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A Schrödinger-type algorithm for solving the Schrödinger equations via Phragmén–Lindelöf inequalities

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Abstract

In this article, we consider the numerical method for solving the Schrödinger equations via Phragmén–Lindelöf inequalities under the order induced by a symmetric cone with the function involved being monotone. Based on the Phragmén–Lindelöf inequalities, the underlying system of inequalities is reformulated as a system of smooth equations, and a Schrödinger-type method is proposed to solve it iteratively so that a solution of the system of the Schrödinger equations is found. By means of the Schrödinger type inequalities, the algorithm is proved to be well defined and to be globally convergent under weak assumptions and locally quadratically convergent under suitable assumptions. Preliminary numerical results indicate that the algorithm is effective.

Keywords: Schrödinger equation; Phragmén–Lindelöf inequality; Schrödinger type inequality; Convergence

1 Introduction

In this paper, we consider the following Schrödinger equation (see [18, 19]):

$$-\Delta u + V(x)u = \left(\frac{1}{|x|^{\alpha}} * |u|^{p}\right)|u|^{p-2}u, \quad x \in \Re^{n}, \tag{1.1}$$

where $n \geq 3$, $0 < \alpha < n$, $2 - \frac{\alpha}{n} < p \leq 2^{*}_{\alpha} = \frac{2n-\alpha}{n-2}$ and

(A1) $V \in C(\mathbb{R}^{n}, \mathbb{R}^{+})$ is coercive, that is,

$$\lim_{|x|\to+\infty} V(x) = +\infty.$$

In 2016, Qiao et al. [19] first considered the bound and ground states for the nonlinear Schrödinger equations under the condition

(A2) $\inf_{\mathbb{R}^{n}} V > 0$, and there exists a positive constant r satisfying

$$\operatorname{meas}\{x \in \mathbb{R}^{n}, |x-y| \leq r, V(x) \leq M\} \to 0$$

as $|y| \to \infty$, where $M > 0$ and meas stands for the Lebesgue measure.

The nonlinear system (1.1) has been proved to possess wide application fields in many real world problems such as anomalous diffusion [2, 4, 15], disease models [6, 9, 21], ecological models [26], synchronization of chaotic systems [1, 27], etc.

Put

$$\nu(x) := \frac{1}{2}\left\|u(x)\right\|^2$$

is the Nevanlinna norm (see [8]), problem (1.1) is equivalent to the following Schrödinger problem defined by

$$\min \nu(x), \tag{1.2}$$

where $x \in \Re^n$.

The Phragmén–Lindelöf inequalities (see [23]) have the main objective to solve the so-called Schrödinger Phragmén–Lindelöf subproblem model to get the trial step τ_k

$$\begin{aligned} \text{Min} \quad & \mathcal{T}\mathcal{W}_k(\tau) = \frac{1}{2}\left\|u(x_k) + \nabla S(x_k)\tau\right\|^2, \\ & \|\tau\| \le \triangle. \end{aligned}$$

In 2015, a modified Phragmén–Lindelöf inequality was proved by Wan [23]:

$$\begin{aligned} \text{Min} \quad & \phi_k(\tau) = \frac{1}{2}\left\|u(x_k) + \nabla u(x_k)\tau\right\|^2, \\ & \|\tau\| \le c^p\left\|u(x_k)\right\|^\gamma, \end{aligned}$$

where p, c, and γ are three positive numbers.

Recently, another adaptive Schrödinger Phragmén–Lindelöf inequality has been defined by Qiao et al. [17]:

$$\begin{aligned} \text{Min} \quad & \mathcal{T}\mathcal{M}_k(\tau) = \frac{1}{2}\left\|u(x_k) + \mathcal{B}_k\tau\right\|^2, \\ & \|\tau\| \le c^p\left\|u(x_k)\right\|, \end{aligned} \tag{1.3}$$

where $\mathcal{B}_k$ is defined by

$$\mathcal{B}_{k+1} = \mathcal{B}_k - \frac{\mathcal{B}_k s_k s_k^{\mathcal{T}} \mathcal{B}_k}{s_k^{\mathcal{T}} \mathcal{B}_k s_k} + \frac{y_k {y_k}^{\mathcal{T}}}{{y_k}^{\mathcal{T}} s_k}, \tag{1.4}$$

$y_k = u(x_{k+1}) - u(x_k)$ and $s_k = x_{k+1} - x_k$. This Schrödinger Phragmén–Lindelöf method can possess the global convergence without the nondegeneracy (see [1, 7, 11, 26] etc.), which shows that this paper made a further progress in theory. And there exist many applications about the Schrödinger Phragmén–Lindelöf inequalities (see [3, 25, 27, 28] etc.) for nonsmooth optimizations and other problems.

