha}(a))^2}{[\alpha+1]_q^2[\alpha+2]_q\Gamma_q^2(\alpha+1)}.\tag{48}$$

Applying Lemma 2 to (48), we deduce the desired inequality in (45). This completes the proof. □

Remark 3 If $\alpha=1$, $q\to1$, and $\omega=0$, then (45) is reduced to the classical Grüss–Čebyšev integral inequality as follows:

$$\biggl|\frac{1}{b-a}\int_a^b f(s)g(s)\,ds-\biggl(\frac{1}{b-a}\int_a^b f(s)\,ds\biggr)\biggl(\frac{1}{b-a}\int_a^b g(s)\,ds\biggr)\biggr|\le\frac{L_1L_2(b-a)^2}{12}.$$

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