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A new discrete Hilbert-type inequality involving partial sums

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Abstract

In this paper, we derive a new discrete Hilbert-type inequality involving partial sums. Moreover, we show that the constant on the right-hand side of this inequality is the best possible. As an application, we consider some particular settings.

MSC: 26D15

Keywords: Hilbert-type inequality; Gamma function; The best possible constant; Conjugate exponents

1 Introduction

The Hilbert inequality [5] asserts that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

holds for non-negative sequences a_m and b_n, provided that $(\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$ and $(\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}} > 0$. The parameters p and q appearing in (1) are mutually conjugate, i.e. $\frac{1}{p}+\frac{1}{q}=1$, where $p>1$. In addition, the constant $\frac{\pi}{\sin(\pi/p)}$ is the best possible in the sense that it can not be replaced with a smaller constant so that (1) still holds.

The Hilbert inequality is one of the most interesting inequalities in mathematical analysis. For a detailed review of the starting development of the Hilbert inequality the reader is referred to monograph [5]. The most important recent results regarding Hilbert-type inequalities are collected in monographs [4] and [7].

In 2006, Krnić and Pečarić [6], obtained the following generalization of classical Hilbert inequality.

Theorem 1 *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and let $2<s\leq 14$. Suppose that $\alpha_1\in[-\frac{1}{q},\frac{1}{q})$, $\alpha_2\in[-\frac{1}{p},\frac{1}{p})$ and $p\alpha_2+q\alpha_1=2-s$. If $\sum_{m=1}^{\infty} m^{pq\alpha_1-1}a_m^p<\infty$ and $\sum_{n=1}^{\infty} n^{pq\alpha_2-1}b_n^q<\infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^s} < B(1-p\alpha_2, p\alpha_2+s-1)\left(\sum_{m=1}^{\infty}m^{pq\alpha_1-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{pq\alpha_2-1}b_n^q\right)^{\frac{1}{q}}, \tag{2}$$

where the constant $B(1-p\alpha_2, p\alpha_2+s-1)$ is the best possible.

In the last few years, considerable attention is given to a class of Hilbert-type inequalities where the functions and sequences are replaced by certain integral or discrete operators. For example: in 2013, Azar [3] introduced a new Hilbert-type integral inequality including functions $F(x) = \int_0^x f(t)\,dt$ and $g(y) = \int_0^y g(t)\,dt$. For some related Hilbert-type inequalities where the functions and sequences are replaced by certain integral or discrete operators, the reader is referred to [1] and [2].

The main objective of this paper is to derive a discrete Hilbert-type inequality involving partial sums, similar to a result of Azar [3]. Such inequality is derived by virtue of inequality (2) and some well-known classical inequalities. As an application, we consider some particular settings.

2 Preliminaries and lemma

Recall that the Gamma function $\Gamma(\theta)$ and the Beta function $B(\mu,\nu)$ are defined, respectively, by

$$\Gamma(\theta) = \int_0^\infty t^{\theta-1}e^{-t}\,dt, \quad \theta > 0,$$

$$B(\mu,\nu) = \int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}\,dt, \quad \mu,\nu > 0,$$

and they satisfy the following relation

$$B(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

By the definition of the Gamma function, the following equality holds:

$$\frac{1}{(m+n)^\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(m+n)t}\,dt. \tag{3}$$

To prove our main results we need the following lemma.

Lemma 2 *Let $a_m > 0$, $a_m \in \ell^1$, $A_m = \sum_{k=1}^m a_k$, then for $t > 0$, we have*

$$\sum_{m=1}^\infty e^{-tm}a_m \le t\sum_{m=1}^\infty e^{-tm}A_m. \tag{4}$$

Proof Using Abel's summation by parts formula and the inequality $1 - \frac{1}{e^t} \le t$, we have

$$\begin{aligned}\sum_{m=1}^\infty e^{-tm}a_m &= \lim_{m\to\infty} A_m e^{-t(m+1)} + \sum_{m=1}^\infty A_m\bigl(e^{-tm} - e^{-t(m+1)}\bigr)\\ &= \left(1 - \frac{1}{e^t}\right)\sum_{m=1}^\infty e^{-tm}A_m\\ &\le t\sum_{m=1}^\infty e^{-tm}A_m.\end{aligned}$$

The lemma is proved. □

3 Main results

Theorem 3 *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $a_m, b_n > 0$, $a_m, b_n \in \ell^1$, *define* $A_m = \sum_{k=1}^m a_k$, $B_n = \sum_{k=1}^n b_k$. *If* $\sum_{m=1}^\infty m^{pq\alpha_1 - 1} A_m^p < \infty$ *and* $\sum_{n=1}^\infty n^{pq\alpha_2 - 1} B_n^q < \infty$, *then*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} < C \left(\sum_{m=1}^\infty m^{pq\alpha_1 - 1} A_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{pq\alpha_2 - 1} B_n^q \right)^{\frac{1}{q}}, \tag{5}$$

where $\alpha_1 \in [-\frac{1}{q}, 0)$, $\alpha_2 \in [-\frac{1}{p}, 0)$ *and* $p\alpha_2 + q\alpha_1 = -\lambda$. *In addition, the constant* $C = pq\alpha_1\alpha_2 B(-p\alpha_2, -q\alpha_1)$ *is the best possible in* (5).