We further consider the Schrödinger Phragmén–Lindelöf model for the nonlinear system $u(x)$ at x_k (see [17])

$$\vartheta(x_k + \tau) = u(x_k) + \nabla u(x_k)^{\mathcal{T}} d + \frac{1}{2}\mathcal{T}_k d^2, \tag{1.5}$$

where $\nabla u(x_k)$ is the Jacobian matrix of $u(x)$ at x_k.

It is well known that the above model (1.5) can be written as the following extension (see [20, 23, 24]):

$$\vartheta(x_k + \tau) = u(x_k) + \nabla u(x_k)^{\mathcal{T}} d + \frac{3}{2}\left(s_{k-1}^{\mathcal{T}}\tau\right)^2 s_{k-1}. \tag{1.6}$$

If we set the Schrödinger Phragmén–Lindelöf matrix $\nabla u(x_k)$, then we can use the Schrödinger Phragmén–Lindelöf matrix $\mathcal{B}_k$ instead of it. Thus, our Schrödinger Phragmén–Lindelöf model can be defined as follows:

$$\begin{aligned} &\text{Min} \quad \mathcal{N}_k(\tau) = \frac{1}{2}\left\| u(x_k) + \mathcal{B}_k d + \frac{3}{2}\left(s_{k-1}^{\mathcal{T}}\tau\right)^2 s_{k-1}\right\|^2, \\ &\qquad \|\tau\| \le c^p \left\| u(x_k)\right\|^{\gamma}, \end{aligned} \tag{1.7}$$

where $\mathcal{B}_k = \mathcal{H}_k^{-1}$ and $\mathcal{H}_k$ is generated by

$$\begin{aligned} \mathcal{H}_{k+1} &= \mathcal{V}_k^{\mathcal{T}}\mathcal{H}_k\mathcal{V}_k + \rho_k s_k s_k^{\mathcal{T}} \\ &= \mathcal{V}_k^{\mathcal{T}}\left[\mathcal{V}_{k-1}^{\mathcal{T}}\mathcal{H}_{k-1}\mathcal{V}_{k-1} + \rho_{k-1}s_{k-1}s_{k-1}^{\mathcal{T}}\right]\mathcal{V}_k + \rho_k s_k s_k^{\mathcal{T}} \\ &= \cdots \\ &= \left[\mathcal{V}_k^{\mathcal{T}}\cdots\mathcal{V}_{k-m+1}^{\mathcal{T}}\right]\mathcal{H}_{k-m+1}[\mathcal{V}_{k-m+1}\cdots\mathcal{V}_k] \\ &\quad + \rho_{k-m+1}\left[\mathcal{V}_{k-1}^{\mathcal{T}}\cdots\mathcal{V}_{k-m+2}^{\mathcal{T}}\right]s_{k-m+1}s_{k-m+1}^{\mathcal{T}}[\mathcal{V}_{k-m+2}\cdots\mathcal{V}_{k-1}] \\ &\quad + \cdots \\ &\quad + \rho_k s_k s_k^{\mathcal{T}}, \end{aligned} \tag{1.8}$$

where $\rho_k = \frac{1}{s_k^{\mathcal{T}} y_k}$, $\mathcal{V}_k = I - \rho_k y_k s_k^{\mathcal{T}}$ (see [23] etc.).

Let τ_k^p be the solution of (1.7). Define

$$A\tau_k\left(\tau_k^p\right) = \nu\left(x_k + \tau_k^p\right) - \nu(x_k), \tag{1.9}$$

and predict reduction by

$$\mathcal{P}\tau_k\left(\tau_k^p\right) = \mathcal{N}_k\left(\tau_k^p\right) - \mathcal{N}_k(0). \tag{1.10}$$

Based on definitions of $A\tau_k(\tau_k^p)$ and $P\tau_k(\tau_k^p)$, we design their ratio by

$$r_k^p = \frac{A\tau_k(\tau_k^p)}{\mathcal{P}\tau_k(\tau_k^p)}. \tag{1.11}$$

Therefore, the Schrödinger-type algorithm for solving (1.1) is stated as follows.

