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A general slicing inequality for measures of convex bodies

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Abstract

We consider the following inequality:

$$\mu(L)^{\frac{n-k}{n}} \leq C^k \max_{H\in Gr_{n-k}} \mu(L \cap H),$$

which is a variant of the notable slicing inequality in convex geometry, where L is an origin-symmetric star body in $\mathbb{R}^n$ and is μ-measurable, μ is a nonnegative measure on $\mathbb{R}^n$, Gr_{n-k} is the Grassmanian of an $n-k$-dimensional subspaces of $\mathbb{R}^n$, and C is a constant. By constructing the generalized k-intersection body with respect to μ, we get some results on this inequality.

Keywords: Convex bodies; Intersection bodies; Generalized measures

1 Introduction

The notable slicing problem in convex geometry asks whether there exists a constant C such that for any positive integer $n \geq 1$ and any origin-symmetric convex body [1] L in $\mathbb{R}^n$

$$|L|^{\frac{n-1}{n}} \leq C^k \max_{\xi\in S^{n-1}} \left|L \cap \xi^\perp\right|, \tag{1.1}$$

where $\xi^\perp$ is the hyperplane in $\mathbb{R}^n$, perpendicular to ξ passing through the origin, and $|L|$ stands for volume of proper dimension. There is a lot of literature focusing on this problem. We refer the reader to [2–5] [6, Theorem 9.4.11] for the history and more results. Iterating (1.1) one gets the lower slicing problem asking whether the inequality

$$|L|^{\frac{n-k}{n}} \leq C^k \max_{H\in Gr_{n-k}} |L \cap H| \tag{1.2}$$

holds with an absolute constant C, where $1 \leq k < n$ and Gr_{n-k} is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$; see [7, 8].

A more general problem considered in [9] is: Does there exist an absolute constant C such that for every positive integer n and every integer $1 \leq k < n$, and for every origin-symmetric convex body L and every measure μ with nonnegative even continuous density in $\mathbb{R}^n$,

$$\mu(L) \le C^k \max_{H \in Gr_{n-k}} \mu(L \cap H) |L|^{\frac{k}{n}}, \tag{1.3}$$

where $|L|$ stands for volume of proper dimension?

This question is an extension to that of (1.2), general measures taking place of volumes, a major open problem in convex; see [4, 10–12]. By this reason, (1.3) is also called slicing inequality in convex geometry; see [9, 13].

In the literature [9] one proved (1.3) for unconditional convex bodies and for duals of bodies with bounded volume ratio. And it also was proved that for every $\lambda \in (0,1)$ there exists a constant $C = C(\lambda)$ such that (1.3) holds for every positive integer n, for every origin-symmetric convex body L, the codimensions of whose sections in $\mathbb{R}^n$ $k \ge \lambda n$, and for every measure μ with continuous density.

Inequality (1.3) gives a link between $\mu(L)$ and $\mu(L \cap H)$, which denote different dimensional measures. Observe (1.3) and we found that there are two kinds of measures with respect to L, $\mu(L)$ and the Lebesgue measure $|L|$. Therefore, we consider a problem that whether the Lebesgue measure $|L|$ in (1.3) could be replaced by the general measure $\mu(L)$, that is, whether the following inequality holds:

$$\mu(L)^{\frac{n-k}{n}} \le C^k \max_{H \in Gr_{n-k}} \mu(L \cap H) \tag{1.4}$$

for some constant C. Inequality (1.4) is a variant of (1.3) and but more concise than (1.3). And inequality (1.4) is the purpose of this article. Next, we introduce the main tool employed in this article and the program of this article.

The k-intersection body, introduced by [7, 14], plays an important role in the solution to the Busemann–Petty problem, which is equivalent to the slicing problem; see [6, 7, 9, 15–17]. In this article, we define the generalized k-intersection body with measure μ, and denote by $\mathcal{BP}^n_{k,\mu}$ the class of generalized k-intersection bodies with measure μ, with $\mathcal{BP}^n_k$ being the class of k-intersection bodies. If μ is the Lebesgue measure on considered set, then the generalized k-intersection body with measure μ becomes the k-intersection body, and $\mathcal{BP}^n_{k,\mu}$ becomes $\mathcal{BP}^n_k$. Using the outer measure ratio distance from μ-measurable set L to the class $\mathcal{BP}^n_{k,\mu}$, denoted by $o.m.r.(L, \mathcal{BP}^n_{k,\mu})$ (see (2.3)), we get inequality (1.4) for some constant C. We also give a comparison of $o.m.r.(L, \mathcal{BP}^n_{k,\mu})$ and $o.v.r.(L, \mathcal{BP}^n_k)$

$$o.m.r.\big(L, \mathcal{BP}^n_{k,\mu}\big) \le C o.v.r.\big(L, \mathcal{BP}^n_k\big),$$

for some constant C, which depends only on μ. Then the results for $o.v.r.(L, \mathcal{BP}^n_k)$ [8, 9, 13] can transfer to that for $o.m.r.(L, \mathcal{BP}^n_{k,\mu})$.

