Hu and Wang *Journal of Inequalities and Applications* (2019) 2019:126
https://doi.org/10.1186/s13660-019-2081-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Estimates for the commutators of operator $V^\alpha\nabla(-\Delta+V)^{-\beta}$

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Abstract

Let a function b belong to the space $\mathrm{BMO}_\theta(\rho)$, which is larger than the space $\mathrm{BMO}(\mathbb{R}^n)$, and let a nonnegative potential V belong to the reverse Hölder class RH_s with $n/2 < s < n$, $n \geq 3$. Define the commutator $[b,T_\beta]f = bT_\beta f - T_\beta(bf)$, where the operator $T_\beta = V^\alpha\nabla\mathcal{L}^{-\beta}$, $\beta-\alpha=\frac{1}{2}$, $\frac{1}{2}<\beta\leq 1$, and $\mathcal{L}=-\Delta+V$ is the Schrödinger operator. We have obtained the L^p-boundedness of the commutator $[b,T_\beta]f$ and we have proved that the commutator is bounded from the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$.

MSC: 42B30; 35J10; 42B35

Keywords: Schrödinger operator; Hardy space; Reverse Hölder class; BMO; Commutator

1 Introduction and results

Let $\mathcal{L}=-\Delta+V$ be the Schrödinger operator, where the nonnegative potential V belongs to the reverse Hölder class RH_s with $s > n/2$, $n \geq 3$. Many papers related to Schrödinger operator have appeared (see [1–5]). In recent years, some researchers have studied the boundedness of the commutators generated by the operators associated with $\mathcal{L}$ and the BMO type space (see [6–9]). In this paper, we investigated the boundedness of the commutator $[b,T_\beta]$, where $T_\beta = V^\alpha\nabla\mathcal{L}^{-\beta}$ and the function $b\in\mathrm{BMO}_\theta(\rho)$. We note that the space $\mathrm{BMO}_\theta(\rho)$ is larger than the space $\mathrm{BMO}(\mathbb{R}^n)$.

For $s > 1$, a nonnegative locally L^s-integrable function V is said to belong to RH_s if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^s\,dy\right)^{1/s} \leq \frac{C}{|B|}\int_B V(y)\,dy$$

holds for every ball $B\subset\mathbb{R}^n$. It is obvious that $RH_{s_1}\subseteq RH_{s_2}$ for $s_1 \geq s_2$.

As in [2], for a given potential $V\in RH_s$ with $s > n/2$, we will use the auxiliary function $\rho(x)$ defined as

$$\rho(x) = \sup\left\{r>0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \leq 1\right\}, \quad x\in\mathbb{R}^n.$$

It is well known that $0<\rho(x)<\infty$ for any $x\in\mathbb{R}^n$.

Let $\mathcal{L} = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, where $V \in RH_s$ with $s > n/2$ and $n \geq 3$. We know $\mathcal{L}$ generates a (C_0) semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$. The maximal function with respect to the semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ is defined by $M^{\mathcal{L}}f(x) = \sup_{t>0} |e^{-t\mathcal{L}}f(x)|$. The Hardy space associated with $\mathcal{L}$ is defined as follows (see [3, 4]).

Definition 1 We say that f is an element of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ if the maximal function $M^{\mathcal{L}}f$ belongs to $L^1(\mathbb{R}^n)$. The quasi-norm of f is defined by

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} = \|M^{\mathcal{L}}f\|_{L^1(\mathbb{R}^n)}.$$

Definition 2 Let $1 < q \leq \infty$. A measurable function a is called an $(1,q)_\rho$-atom related to the ball $B(x_0, r)$ if $r < \rho(x_0)$ and the following conditions hold:

(1) $\operatorname{supp} a \subset B(x_0, r)$;
(2) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(x_0, r)|^{1/q-1}$;
(3) $\int_{B(x_0,r)} a(x)\,dx = 0$ if $r < \rho(x_0)/4$.

The space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ admits the following atomic decomposition (see [3, 4]).

Proposition 1 *Let $f \in L^1(\mathbb{R}^n)$. Then $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $(1,q)_\rho$- atoms, $\sum_j |\lambda_j| < \infty$, and the sum converges in the $H^1_{\mathcal{L}}(\mathbb{R}^n)$ quasi-norm. Moreover*

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} \sim \inf\left\{\sum_j |\lambda_j|\right\},$$

where the infimum is taken over all atomic decompositions of f into $(1,q)_\rho$- atoms.