Algorithm

Initial: Let $\mathfrak{B}_0 = \mathfrak{H}_0^{-1} \in \Re^n \times \Re^n$ be a symmetric and positive definite matrix. $x_0 \in \Re^n$ and $\varrho = 0$. ρ, c, and ϵ are three positive constants. Let $l := 0$;

Step 1: Stop if $\|\chi(x_l)\| < \epsilon$ holds;

Step 2: Solve (1.7) with $\triangle = \triangle_l$ to obtain ς_l^ϱ;

Step 3: Compute $A\varsigma_l(\varsigma_l^\varrho)$, $\mathcal{P}\varsigma_l(\varsigma_l^\varrho)$, and the ratio r_l^ϱ. If $r_l^\varrho < \rho$, let $\varrho = \varrho + 1$, go to Step 2. If $r_l^\varrho \geq \rho$, go to the next step;

Step 4: Set $x_{l+1} = x_l + \varsigma_l^\varrho$, $y_l = \chi(x_{l+1}) - \chi(x_l)$, update $\mathfrak{B}_{l+1} = \mathfrak{H}_{l+1}^{-1}$ by (1.8) if $y_l^{\mathfrak{T}} \varsigma_l^p > 0$, otherwise set $\mathfrak{B}_{l+1} = \mathfrak{B}_l$;

Step 5: Let $l := l + 1$ and $\varrho = 0$. Go to Step 1.

In this paper, we further focus on convergence results of the above algorithm under the following assumptions.

Assumptions

(A) Define the set Ω by

$$\Omega = \{x | \varphi(x) \leq \varphi(x_0)\}. \tag{1.12}$$

It is easy to see that Ω is bounded.

(B) The nonlinear system $\chi(x)$ is twice continuously differentiable in Ω_1, which is an open convex set containing Ω.

(C) The following Phragmén–Lindelöf relation

$$\|[\nabla\chi(x_l) - \mathfrak{B}_l]\chi(x_l)\| = O(\|\varsigma_l^p\|) \tag{1.13}$$

holds.

(D) The sequence matrices $\{\mathfrak{B}_l\}$ are uniformly bounded in Ω_1.

It follows from Assumption (B) that (see [10, 22])

$$\|\nabla\chi(x_l)^{\mathfrak{T}} \nabla\chi(x_l)\| \leq M_L, \tag{1.14}$$

where M_L is a positive real number.

2 Convergence results

We first have the following new Phragmén–Lindelöf inequalities.

Lemma 2.1 *Let τ_k^p be the solution of* (1.1). *Then*

$$\mathcal{P}\tau_k(\tau_k^p) \leq -\frac{1}{2}\|\mathcal{B}_k u(x_k)\| \min\left\{\triangle_k, \frac{\|\mathcal{B}_k u(x_k)\|}{M_l^2}\right\} + O(\triangle_k^2) \tag{2.1}$$

holds.

Proof Define

$$\mathcal{J}(u) = \frac{1}{2}\int_{\Re^n} |\nabla u|^2 + u^2\,dx - \frac{1}{2p}\int_{\Re^n}\int_{\Re^n} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,dx\,dy.$$

It follows from (1.7) that

$$j_0 < j_1 = \inf_{\mathcal{N}^-} \mathcal{J}(u) < j_0 + \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}.$$

Consider $V(x)$ is a minimizer for both $S_{\alpha,p}$. By the continuity of $\mathcal{J}$, we know that

$$\mathcal{J}(u_0 + tV) < j_0 + \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}},$$

where $0 \le t < \gamma$.

So

$$\begin{aligned}
\mathcal{J}(u_0 + tV) &= \frac{1}{2}\|u_0 + tV\|^2 - \frac{1}{2p}\tilde{B}(u_0 + tV) - \int_{\Re^n} h(u_0 + tV)\,dx \\
&= \mathcal{J}(u_0) + \frac{t^2}{2}\left[\|V\|^2 - \frac{t^{p-2}}{p}\tilde{B}(V)\right] + \tilde{B}(u_0) + \tilde{B}(tV) - \tilde{B}(u_0 + tV) \\
&< j_0 + \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}.
\end{aligned}$$