This article is arranged naturally. In Sect. 2, we define the class of generalized k-intersection body with measure μ, $\mathcal{BP}^n_{k,\mu}$, and the outer measure ratio distance from μ-measurable set L to the class $\mathcal{BP}^n_{k,\mu}$. The main results are exposed in Sect. 3. Section 4 contains the proof of the main results. Finally, in the last section, we give the conclusions and some remarks.

2 Preliminaries

The k-intersection body plays an important role in the solution to the Busemann–Petty problem, which is equivalent to the slicing problem. An analog to the k-intersection body plays an important role in the solution of our problem. In this section, we will construct the class of generalized k-intersection body with measure μ, $\mathcal{BP}^n_{k,\mu}$ and define the outer measure ratio distance from μ-measurable set L to the class $\mathcal{BP}^n_{k,\mu}$.

Let g be a continuous nonnegative even function on $\mathbb{R}^n$ ($g(x) = g(-x)$ for $x \in \mathbb{R}^n$; see [9]). For any Lebesgue measurable set A in $\mathbb{R}^n$, let

$$\mu(A) = \int_A g(x)\,dx,$$

where dx is the Lebesgue measure. Then μ is a nonnegative measure on $\mathbb{R}^n$, and function g is called the density of measure μ [9]. At this time, A is called μ-measurable, and $\mu(A)$ is the μ-measure of A.

A closed bounded set L in $\mathbb{R}^n$ is called a star body if every straight line passing through the origin crosses the boundary of L at exactly two points different from the origin and the origin is an interior point of L, where the Minkowski functional of L defined by

$$\|x\|_L = \min\{a \geq 0 : x \in aL\}$$

is a continuous function on $\mathbb{R}^n$; see [9]. The radial function of a star body L is defined by

$$\rho_L(x) = \|x\|_L^{-1} \quad \text{for } x \in \mathbb{R}^n \text{ and } x \neq 0;$$

see [9]. If $x \in \mathbb{S}^{n-1}$ then $\rho_L(x)$ is the radius of L in the direction x.

The generalized k-intersection body was introduced in [9]. An origin-symmetric star body L in $\mathbb{R}^n$ is a generalized k-intersection body, and write $L \in \mathcal{BP}^n_k$, if there exists a finite Borel nonnegative measure μ_1 on Gr_{n-k} so that for every $\varphi \in C(S^{n-1})$ (class of continuous functions on S^{n-1})

$$\int_{S^{n-1}} \|\theta\|_L^{-k}\varphi(\theta)\,d\theta = \int_{Gr_{n-k}} R_{n-k}\varphi(H)\,d\mu_1(H),$$

where $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is the $(n-k)$-dimensional spherical Radon transform, defined by

$$R_{n-k}\varphi(H) = \int_{S^{n-1}\cap H} \varphi(x)\,dx$$

for every function $\varphi \in C(S^{n-1})$ and for every $H \in Gr_{n-k}$.

Putting the measure μ into the generalized k-intersection body, we define the generalized k-intersection body with respect to measure μ. An origin-symmetric star body K in $\mathbb{R}^n$ is called a generalized k-intersection body with respect to measure μ, denoted by $K \in \mathcal{BP}^n_{k,\mu}$ (or $K \in \mathcal{BP}^n_{k,g}$), if there exists a nonnegative finite Borel measure μ_1 on Gr_{n-k} such that for every φ in $C(S^{n-1})$

$$\int_{S^{n-1}} \left(n\int_0^{\|\theta\|_K^{-1}} r^{n-1}g(r\theta)\,dr \right)^{\frac{k}{n}} \varphi(\theta)\,d\theta = \int_{Gr_{n-k}} R_{n-k}\varphi(H)\,d\mu_1(H), \tag{2.1}$$

where g is the density of measure μ.

Note that when $g \equiv 1$ (namely μ is the Lebesgue measure), $\mathcal{BP}^n_{k,g}$ becomes $\mathcal{BP}^n_k$.