Following [10], the space $\mathrm{BMO}_\theta(\rho)$ with $\theta \geq 0$ is defined as the set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \left|b(y) - b_B\right|\,dy \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)\,dy$. A norm for $b \in \mathrm{BMO}_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $\mathrm{BMO} \subset \mathrm{BMO}_\theta(\rho)$.

We consider the operator

$$T_\beta = V^\alpha \nabla \mathcal{L}^{-\beta}, \quad \frac{1}{2} \leq \beta \leq 1, \beta - \alpha = \frac{1}{2}.$$

The boundedness of operator $T_{1/2}$ and its commutator have been researched under the condition $V \in RH_s$ for $n/2 < s < n$. In [2], Shen showed that $T_{1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < p_0$, $\frac{1}{p_0} = \frac{1}{s} - \frac{1}{n}$. For $b \in \mathrm{BMO}(\mathbb{R}^n)$, Guo, Li and Peng [11] investigated the L^p-boundedness of commutator $[b, T_{1/2}]$ for $1 < p < p_0$; Li and Peng [12] studied the boundedness of $[b, T_{1/2}]$ from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$. When $b \in \mathrm{BMO}_\theta(\rho)$, Bongioanni, Harboure and Salinas [10] obtained the L^p-boundedness of $[b, T_{1/2}]$ and Liu, Sheng and Wang [13] proved that $[b, T_{1/2}]$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. More boundedness of commutator $[b, T_{1/2}]$ can be found in [14] and [15].

For $1/2 < \beta \leq 1$, $\beta - \alpha = 1/2$, $n/2 < s < n$, Sugano [5] established the estimate for T_β^* (the adjoint operator of T_β), and proved that there exists a constant C such that

$$\left|T_\beta^* f(x)\right| \leq CM\left(|f|^{p'_\alpha}\right)(x)^{1/p'_\alpha}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where $\frac{1}{p_\alpha} = \frac{\alpha+1}{s} - \frac{1}{n}$, and $\frac{1}{p_\alpha} + \frac{1}{p'_\alpha} = 1$. Then, by the boundedness of maximal function, we get

Theorem 1 *Suppose $V \in RH_s$ with $n/2 < s < n$. Let $1/2 < \beta \leq 1$, $\frac{1}{p_\alpha} = \frac{\alpha+1}{s} - \frac{1}{n}$. Then*

$$\left\|T_\beta^* f\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $p'_\alpha < p \leq \infty$, and by duality we get

$$\|T_\beta f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $1 \leq p < p_\alpha$.

Inspired by the above results, in the present work, we are interested in the boundedness of $[b, T_\beta]$. Our main results are as follows.

Theorem 2 *Suppose $V \in RH_s$ with $n/2 < s < n$. Let $1/2 < \beta \leq 1$, $b \in \mathrm{BMO}_\theta(\rho)$. Then,*

$$\left\|\left[b, T_\beta^*\right](f)\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $p'_\alpha < p < \infty$, and

$$\left\|[b, T_\beta](f)\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $1 < p < p_\alpha$, where $\frac{1}{p_\alpha} = \frac{\alpha+1}{s} - \frac{1}{n}$.

Theorem 3 *Suppose $V \in RH_s$ with $n/2 < s < n$. Let $1/2 < \beta \leq 1$, $b \in \mathrm{BMO}_\theta(\rho)$. Then,*

$$\left\|[b, T_\beta](f)\right\|_{WL^1(\mathbb{R}^n)} \leq C\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}.$$

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant c, independent of all important parameters, such that $A \leq cB$. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some preliminaries

We recall some important properties concerning the auxiliary function $\rho(x)$ which have been proved by Shen [2]. Throughout this section we always assume $V \in RH_s$ with $n/2 < s < n$.

Proposition 2 *There exist constants C and $k_0 \geq 1$ such that*

$$C^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}$$

for all $x, y \in \mathbb{R}^n$.

Assume that $Q = B(x_0, \rho(x_0))$, for any $x \in Q$, then Proposition 2 tells us that $\rho(x) \sim \rho(y)$, if $|x - y| < C\rho(x)$. It is easy to get the following result from Proposition 2.

Lemma 1 *Let $k \in \mathbb{N}$ and $x \in 2^{k+1}B(x_0, r) \setminus 2^k B(x_0, r)$. Then we have*

$$\frac{1}{(1 + \frac{2^k r}{\rho(x)})^N} \lesssim \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^{N/(k_0+1)}}.$$

Lemma 2 *There exists a constant $l_0 > 0$ such that*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\, dy \lesssim \left(1 + \frac{r}{\rho(x)}\right)^{l_0}.$$

The following finite overlapping property was given by Dziubański and Zienkiewicz in [3].