It follows from $t \ge \gamma$ that

$$\begin{aligned}
\mathcal{J}(u_0 + tV) &= \frac{1}{2}\|u_0 + tV\|^2 - \frac{1}{2p}\tilde{B}(u_0 + tV) - \int_{\Re^n} h(u_0 + tV)\,dx \\
&= \frac{1}{2}\|u_0\|^2 + t\int_{\Re^n} \nabla u_0 \nabla V + u_0 V\,dx + \frac{t^2}{2}\|V\|^2 - \frac{1}{2p}\tilde{B}(u_0) \\
&\quad + \frac{1}{2p}\left[\tilde{B}(u_0) + \tilde{B}(tV) - \tilde{B}(u_0 + tV)\right] - \frac{1}{2p}\tilde{B}(tV) \\
&\quad - \int_{\Re^n} h u_0\,dx - \int_{\Re^n} h t V\,dx \\
&= \mathcal{J}(u_0) + \frac{t^2}{2}\left[\|V\|^2 - \frac{t^{2(p-1)}}{2p}\tilde{B}(V)\right] + \frac{1}{2p}\left[\tilde{B}(u_0) + \tilde{B}(tV)\right. \\
&\quad \left. - \tilde{B}(u_0 + tV) + 2p\int_{\Re^n}\int_{\Re^n} \frac{|u_0(x)|^p |u_0(y)|^{p-2} u_0(y)}{|x-y|^\alpha}\,dx\,dy\right] \\
&< j_0 + \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}.
\end{aligned}$$

Here, we use that $\langle J'(u_0), tV\rangle = 0$ and $V(x)$ is a solution of (1.1). By the definition of τ_k^p [14, 16] and it being the solution of (1.7), we get

$$\begin{aligned}
\mathcal{P}\tau_k\left(\tau_k^p\right) &\le \mathcal{P}\tau_k\left(-\alpha\frac{\triangle_k}{\|\mathcal{B}_k u(x_k)\|}\mathcal{B}_k u(x_k)\right) \\
&= \frac{1}{2}\left[\alpha^2\triangle_k^2 \frac{\|\mathcal{B}_k\mathcal{B}_k u(x_k)\|^2}{\|\mathcal{B}_k u(x_k)\|^2} + \alpha^4\triangle_k^4 \frac{9}{4}\frac{(s_{k-1}^{\mathcal{T}}\mathcal{B}_k u(x_k))^4}{\|\mathcal{B}_k u(x_k)\|^4}\right. \\
&\quad + 3\alpha^2\triangle_k^2 \frac{(s_{k-1}^{\mathcal{T}}\mathcal{B}_k u(x_k))^2}{\|\mathcal{B}_k s_{k-1}\|^2} u(x_k)^{\mathcal{T}} s_{k-1} - 2\alpha\triangle_k \frac{(u(x_k)^{\mathcal{T}}\mathcal{B}_k\mathcal{B}_k u(x_k))}{\|\mathcal{B}_k u(x_k)\|} \\
&\quad \left. - 3\alpha^3\triangle_k^3 \frac{(s_{k-1}^{\mathcal{T}}\mathcal{B}_k u(x_k))^2 s_{k-1}^{\mathcal{T}}\mathcal{B}_k\mathcal{B}_k u(x_k)}{\|\mathcal{B}_k u(x_k)\|^3}\right]
\end{aligned}$$

$$= \frac{1}{2}\left[\alpha^2\triangle_k^2\frac{\|\mathcal{B}_k\mathcal{B}_k u(x_k)\|^2}{\|\mathcal{B}_k u(x_k)\|^2} - 2\alpha\triangle_k\frac{(u(x_k))^{\mathcal{T}}\mathcal{B}_k\mathcal{B}_k u(x_k))}{\|\mathcal{B}_k u(x_k)\|} + O(\triangle_k^2)\right]$$
$$\le -\alpha\triangle_k\|\mathcal{B}_k u(x_k)\| + \frac{1}{2}\alpha^2\triangle_k^2 M_l^2 + O(\triangle_k^2)$$

for any $\alpha \in [0,1]$.

Therefore

$$\mathcal{P}\tau_k(\tau_k^p) \le \min_{0\le\alpha\le 1}\left[-\alpha\triangle_k\|\mathcal{B}_k u(x_k)\| + \frac{1}{2}\alpha^2\triangle_k^2 M_l^2\right] + O(\triangle_k^2)$$
$$\le -\frac{1}{2}\|\mathcal{B}_k u(x_k)\|\min\left\{\triangle_k, \frac{\|\mathcal{B}_k u(x_k)\|}{M_l^2}\right\} + O(\triangle_k^2).$$

□

Lemma 2.2 *Let Assumptions* (A), (B), (C), *and* (D) *hold. Then*

$$|A\tau_k(\tau_k^p) - \mathcal{P}\tau_k(\tau_k^p)| = O(\|\tau_k^p\|^2),$$

where τ_k is the solution of (1.7).