For a convex body L in $\mathbb{R}^n$, the outer volume ratio distance from L to $\mathcal{BP}^n_k$ is defined by

$$o.v.r.(L, \mathcal{BP}^n_k) = \inf\left\{ \left(\frac{|K|}{|L|} \right)^{\frac{1}{n}} : L \subset K, K \in \mathcal{BP}^n_k \right\}; \tag{2.2}$$

see [9].

Similarly, for a Lebesgue measurable convex body L in $\mathbb{R}^n$, we define the outer measure ratio distance with respect to measure μ from L to the class $\mathcal{BP}^n_{k,\mu}$ by

$$o.m.r.(L, \mathcal{BP}^n_{k,\mu}) = \inf\left\{ \left(\frac{\mu(K)}{\mu(L)} \right)^{\frac{1}{n}} : L \subset K, K \in \mathcal{BP}^n_{k,\mu} \right\}. \tag{2.3}$$

Now we turn to the main results in the next section.

3 Main results and discussion

In this section, the goal is to establish the inequalities, namely that similar to (1.4), which reveals the relationship of the general measures of origin-symmetric star bodies and their $n-k$ ($1 \le k \le n-1$) dimensional intersection bodies, and is a variant of the classical slicing problems (1.3) in convex geometry.

Theorem 3.1 *Let L be an origin-symmetric star body in $\mathbb{R}^n$ and μ-measurable with density g, and $m = \inf_{x\in K}\{g(x)\} > 0$. Then, for $1 \le k \le n-1$,*

$$\big(\mu(L)\big)^{\frac{n-k}{n}} \le \big(o.m.r.(L, \mathcal{BP}^n_{k,\mu})\big)^k \left(\frac{n}{m} \right)^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \max_{H \in Gr_{n-k}} \mu(L \cap H). \tag{3.1}$$

Theorem 3.2 *Let L be an origin-symmetric star body in $\mathbb{R}^n$ and μ-measurable with density g, $R = \frac{1}{2}\operatorname{diam}(L)$, and*

$$\int_0^{\|\theta\|_L^{-1}} r^{n-1} g(r\theta)\, dr \ge \frac{1}{n} \tag{3.2}$$

for all $\theta \in S^{n-1}$. Then, for $1 \le k \le n-1$,

$$\big(\mu(L)\big)^{\frac{n-k}{n}} \le \big(o.m.r.(L, \mathcal{BP}^n_{k,\mu})\big)^k R^k n^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \max_{H \in Gr_{n-k}} \mu(L \cap H). \tag{3.3}$$

By Theorem 3.2 and the relationship (Proposition 4.7) between $o.m.r.(L, \mathcal{BP}^n_{k,\mu})$ and $o.v.r.(L, \mathcal{BP}^n_k)$, we have the following.

Theorem 3.3 *Under the assumptions of Theorem 3.2,*

$$\big(\mu(L)\big)^{\frac{n-k}{n}} \le \big(o.v.r.(L, \mathcal{BP}^n_k)\big)^k \left(\frac{M}{m} \right)^{\frac{2k}{n}} R^k n^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \max_{H \in Gr_{n-k}} \mu(L \cap H), \tag{3.4}$$

where $m = \inf_{x\in\mathbb{R}^n} g(x) > 0$ and $M = \sup_{x\in\mathbb{R}^n} g(x) < +\infty$.

Theorems 3.1 and 3.2 both contain the coefficient: the outer measure ratio distance with respect to measure μ from L to the class $\mathcal{BP}^n_{k,\mu}$. Moreover, the coefficient in Theorem 3.1 is also relevant to the measure μ, the coefficient in Theorem 3.2 is also relevant to the diameter of L.

Under the assumptions of Theorem 3.2, the outer measure ratio distance with respect to measure μ from L to the class $\mathcal{BP}^n_{k,\mu}$, $o.m.r.(L,\mathcal{BP}^n_{k,\mu})$ can be replaced by the outer volume ratio distance from L to $\mathcal{BP}^n_k$, $o.v.r.(L,\mathcal{BP}^n_k)$, which essentially is $o.m.r.(L,\mathcal{BP}^n_{k,\mu})$ specialized by letting μ be the Lebesgue measure. This result is stated as Theorem 3.3. The coefficient in Theorem 3.3 is also relevant to the measure μ.

4 Proof of the main results

First, using the polar formula for volume of a star body L we get a useful formula

$$|L| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\theta\|_L^{-n}\,d\theta. \tag{4.1}$$

To prove Theorem 3.1, we first give the following result.