Proof Using (3), the left-hand side of inequality (5) can be expressed in the following form:

$$\begin{aligned} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} &= \frac{1}{\Gamma(\lambda)} \sum_{m=1}^\infty \sum_{n=1}^\infty a_m b_n \left(\int_0^\infty t^{\lambda-1} e^{-(m+n)t}\, dt \right) \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \left(\sum_{m=1}^\infty e^{-tm} a_m \right) \left(\sum_{n=1}^\infty e^{-tn} b_n \right) dt. \end{aligned} \tag{6}$$

Now, by applying inequality (4) and equality (3) to the previous equality, we have

$$\begin{aligned} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} &\le \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+1} \left(\sum_{m=1}^\infty e^{-tm} A_m \right) \left(\sum_{n=1}^\infty e^{-tn} B_n \right) dt \\ &= \frac{1}{\Gamma(\lambda)} \sum_{m=1}^\infty \sum_{n=1}^\infty A_m B_n \left(\int_0^\infty t^{\lambda+1} e^{-(m+n)t}\, dt \right) \\ &= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{A_m B_n}{(m+n)^{\lambda+2}}. \end{aligned} \tag{7}$$

Moreover, the last double series represents the left-hand side of the Hilbert-type inequality (2) for $s = 2 + \lambda$, that is, we have the inequality

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{A_m B_n}{(m+n)^{\lambda+2}} < B(1 - p\alpha_2, p\alpha_2 + \lambda + 1) \left(\sum_{m=1}^\infty m^{pq\alpha_1 - 1} A_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{pq\alpha_2 - 1} B_n^q \right)^{\frac{1}{q}},$$

so by (7) we get

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} < C \left(\sum_{m=1}^\infty m^{pq\alpha_1 - 1} A_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{pq\alpha_2 - 1} B_n^q \right)^{\frac{1}{q}}.$$

Now we shall prove that the constant factor is the best possible. Assuming that the constant C is not the best possible, then there exists a positive constant K such that $K < C$ and (5) still remains valid if C is replaced by K. Further, consider the $\tilde{a}_m = m^{-q\alpha_1 - 1 - \frac{\varepsilon}{p}}$ and $\tilde{b}_n = n^{-p\alpha_2 - 1 - \frac{\varepsilon}{q}}$, where $\varepsilon > 0$ is sufficiently small number. Then, we have

$$\tilde{A}_m = \sum_{k=1}^m \tilde{a}_k = \sum_{k=1}^m m^{-q\alpha_1 - 1 - \frac{\varepsilon}{p}} \le \int_0^m x^{-q\alpha_1 - 1 - \frac{\varepsilon}{p}}\, dx = \frac{m^{-q\alpha_1 - \frac{\varepsilon}{p}}}{-q\alpha_1 - \frac{\varepsilon}{p}},$$

and similarly

$$\tilde{B}_n = \sum_{k=1}^{n} \tilde{b}_k \le \frac{n^{-p\alpha_2-\frac{\varepsilon}{q}}}{-p\alpha_2-\frac{\varepsilon}{q}}.$$

Inserting the above sequences in (5), the right-hand side of (5) becomes

$$\begin{aligned}
&K\left(\sum_{m=1}^{\infty} m^{pq\alpha_1-1}\tilde{A}_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{pq\alpha_2-1}\tilde{B}_n^q\right)^{\frac{1}{q}} \\
&\quad\le K\left(\sum_{m=1}^{\infty} m^{pq\alpha_1-1}\frac{m^{-pq\alpha_1-\varepsilon}}{(-q\alpha_1-\frac{\varepsilon}{p})^p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{pq\alpha_2-1}\frac{n^{-pq\alpha_2-\varepsilon}}{(-p\alpha_2-\frac{\varepsilon}{q})^q}\right)^{\frac{1}{q}} \\
&\quad= \frac{K}{(-q\alpha_1-\frac{\varepsilon}{p})(-p\alpha_2-\frac{\varepsilon}{q})}\left(\sum_{m=1}^{\infty} m^{-1-\varepsilon}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-1-\varepsilon}\right)^{\frac{1}{q}} \\
&\quad= \frac{K}{(-q\alpha_1-\frac{\varepsilon}{p})(-p\alpha_2-\frac{\varepsilon}{q})}\left(1+\sum_{m=2}^{\infty} m^{-1-\varepsilon}\right) \\
&\quad\le \frac{K}{(-q\alpha_1-\frac{\varepsilon}{p})(-p\alpha_2-\frac{\varepsilon}{q})}\left(1+\int_1^{\infty} x^{-1-\varepsilon}\,dx\right) \\
&\quad= \frac{K(1+\varepsilon)}{\varepsilon(-q\alpha_1-\frac{\varepsilon}{p})(-p\alpha_2-\frac{\varepsilon}{q})}.
\end{aligned} \tag{8}$$