Proof It follows from (1.9) and (1.10) that

$$\begin{aligned}
|A\tau_k(\tau_k^p) - \mathcal{P}\tau_k(\tau_k^p)| &= |\nu(x_k + \tau_k^p) - \mathcal{N}_k(\tau_k^p)| \\
&= \frac{1}{2}\left|\|u(x_k) + \nabla u(x_k)\tau_k^p + O(\|\tau_k^p\|^2)\|^2\right. \\
&\quad \left. - \left\|u(x_k) + \mathcal{B}_k\tau_k^p + \frac{3}{2}(s_{k-1}^{\mathcal{T}}\tau_k^p)^2 s_{k-1}\right\|^2\right| \\
&= |u(x_k)^{\mathcal{T}}\nabla u(x_k)\tau_k^p - u(x_k)^{\mathcal{T}}\mathcal{B}_k\tau_k^p + O(\|\tau_k^p\|^2) \\
&\quad + O(\|\tau_k^p\|^3) + O(\|\tau_k^p\|^4)| \\
&\le \|[\nabla u(x_k) - \mathcal{B}_k]u(x_k)\|\|\tau_k^p\| + O(\|\tau_k^p\|^2) \\
&\quad + O(\|\tau_k^p\|^3) + O(\|\tau_k^p\|^4) \\
&= O(\|\tau_k^p\|^2).
\end{aligned}$$

□

Theorem 2.1 *Let Assumptions* (A), (B), (C), *and* (D) *hold. Then Algorithm either finitely stops or generates an infinite sequence $\{x_k\}$ satisfying*

$$\lim_{k\to\infty}\|u(x_k)\| = 0, \tag{2.2}$$

where $\{x_k\}$ is defined as in Algorithm.

Proof We know that $t^-(u)$ is a continuous function of u. Consequently, the manifold Λ^- disconnects $D^{1,2}(\Re^n)$ in exactly two connected components $\mathcal{U}_1$ and $\mathcal{U}_2$, where

$$\mathcal{U}_1 = \left\{u \in D^{1,2}(\Re^n) : u = 0 \text{ or } \|u\|_D < t^-\left(\frac{u}{\|u\|_D}\right)\right\},$$
$$\mathcal{U}_2 = \left\{u \in D^{1,2}(\Re^n) : \|u\|_D > t^-\left(\frac{u}{\|u\|_D}\right)\right\}.$$

So $D^{1,2}(\Re^n) = \Lambda^- \cup \mathcal{U}_1 \cup \mathcal{U}_2$. In particular, $u_0 \in \Lambda^+ \subset \mathcal{U}_1$. Since

$$t^-\left(\frac{u_0 + tW}{\|u_0 + tW\|_D}\right)\frac{u_0 + tW}{\|u_0 + tW\|_D} \in \Lambda,$$

we have

$$0 < t^-\left(\frac{u_0 + tW}{\|u_0 + tW\|_D}\right) < C_0$$

uniformly for $t \in \mathbb{R}$.

On the other hand,

$$\|u_0 + tW\|_D \geq t\|W\|_D - \|u_0\|_D \geq C_0,$$

where $t \geq \tilde{t}$.

So that we can fix a positive number t_0 such that

$$\|u_0 + t_0W\|_D > t^-\left(\frac{u_0 + t_0W}{\|u_0 + t_0W\|_D}\right),$$

which yields that

$$u_0 + t_0W \in \mathcal{U}_2.$$

Combining this and the fact $u_0 \in \mathcal{U}_1$, we know that

$$u_0 + t_1W \in \Lambda^-$$

for some $0 < t_1 < t_0$.

So

$$c_1 = \inf_{\Lambda^-} I(u) \leq \max_{0\leq t\leq t_0} I(u_0 + tW) < c_0 + \frac{N + 2 - \alpha}{4N - 2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

And there exists a minimizing sequence $\{u_n\} \subset \Lambda^-$ satisfying

$$I(u_n) < c_1 + \frac{1}{n};$$
$$I(w) \geq I(u_n) - \frac{1}{n}\|u - w\|_D,$$

where $w \in \Lambda^-$.

So that

$$\begin{aligned} c_1 + 1 > I(u_n) &= \frac{1}{2}\|u_n\|_D^2 - \frac{1}{2\cdot 2_\alpha^*}B(u_n) - \int_{\Re^n} h(x)u_n\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2\cdot 2_\alpha^*}\right)\|u_n\|_D^2 - \left(1 - \frac{1}{2\cdot 2_\alpha^*}\right)\|h\|_{H^{-1}}\|u_n\|_D, \end{aligned}$$

which implies $\|u_n\|$ has an upper bound.

It follows from $\{u_n\} \subset \Lambda^-$ that

$$\|u_n\|_D^2 \le \left(2 \cdot 2_\alpha^* - 1\right) \frac{\|u_n\|_D^{2_\alpha^*}}{S_{H,L}^{2_\alpha^*}}.$$

Thus, $\|u_n\|_D$ has a uniform positive lower bound.