Lemma 4.1 *Let K be in $\mathcal{BP}^n_{k,g}$ and μ-measurable with density g, and $m = \inf_{x\in K} g(x) > 0$. Assume that f is a continuous nonnegative even function on K, and $\varepsilon > 0$. If for every $H \in Gr_{n-k}$,*

$$\int_{K\cap H} f(x)\,dx \le \varepsilon,$$

then, for $1 \le k \le n-1$,

$$\int_K f(x)\,dx \le \left(\frac{n}{m}\right)^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|}\big(\mu(K)\big)^{\frac{k}{n}}\varepsilon. \tag{4.2}$$

Proof Writing the integrals in spherical coordinates, we get

$$\int_K f(x)\,dx = \int_{S^{n-1}}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\,dr\right)d\theta$$

and

$$\begin{aligned}\int_{K\cap H} f(x)\,dx &= \int_{S^{n-1}\cap H}\left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr\right)d\theta\\ &= R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\cdot)\,dr\right)(H).\end{aligned}$$

So the condition of the lemma can be written as

$$R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\cdot)\,dr\right)(H) \le \varepsilon, \quad \text{for all } H \in Gr_{n-k}.$$

Integrate both sides with respect to the measure μ_1 that corresponds to K as a generalized k-intersection body with respect to measure μ by (2.1). We get

$$\int_{Gr_{n-k}} R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\cdot)\,dr\right)(H)\,d\mu_1(H) \le \varepsilon\mu_1(Gr_{n-k}). \tag{4.3}$$

Estimate the integral in the left-hand side of (4.3) using $K \in \mathcal{BP}_{k,g}^n$ and $m > 0$, then we have

$$\begin{aligned}
&\int_{Gr_{n-k}} R_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\cdot)\,dr\right)(H)\,d\mu_1(H) \\
&\quad = \int_{S^{n-1}} \left(n\int_0^{\|\theta\|_K^{-1}} r^{n-1} g(r\theta)\,dr\right)^{\frac{k}{n}} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr\,d\theta \\
&\quad \ge m^{\frac{k}{n}} \int_{S^{n-1}} \|\theta\|_K^{-k} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr\,d\theta.
\end{aligned} \tag{4.4}$$

Noting that $\|\theta\|_K^{-1} \ge r$ in the right-hand side of (4.4), we get

$$\int_{S^{n-1}} \|\theta\|_K^{-k} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr\,d\theta \ge \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\,dr\,d\theta = \int_K f(x)\,dx. \tag{4.5}$$

Now we estimate $\mu_1(Gr_{n-k})$ in the right-hand side of (4.3).

By the assumptions of Lemma 4.1 and the integral transform of spherical coordinates, we get

$$\mu(K) = \int_K g(x)\,dx = \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} r^{n-1} g(r\theta)\,dr\,d\theta. \tag{4.6}$$

Using $1 = R_{n-k}1(H)/|S^{n-k-1}|$ for every $H \in Gr_{n-k}$, definition (2.1) and Hölder's inequality, we have

$$\begin{aligned}
\mu_1(Gr_{n-k}) &= \frac{1}{|S^{n-k-1}|}\int_{Gr_{n-k}} R_{n-k}1(H)\,d\mu_1(H) \\
&= \frac{1}{|S^{n-k-1}|}\int_{S^{n-1}} \left(n\int_0^{\|\theta\|_K^{-1}} r^{n-1} g(r\theta)\,dr\right)^{\frac{k}{n}} d\theta \\
&\le \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|}\left(n\int_{S^{n-1}}\int_0^{\|\theta\|_K^{-1}} r^{n-1} g(r\theta)\,dr\,d\theta\right)^{\frac{k}{n}}.
\end{aligned} \tag{4.7}$$

Putting (4.6) into the right-hand side of (4.7), we have

$$\mu_1(Gr_{n-k}) \le \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} n^{\frac{k}{n}} \left(\mu(K)\right)^{\frac{k}{n}}. \tag{4.8}$$

Combination of (4.3),(4.4),(4.5) and (4.8) gives

$$\int_K f(x)\,dx \le \left(\frac{n}{m}\right)^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \left(\mu(K)\right)^{\frac{k}{n}} \varepsilon,$$

which completes the proof Lemma 4.1. □

Next let us prove Theorem 3.1.

Proof of Theorem 3.1 Set constant $C > o.m.r.(L, \mathcal{BP}^n_{k,\mu})$. Then there exists a star body $K \in \mathcal{BP}^n_{k,\mu}$ such that $L \subset K$ and $(\mu(K))^{\frac{1}{n}} \le C(\mu(L))^{\frac{1}{n}}$

Let $f = g\chi_L$, where χ_L is the indicator function of L, then f is nonnegative on K.