Now, let us estimate the left-hand side of inequality (5). Namely, by inserting the above defined sequences $\tilde{a}_m$ and $\tilde{b}_n$ in the left-hand side of inequality (5), we get the inequality

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(m+n)^\lambda} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{-q\alpha_1-1-\frac{\varepsilon}{p}}n^{-p\alpha_2-1-\frac{\varepsilon}{q}}}{(m+n)^\lambda} \\
&\ge \int_1^{\infty}\int_1^{\infty}\frac{x^{-q\alpha_1-1-\frac{\varepsilon}{p}}y^{-p\alpha_2-1-\frac{\varepsilon}{q}}}{(x+y)^\lambda}\,dx\,dy \\
&= \int_1^{\infty} x^{-1-\varepsilon}\int_{1/x}^{\infty}\frac{u^{-p\alpha_2-1-\frac{\varepsilon}{q}}}{(1+u)^\lambda}\,du\,dx \\
&= \int_1^{\infty} x^{-1-\varepsilon}\left(\int_0^{\infty}\frac{u^{-p\alpha_2-1-\frac{\varepsilon}{q}}}{(1+u)^\lambda}\,du-\int_0^{1/x}\frac{u^{-p\alpha_2-1-\frac{\varepsilon}{q}}}{(1+u)^\lambda}\,du\right)dx \\
&\ge \int_1^{\infty} x^{-1-\varepsilon}\left(\int_0^{\infty}\frac{u^{-p\alpha_2-1-\frac{\varepsilon}{q}}}{(1+u)^\lambda}\,du-\int_0^{1/x}u^{-p\alpha_2-1-\frac{\varepsilon}{q}}\,du\right)dx \\
&= \frac{1}{\varepsilon}\cdot B\left(-p\alpha_2-\frac{\varepsilon}{q},-q\alpha_1+\frac{\varepsilon}{q}\right)-\frac{1}{(p\alpha_2+\frac{\varepsilon}{q})(p\alpha_2-\frac{\varepsilon}{p})}.
\end{aligned} \tag{9}$$

It follows from inequalities (8) and (9) that

$$B\left(-p\alpha_2-\frac{\varepsilon}{q},-q\alpha_1+\frac{\varepsilon}{q}\right)-\frac{\varepsilon}{(p\alpha_2+\frac{\varepsilon}{q})(p\alpha_2-\frac{\varepsilon}{p})} \le \frac{K(1+\varepsilon)}{(-q\alpha_1-\frac{\varepsilon}{p})(-p\alpha_2-\frac{\varepsilon}{q})}. \tag{10}$$

Now, letting $\varepsilon \to 0+$, relation (10) yields a contradiction with the assumption $K < C$. So the constant C, in inequality (5) is the best possible. □

Considering Theorem 3, equipped with parameters $\lambda = 1$, $\alpha_1 = -\frac{1}{q^2}$, $\alpha_2 = -\frac{1}{p^2}$, we obtain the following result.

Corollary 4 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $a_m, b_n > 0$ with $a_m, b_n \in \ell^1$. If $\sum_{m=1}^{\infty} m^{-p} A_m^p < \infty$ and $\sum_{n=1}^{\infty} n^{-q} B_n^q < \infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{pq\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\left(\frac{A_m}{m}\right)^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^q\right)^{\frac{1}{q}}. \tag{11}$$

where the constant $\frac{\pi}{pq\sin(\pi/p)}$ is the best possible.

Remark 5 It should be noticed here that if sequences $a_m, b_n \in \ell^1$ such that $\sum_{m=1}^{\infty} a_m^p < \infty$ and $\sum_{n=1}^{\infty} b_n^q < \infty$, inequality (11) provides refinement of the Hilbert inequality. Indeed, by Hardy's inequality, the series $\sum_{m=1}^{\infty}(\frac{A_m}{m})^p$ and $\sum_{n=1}^{\infty}(\frac{B_n}{n})^q$ are converge. So, inequality (11) holds. The Hilbert inequality becomes after applying Hardy's inequality on the right-hand side of inequality (11).

Letting $\alpha_1 = \alpha_2 = \frac{-\lambda}{pq}$ in Theorem 3, we can obtain the following Hilbert-type inequality.

Corollary 6 *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $0 < \lambda \le \min\{p, q\}$, $a_m, b_n > 0$ with $a_m, b_n \in \ell^1$. If $\sum_{m=1}^{\infty} m^{-\lambda-1} A_m^p < \infty$ and $\sum_{n=1}^{\infty} n^{-\lambda-1} B_n^q < \infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)\left(\sum_{m=1}^{\infty} m^{-\lambda-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-\lambda-1}B_n^q\right)^{\frac{1}{q}}, \tag{12}$$

where the constant $\frac{\lambda^2}{pq}B(\frac{\lambda}{q}, \frac{\lambda}{p})$ is the best possible.

4 Conclusion

In the present study, we have established a discrete Hilbert-type inequality involving partial sums. Moreover, we have proved that the constant on the right-hand side of this inequality is the best possible. As an application, we considered some particular settings.

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Authors' contributions
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