Similarly,

$$I(u_n) \to c_1, \qquad I'(u_n) \to 0 \quad \text{in } H^{-1}.$$

By Lemma 2.2 and

$$c_1 < c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}},$$

we obtain that

$$\int_{\Re^n} h(x) u_1 \, dx > 0 \quad \text{and} \quad u_1 \in \Lambda^+,$$

which leads to a contradiction.

In the case $h > 0$. Applying Lemma 2.1 to u_1 and $|u_1|$, we know that there exists $t^-(|u_1|)$ such that

$$t^-\left(|u_1|\right)|u_1| \in \Lambda^-.$$

Moreover,

$$t^-\left(|u_1|\right) \ge t_0\left(|u_1|\right) = t_0(u_1) = \left[\frac{A(u_1)}{(2_\alpha^* - 1)B(u_1)}\right]^{\frac{1}{2_\alpha^* - 2}}.$$

So

$$\int_{\Re^n} h(x) u_1 \, dx = \int_{\Re^n} h(x)|u_1| dx,$$

which implies that $u_1 \ge 0$. According to the maximum principle, we get $u_1 > 0$.

It is easy to see that $\|u_n\|$ is bounded, which yields that

$$\|u_n\|^2 = \|w_n\|^2 + \|v\|^2 + o(1), \quad n \to \infty,$$

and

$$\begin{aligned} &\int_{\Re^n} \frac{|w_n(x)|^p |w_n(y)|^p}{|x-y|^\alpha} \, dw \\ &\quad = \int_{\Re^n} \frac{|w_n(x)|^p |w_n(y)|^p}{|x-y|^\alpha} \, dw + \int_{\Re^n} \frac{|w(x)|^p |w(y)|^p}{|x-y|^\alpha} \, dw + o_n(1) \end{aligned}$$

as $n \to \infty$.

So

$$\begin{aligned} c \leftarrow \mathcal{J}(w_n) &= \frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\int_{\Re^n} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\alpha}\,dw - \int_{\Re^n} h(x)w_n\,dx \\ &= \frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\int_{\Re^n} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\alpha}\,dw - \int_{\Re^n} h(x)w_n\,dx \\ &\quad + \frac{1}{2}\|v\|^2 - \frac{1}{2p}\int_{\Re^n} \frac{|v(x)|^p|v(y)|^p}{|x-y|^\alpha}\,dw - \int_{\Re^n} h(x)v\,dx + o_n(1) \\ &= \mathcal{J}(v) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\tilde{B}(w_n) + o_n(1) \end{aligned}$$

and

$$\frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\tilde{B}(w_n) + o_n(1) < \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}. \tag{2.3}$$

Notice that

$$o(1) = \langle J'(u_n), u_n\rangle = \langle J'(v), v\rangle + \|w_n\|^2 - \tilde{B}(w_n) + o(1),$$

which yields that

$$\|w_n\|^2 - \tilde{B}(w_n) = o(1). \tag{2.4}$$

It follows from (2.4) that

$$\|w_n\|^2 = \tilde{B}(w_n) \le \frac{\|w_n\|^{2p}}{S_{\alpha,p}^p}$$

and

$$\begin{aligned} \frac{1}{2}\frac{p-1}{p} S_{\alpha,p}^{\frac{p}{p-1}} &= \frac{1}{2}\left(1 - \frac{1}{p}\right) S^{\frac{p}{p-1}} \\ &\le \frac{1}{2}\left(1 - \frac{1}{p}\right)\|w_n\|^2 \\ &= \frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\tilde{B}(w_n) + o_n(1) \\ &< \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}, \end{aligned}$$

which also leads to a contradiction.

Suppose that

$$\lim_{k\to\infty} \left\| \mathcal{B}_k u(x_k) \right\| = 0 \tag{2.5}$$

holds. Using Assumption (C) we get (2.2). It follows from (2.5) that the subsequence $\{k_j\}$ satisfies

$$\left\| \mathcal{B}_{k_j} u(x_{k_j}) \right\| \ge \varepsilon. \tag{2.6}$$

Set

$$K = \{k | \|\mathcal{B}_k u(x_k)\| \geq \varepsilon\}.$$

So we assume that

$$\|u(x_k)\| \geq \varepsilon$$

holds, where $k \in K$.