Put $\varepsilon = \max_{H\in Gr_{n-k}} \int_{K\cap H} f(x)\,dx = \max_{H\in Gr_{n-k}} \int_{L\cap H} g(x)\,dx = \max_{H\in Gr_{n-k}} \mu(L\cap H)$. Apply Lemma 4.1 to f and K (f may be not continuous, but we do an easy approximation) and we have

$$\begin{aligned}\mu(L) = \int_L g(x)\,dx = \int_K f(x)\,dx &\le \left(\frac{n}{m}\right)^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \big(\mu(K)\big)^{\frac{k}{n}} \max_{H\in Gr_{n-k}} \mu(L\cap H)\\ &\le C^k \left(\frac{n}{m}\right)^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \big(\mu(L)\big)^{\frac{k}{n}} \max_{H\in Gr_{n-k}} \mu(L\cap H).\end{aligned} \tag{4.9}$$

Let $C \to o.m.r.(L, \mathcal{BP}^n_{k,\mu})$ in (4.9). Then

$$\mu(L) \le \big(o.m.r.(L, \mathcal{BP}^n_{k,\mu})\big)^k \left(\frac{n}{m}\right)^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \big(\mu(L)\big)^{\frac{k}{n}} \max_{H\in Gr_{n-k}} \mu(L\cap H),$$

i.e.,

$$\big(\mu(L)\big)^{\frac{n-k}{n}} \le \big(o.v.r.(L, \mathcal{BP}^n_{k,\mu})\big)^k \left(\frac{n}{m}\right)^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \max_{H\in Gr_{n-k}} \mu(L\cap H),$$

which completes the proof of Theorem 3.1. □

To prove Theorem 3.2, we need to apply the following lemma which comes from the proof of Lemma 4.1.

Lemma 4.2 *Assume that K is in $\mathcal{BP}^n_{k,g}$ and μ-measurable with density g, $m = \inf_{x\in K} g(x) > 0$. Let f be continuous nonnegative even function on K, $1 \le k \le n-1$ and $\varepsilon > 0$. If*

$$\int_{K\cap H} f(x)\,dx \le \varepsilon \quad \textit{for all } H \in Gr_{n-k},$$

then

$$\begin{aligned}&\int_{S^{n-1}} \left(n\int_0^{\|\theta\|_K^{-1}} r^{n-1} g(r\theta)\,dr\right)^{\frac{k}{n}} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\,dr\,d\theta\\ &\quad \le n^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \big(\mu(K)\big)^{\frac{k}{n}} \varepsilon.\end{aligned} \tag{4.10}$$

Now we use Lemma 4.2 to prove Theorem 3.2.

Proof of Theorem 3.2 Set constant $C > o.m.r.(L, \mathcal{BP}^n_{k,\mu})$. Then there exists a star body $K \in \mathcal{BP}^n_{k,\mu}$ such that $L \subset K$ and

$$\big(\mu(K)\big)^{\frac{1}{n}} \le C\big(\mu(L)\big)^{\frac{1}{n}}. \tag{4.11}$$

Put $f = g\chi_L$, where χ_L is the indicator function of L, then f is nonnegative on K. Let

$$\varepsilon = \max_{H\in Gr_{n-k}} \int_{K\cap H} f(x)\,dx = \max_{H\in Gr_{n-k}} \int_{L\cap H} g(x)\,dx = \max_{H\in Gr_{n-k}} \mu(L\cap H). \tag{4.12}$$

Apply Lemma 4.2 to f and K (f may be not continuous, but we can do an easy approximation) and we have

$$\begin{aligned}&\int_{S^{n-1}} \left(n\int_0^{\|\theta\|_K^{-1}} r^{n-1}g(r\theta)\,dr\right)^{\frac{k}{n}} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1}f(r\theta)\,dr\,d\theta\\ &\quad\le n^{\frac{k}{n}}\frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|}\big(\mu(K)\big)^{\frac{k}{n}}\varepsilon.\end{aligned} \tag{4.13}$$