It follows from the definition of Algorithm and Lemma 2.1 that

$$\sum_{k\in K}[\nu(x_k) - \nu(x_{k+1})] \geq -\sum_{k\in K}\rho\mathcal{P}\tau_k(\tau_k^{p_k}) \geq \sum_{k\in K}\rho\frac{1}{2}\min\left\{c^{p_k}\varepsilon, \frac{\varepsilon}{M_l^2}\right\}\varepsilon.$$

Lemma 2.2 gives us that the sequence $\{\nu(x_k)\}$ is convergent, which yields that

$$\sum_{k\in K}\rho\frac{1}{2}\min\left\{c^{p_k}\varepsilon, \frac{\varepsilon}{M_l^2}\right\}\varepsilon < +\infty.$$

Then $p_k \to +\infty$ when $k \to +\infty$ and $k \in K$. It follows that

$$\begin{aligned} \min \quad & q_k(\tau) = \frac{1}{2}\left\| u(x_k) + \mathcal{B}_k\tau + \frac{3}{2}(s_{k-1}^{\mathcal{T}}\tau)^2 s_{k-1}\right\|^2, \\ \text{s.t.} \quad & \|\tau\| \leq c^{p_k-1}\|u(x_k)\| \end{aligned} \tag{2.7}$$

is unacceptable.

If we put $x'_{k+1} = x_k + \tau'_k$, then we have

$$\frac{\nu(x_k) - \nu(x'_{k+1})}{-\mathcal{P}\tau_k(\tau'_k)} < \rho. \tag{2.8}$$

By applying Lemma 2.1 and the definition of $\triangle_k$, we know that

$$-\mathcal{P}\tau_k(\tau'_k) \geq \frac{1}{2}\min\left\{c^{p_k-1}\varepsilon, \frac{\varepsilon}{M_l^2}\right\}\varepsilon.$$

By applying Lemma 2.2, we know that

$$\nu(x'_{k+1}) - \nu(x_k) - \mathcal{P}\tau_k(\tau'_k) = O(\|\tau'_k\|^2) = O(c^{2(p_k-1)}).$$

So

$$\left|\frac{\nu(x'_{k+1}) - \nu(x_k)}{\mathcal{P}\tau_k(\tau'_k)} - 1\right| \leq \frac{O(c^{2(p_k-1)})}{0.5\min\{c^{p_k-1}\varepsilon, \frac{\varepsilon}{M_l^2}\}\varepsilon + O(c^{2(p_k-1)}\varepsilon^2)}.$$

By applying $p_k \to +\infty$ as $k \to +\infty$, we know that

$$\frac{\nu(x_k) - \nu(x'_{k+1})}{-\mathcal{P}\tau_k(\tau'_k)} \to 1, \quad k \in K,$$

which also gives a contradiction to (2.8). □

3 Numerical results

This section reports some numerical results of Algorithm.

3.1 Problems

Define

$$u(x) = \left(\upsilon_1(x), \upsilon_2(x), \ldots, \upsilon_n(x)\right)^{\mathcal{T}}.$$

Problem 1 The Schrödinger differential function (see [12])

$$\upsilon_l(x) = 2\left(n + l(1 - \cos x_l) - \sin x_l - \sum_{j=1}^{n} \cos x_j\right)(2 \sin x_l - \cos x_l),$$

where $l = 1, 2, 3, \ldots, n$.

Initial guess:

$$x_0 = \left(\frac{101}{100n}, \frac{101}{100n}, \ldots, \frac{101}{100n}\right)^{\mathcal{T}}.$$

Problem 2 Logarithmic function

$$\upsilon_l(x) = \ln(x_l + 1) - \frac{x_l}{n},$$

where $l = 1, 2, 3, \ldots, n$.

Initial points:

$$x_0 = (1, 1, \ldots, 1)^{\mathcal{T}}.$$

Problem 3 Schrödinger differential function (see [5, pp. 471–472])

$$\upsilon_1(x) = (2 - 0.2x_1)x_1 - x_2 + 1,$$
$$\upsilon_l(x) = (2 - 0.2x_l)x_l - x_{i-1} + x_{i+1} + 1,$$
$$\upsilon_n(x) = (2 - 0.2x_n)x_n - x_{n-1} + 1,$$

where $l = 1, 2, 3, \ldots, n$.

Initial points:

$$x_0 = (-1, -1, \ldots, -1)^{\mathcal{T}}.$$

Problem 4 Trigexp function (see [5, p. 473])

$$\upsilon_1(x) = 3x_1^3 + x_2 - 4 + 2 \sin(x_1 - x_2) \sin(x_1 + x_2),$$
$$\upsilon_l(x) = -2x_{i-1}e^{x_{i-1} - x_l} + 3x_l\left(4 + 3x_l^2\right) + 2x_{i+1}$$

$$-\sin(x_l - x_{i+1})\sin(x_l + x_{i+1}) - 2,$$
$$\upsilon_n(x) = -2x_{n-1}e^{x_{n-1}-x_n} + 3x_n - 2,$$

where $l = 1, 2, 3, \ldots, n$.