By $L\subset K$, we get

$$\|\theta\|_L^{-1} \le \|\theta\|_K^{-1} \tag{4.14}$$

for every $\theta\in S^{n-1}$. Then (3.2) and (4.14) implies

$$\begin{aligned}&\int_{S^{n-1}} \left(n\int_0^{\|\theta\|_K^{-1}} r^{n-1}g(r\theta)\,dr\right)^{\frac{k}{n}} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1}f(r\theta)\,dr\,d\theta\\ &\quad\ge \int_{S^{n-1}} \left(n\int_0^{\|\theta\|_L^{-1}} r^{n-1}g(r\theta)\,dr\right)^{\frac{k}{n}} \int_0^{\|\theta\|_L^{-1}} r^{n-k-1}g(r\theta)\,dr\,d\theta\\ &\quad\ge \frac{1}{nR^k}\int_{S^{n-1}} \left(n\int_0^{\|\theta\|_L^{-1}} r^{n-1}g(r\theta)\,dr\right)^{1+\frac{k}{n}} d\theta\\ &\quad\ge \frac{1}{R^k}\mu(L).\end{aligned} \tag{4.15}$$

Combination of (4.11), (4.12), (4.13) and (4.15) gives

$$\frac{1}{R^k}\mu(L) \le C^k n^{\frac{k}{n}}\frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|}\big(\mu(L)\big)^{\frac{k}{n}}\max_{H\in Gr_{n-k}}\mu(L\cap H). \tag{4.16}$$

Let $C\to o.m.r.(L,\mathcal{BP}^n_{k,\mu})$ in (4.16). Then

$$\big(\mu(L)\big)^{\frac{n-k}{n}} \le \big(o.m.r.(L,\mathcal{BP}^n_{k,\mu})\big)^k R^k n^{\frac{k}{n}}\frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|}\max_{H\in Gr_{n-k}}\mu(L\cap H),$$

which completes the proof of Theorem 3.2. □

In order to prove Theorem 3.3, we need using Proposition 4.3∼4.7 below.

First, in $\mathbb{R}^n$ the dialation of a generalized k-intersection body with respect to measure μ is also a generalized k-intersection body with respect to measure μ.

Proposition 4.3 *If $K\in\mathcal{BP}^n_{k,g}$ then $TK\in\mathcal{BP}^n_{k,g(T^{-1}\cdot)}$, where T is a dialation in $\mathbb{R}^n$.*

Proof Suppose that $Tx = ax$ $(a > 0)$ for all $x \in \mathbb{R}^n$, where a is a constant. By the definition of $K \in \mathcal{BP}^n_{k,g}$ (see (2.1)), there exists a nonnegative finite Borel measure μ_1 on Gr_{n-k} such that, for every φ in $C(S^{n-1})$,

$$\int_{S^{n-1}} \left(n \int_0^{\|\theta\|_K^{-1}} r^{n-1} g(r\theta)\, dr \right)^{\frac{k}{n}} \varphi(\theta)\, d\theta = \int_{Gr_{n-k}} R_{n-k}\varphi(H)\, d\mu_1(H). \tag{4.17}$$

Then from

$$\|\theta\|_{aK}^{-1} = a\|\theta\|_K^{-1} \quad \text{for every } \theta \in S^{n-1},$$

and (4.17) it follows that

$$\begin{aligned} & a^k \int_{S^{n-1}} \left(n \int_0^{\|\theta\|_K^{-1}} r^{n-1} g(r\theta)\, dr \right)^{\frac{k}{n}} \varphi(\theta)\, d\theta \\ & = \int_{S^{n-1}} \left(n \int_0^{\|\theta\|_{aK}^{-1}} r^{n-1} g\left(\frac{r}{a}\theta \right) dr \right)^{\frac{k}{n}} \varphi(\theta)\, d\theta \\ & = \int_{Gr_{n-k}} R_{n-k}\varphi(H)\, d\mu_2(H), \end{aligned}$$

where $\mu_2 = a^k \mu_1$ is a nonnegative finite Borel measure on Gr_{n-k}. This implies that $aK \in \mathcal{BP}^n_{k,g(a^{-1}\cdot)}$, i.e. $TK \in \mathcal{BP}^n_{k,g(T^{-1}\cdot)}$. □

For given generalized k-intersection body, we can construct a generalized k-intersection body with respect to some measure μ.

Proposition 4.4 *Suppose that $K \in \mathcal{BP}^n_k$ and*

$$\|\theta\|_K^{-1} \leq \left(n \int_0^{+\infty} r^{n-1} g(r\theta)\, dr \right)^{\frac{1}{n}} \quad \textit{for every } \theta \in S^{n-1}.$$

Let the star body D satisfy

$$n \int_0^{\|\theta\|_D^{-1}} r^{n-1} g(r\theta)\, dr = \|\theta\|_K^{-n} \quad \textit{for every } \theta \in S^{n-1}. \tag{4.18}$$

Then $D \in \mathcal{BP}^n_{k,g}$.

Proof This result follows from the definitions of $\mathcal{BP}^n_k$ and $\mathcal{BP}^n_{k,g}$. □

Proposition 4.5 *Under the assumptions of Proposition 4.4, let $m = \inf_{x\in\mathbb{R}^n} g(x) > 0$ and $M = \sup_{x\in\mathbb{R}^n} g(x) < +\infty$. Then*

$$\frac{1}{M^{1/n}} K \subset D \subset \frac{1}{m^{1/n}} K.$$

Proof By (4.18) for every $\theta \in S^{n-1}$

$$m\|\theta\|_D^{-n} \leq \|\theta\|_K^{-n} \leq M\|\theta\|_D^{-n},$$

which implies

$$\frac{1}{M^{1/n}}\|\theta\|_K^{-1} \le \|\theta\|_D^{-1} \le \frac{1}{m^{1/n}}\|\theta\|_K^{-1},$$

i.e.,

$$\frac{1}{M^{1/n}}\rho_K(\theta) \le \rho_D(\theta) \le \frac{1}{m^{1/n}}\rho_K(\theta),$$

where ρ_K is the radial function of K. Therefore,

$$\frac{1}{M^{1/n}}K \subset D \subset \frac{1}{m^{1/n}}K. \qquad \square$$

Proposition 4.6 *Suppose that* $K \in \mathcal{BP}_k^n$, $m = \inf_{x\in\mathbb{R}^n} g(x) > 0$ *and* $M = \sup_{x\in\mathbb{R}^n} g(x) < +\infty$. *Let the star body* D *satisfy*

$$n\int_0^{\|\theta\|_D^{-1}} r^{n-1} g\bigl(M^{\frac{1}{n}} r\theta\bigr)\,dr = \|\theta\|_K^{-n} \quad \textit{for every } \theta \in S^{n-1}. \tag{4.19}$$

Then $K \subset M^{\frac{1}{n}} D \in \mathcal{BP}_{k,g}^n$ *and*

$$\frac{\mu(M^{\frac{1}{n}}D)}{\mu(K)} \le \frac{M}{m}. \tag{4.20}$$

Proof Similarly to the proof of Proposition 4.5 and Proposition 4.4, we get

$$K \subset M^{\frac{1}{n}} D \in \mathcal{BP}_{k,g}^n.$$

By the polar formula for integrals and (4.19),

$$\begin{aligned}
\mu\bigl(M^{\frac{1}{n}}D\bigr) &= \int_{M^{\frac{1}{n}}D} g(x)\,dx \\
&= \int_{S^{n-1}} \int_0^{\|\theta\|_{M^{1/n}D}^{-1}} r^{n-1} g(r\theta)\,dr\,d\theta \\
&= M\int_{S^{n-1}} \int_0^{\|\theta\|_D^{-1}} r^{n-1} g\bigl(M^{\frac{1}{n}} r\theta\bigr)\,dr\,d\theta \\
&= \frac{M}{n}\int_{S^{n-1}} \|\theta\|_K^{-n}\,d\theta \\
&= M|K|,
\end{aligned}$$

which implies

$$\frac{\mu(M^{\frac{1}{n}}D)}{\mu(K)} = \frac{M|K|}{\int_K g(x)\,dx} \le \frac{M}{m}. \qquad \square$$

Next using Proposition 4.6, for given measure μ with density function g, we have a result on $o.m.r.(L, \mathcal{BP}_{k,\mu}^n)$ and $o.v.r.(L, \mathcal{BP}_k^n)$.

Proposition 4.7 *Let L be an origin-symmetric star body in $\mathbb{R}^n$ and μ-measurable with density g,*

$$m = \inf_{x\in\mathbb{R}^n} g(x) > 0, \qquad M = \sup_{x\in\mathbb{R}^n} g(x) < +\infty \quad \textit{and} \quad 1 \le k \le n-1.$$

Then

$$o.m.r.\bigl(L, \mathcal{BP}^n_{k,\mu}\bigr) \le \left(\frac{M}{m}\right)^{\frac{2}{n}} o.v.r.\bigl(L, \mathcal{BP}^n_k\bigr). \tag{4.21}$$

Proof By Proposition 4.6 and the definition of $o.m.r.(L, \mathcal{BP}^n_{k,\mu})$ (see (2.1)) and (2.2)), we can get Proposition 4.7. □