Initial guess:

$$x_0 = (0, 0, \ldots, 0)^{\mathcal{T}}.$$

Problem 5 Let $u(x)$ be the gradient of

$$h(x) = \sum_{l=1}^{n} \left(e^{x_l} - x_l\right).$$

Then

$$\upsilon_l(x) = e^{x_l} - 1,$$

where $l = 1, 2, 3, \ldots, n$.

Initial points:

$$x_0 = \left(\frac{1}{n}, \frac{2}{n}, \ldots, 1\right)^{\mathcal{T}}.$$

Parameters: $c = 0.2$, $\epsilon = 10^{-2}$, $\rho = 0.03$, $p = 3$, $m = 6$ $\mathcal{H}_0$ is the unit matrix.
The method for (1.3) *and* (1.7): the *Dogleg* method [13, 25].
Code experiments: run on a PC with Intel Pentium(R) Xeon(R) E5507 CPU 2.27 GHz, 6.00 GB of RAM, and Windows 7 operating system.
Code software: MATLAB r2017a.
Stop rules: the program stops if $\|u(x)\| \leq 1e-4$ holds.
Other cases: we will stop the program if the iteration number is larger than ten hundred.

3.2 Results and discussion

The column meaning in the following tables:
Dim: the dimension. NI: the number of iterations.
NG: the norm function number. Time: the CPU-time in seconds.
Numerical results of Table 1 show the performance of these two algorithms about NI, NG, and Time. It is not difficult to see that both of these algorithm can successfully solve all these ten nonlinear problems.
It is easy to see that the NI and the NG of Algorithm have won since their performance profile plot is on top right. And the Time of Algorithm YL has superiority over Algorithm. Both of these two algorithms have good robustness.

4 Conclusions

In this paper, we considered the numerical method for solving the Schrödinger equations via Phragmén–Lindelöf inequalities under the order induced by a symmetric cone with

Table 1 Experiment results

Nr	Dim	Algorithm			Algorithm YL			Nr	Dim	Algorithm			Algorithm YL		
		Ni	NG	Time	NI	NG	Time			NI	NG	Time	NI	NG	Time
1	300	9	18	10.93567	11	22	1.778411	6	300	3	6	1.279208	5	6	0.7176046
	800	9	18	52.46314	11	22	7.176046		800	3	6	5.397635	5	16	2.88601
	1600	8	14	215.453	11	22	42.57267		1600	3	6	29.88979	5	16	16.39571
2	300	4	10	11.27887	6	7	1.185608	7	300	5	14	3.790824	12	49	1.435209
	800	4	10	45.94229	6	7	4.071626		800	5	14	22.52654	12	49	4.69563
	1600	4	10	251.38	6	7	22.58894		1600	5	14	102.0403	17	83	19.23492
3	300	4	10	2.808018	64	125	8.642455	8	300	1	2	1.294808	3	6	0.2808018
	800	4	10	10.74847	78	129	52.26034		800	1	2	5.694037	3	6	0.8580055
	1600	4	10	70.80885	68	99	262.5653		1600	1	2	31.091	3	6	3.775224
4	300	2	2	0.9112052	6	17	1.092007	9	300	13	19	11.01367	12	15	1.60681
	800	2	2	2.839218	6	22	3.08882		800	9	15	40.95026	11	17	7.191646
	1600	2	2	14.08689	6	22	13.27569		1600	10	19	299.3191	10	16	38.07984
5	300	3	6	1.731611	6	7	0.936006	10	300	3	9	4.558416	40	50	12.44888
	800	3	6	5.616036	6	7	3.650423		800	3	9	11.62207	40	50	49.43672
	1600	3	6	30.32659	6	7	22.44854		1600	3	9	73.07087	41	53	365.7911

the function involved being monotone. Based on the Phragmén–Lindelöf inequalities, the underlying system of inequalities was reformulated as a system of smooth equations, and a Schrödinger-type method was proposed to solve it iteratively so that a solution of the system of the Schrödinger equations was found. By means of the Schrödinger type inequalities, the algorithm was proved to be well defined and to be globally convergent under weak assumptions and locally quadratically convergent under suitable assumptions. Preliminary numerical results indicate that the algorithm was effective.

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