Proof of Theorem 3.3 Combining Theorem 3.2 and Proposition 4.7, we can get Theorem 3.3. □

5 Conclusions

This article discusses the following inequality:

$$\mu(L)^{\frac{n-k}{n}} \le C^k \max_{H\in Gr_{n-k}} \mu(L\cap H), \tag{5.1}$$

which is a variant of the notable slicing inequality in convex geometry but more concise, for some constant C, for every positive integer n and every integer $1 \le k < n$, and for every origin-symmetric convex body L and every measure μ with nonnegative even continuous density in $\mathbb{R}^n$. The k-intersection body, introduced by [7], plays an important role in the solution to the Busemann–Petty problem, which is equivalent to the slicing problem. By constructing the tool of generalized k-intersection body with measure μ and relevant concepts, we get (5.1) for some constant C.

Next, we give some remarks as the end of this article. Equation (4.21) gives a relationship between $o.m.r.(L, \mathcal{BP}^n_{k,\mu})$ and $o.v.r.(L, \mathcal{BP}^n_k)$. Combining this result with Theorem 3.1 or Theorem 3.2, we get estimates for $\mu(L)$ and $\max_{H\in Gr_{n-k}} \mu(L\cap H)$, in which $o.m.r.(L, \mathcal{BP}^n_{k,\mu})$ is replaced by $o.v.r.(L, \mathcal{BP}^n_k)$. Theorem 3.3 can also follows from the combination of (4.21) and Theorem 3.2.

The combination of (4.21) and Theorem 3.1 yields the following theorem.

Theorem 5.1 *Let L be an origin-symmetric star body in $\mathbb{R}^n$ and μ-measurable with density g,*

$$m = \inf_{x\in\mathbb{R}^n} g(x) > 0, \qquad M = \sup_{x\in\mathbb{R}^n} g(x) < +\infty \quad \textit{and} \quad 1 \le k \le n-1.$$

Then

$$\bigl(\mu(L)\bigr)^{\frac{n-k}{n}} \le \bigl(o.v.r.\bigl(L, \mathcal{BP}^n_k\bigr)\bigr)^k \frac{M^{\frac{2k}{n}}}{m^{\frac{3k}{n}}} n^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \max_{H\in Gr_{n-k}} \mu(L\cap H). \tag{5.2}$$

It is worth noting that Theorem 5.1 can also be derived from Corollary 1 in [9].

Another point that needs noticing is that all the estimates for $o.v.r.(L, \mathcal{BP}_k^n)$ (for example, [8, 9, 13]) can lead to different results on $\mu(L)$ and $\max_{H\in Gr_{n-k}} \mu(L\cap H)$, just using Theorem 3.3 or Theorem 5.1.

For example, there is an estimate [8] for $o.v.r.(L, \mathcal{BP}_k^n)$ as follows.

Theorem 5.2 (*see* [8]) *Let K be a symmetric convex body in $\mathbb{R}^n$ and $1 \le k \le n-1$. Then*

$$o.v.r.\big(K, \mathcal{BP}_k^n\big) \le c\sqrt{\frac{n\log\frac{en}{k}}{k}}, \tag{5.3}$$

where $c > 0$ is an absolute constant.

Then this result united with Theorem 3.2 can yields the following.

Corollary 5.3 *Under the assumptions of Theorem 3.2,*

$$\big(\mu(L)\big)^{\frac{n-k}{n}} \le c^k\left(\frac{n\log\frac{en}{k}}{k}\right)^{\frac{k}{2}}\left(\frac{M}{m}\right)^{\frac{2k}{n}} R^k n^{\frac{k}{n}} \frac{|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \max_{H\in Gr_{n-k}} \mu(L\cap H), \tag{5.4}$$

where $m = \inf_{x\in\mathbb{R}^n} g(x) > 0$, $M = \sup_{x\in\mathbb{R}^n} g(x) < +\infty$, and $c > 0$ is an absolute constant.

The shortcoming of this paper is that the direct estimate of the outer measure ratio distance with respect to measure μ from L to the class $\mathcal{BP}_{k,\mu}^n$, $o.m.r.(L, \mathcal{BP}_{k,\mu}^n)$ is not offered, and Theorems 3.1–3.3 contain the coefficient relevant to the measure μ or the diameter of the star body L.

We want to obtain the best result as follows:

$$\mu(L)^{\frac{n-k}{n}} \le C^k \max_{H\in Gr_{n-k}} \mu(L\cap H),$$

where C is an absolute constant, irrelevant to μ and L. But we have not realized it at present.

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