Proposition 3 *There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$, so that the family of critical balls $Q_k = B(x_k, \rho(x_k))$, $k \geq 1$, satisfies*

(i) $\bigcup_k Q_k = \mathbb{R}^n$.
(ii) *There exists $N = N(\rho)$ such that for every $k \in N$,* $\operatorname{card}\{j : 4Q_j \cap 4Q_k\} \leq N$.

For $\alpha > 0$, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we introduce the following maximal functions:

$$M_{\rho,\alpha} g(x) = \sup_{x \in B \in \mathcal{B}_{\rho,\alpha}} \frac{1}{|B|} \int_B \bigl|g(y)\bigr|\, dy,$$

and

$$M^{\sharp}_{\rho,\alpha} g(x) = \sup_{x \in B \in \mathcal{B}_{\rho,\alpha}} \frac{1}{|B|} \int_B \bigl|g(y) - g_B\bigr|\, dy,$$

where $\mathcal{B}_{\rho,\alpha} = \{B(z,r) : z \in \mathbb{R}^n \text{ and } r \leq \alpha\rho(y)\}$.

The following Fefferman–Stein type inequality can be found in [10].

Proposition 4 *For $1 < p < \infty$, then there exist δ and γ such that if $\{Q_k\}_k$ is a sequence of balls as in Proposition 3 then*

$$\int_{\mathbb{R}^n} \bigl|M_{\rho,\delta} g(x)\bigr|^p\, dx \lesssim \int_{\mathbb{R}^n} \bigl|M^{\sharp}_{\rho,\gamma} g(x)\bigr|^p\, dx + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |g|\right)^p$$

for all $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We have the following result for the function $b \in \mathrm{BMO}_\theta(\rho)$.

Lemma 3 ([10]) *Let $1 \leq s < \infty$, $b \in \mathrm{BMO}_\theta(\rho)$, and $B = B(x, r)$. Then*

$$\left(\frac{1}{|2^k B|} \int_{2^k B} \bigl|b(y) - b_B\bigr|^s\, dy\right)^{1/s} \lesssim [b]_\theta k \left(1 + \frac{2^k r}{\rho(x)}\right)^{\theta'}$$

for all $k \in \mathbb{N}$, with $r > 0$, where $\theta' = (k_0 + 1)\theta$ and k_0 is the constant appearing in Proposition 2.

We give an estimate of fundamental solutions; this result can be found in [2]. We denote by $\Gamma(x,y,\lambda)$ the fundamental solution of $-\Delta + (V(x) + i\lambda)$, and then $\Gamma(x,y,\lambda) = \Gamma(y,x,-\lambda)$.

Lemma 4 *Assume that $-\Delta u + (V(x) + i\lambda)u = 0$ in $B(x_0, 2R)$ for some $x_0 \in \mathbb{R}^n$. Then, there exists a k_0' such that*

$$\left(\int_{B(x_0,R)} |\nabla u|^t \, dx\right)^{1/t} \lesssim R^{n/s-2}\left(1 + \frac{R}{\rho(x_0)}\right)^{k_0'} \sup_{B(x_0,2R)} |u|,$$

where $1/t = 1/s - 1/n$.

Suppose $\mathcal{W}_\beta = \nabla\mathcal{L}^{-\beta}$. Let $\mathcal{W}_\beta^*$ be the adjoint operator of $\mathcal{W}_\beta$, K and K^* be the kernels of $\mathcal{W}_\beta$ and $\mathcal{W}_\beta^*$ respectively, then $K(x,z) = K^*(z,x)$, and we have the following estimates.

Lemma 5 *Suppose $1/2 < \beta \leq 1$.*

(i) *For every N there exists a constant C_N such that*

$$\left|K^*(x,z)\right| \leq \frac{C_N}{(1 + \frac{|x-z|}{\rho(x)})^N} \frac{1}{|x-z|^{n-2\beta}} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}} \, d\xi + \frac{1}{|x-z|}\right).$$

Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

(ii) *For every N and $0 < \delta < \min\{1, 2 - n/q_0\}$ there exists a constant C_N such that*

$$\begin{aligned}\left|K^*(x,z) - K^*(y,z)\right| \leq{}& \frac{C_N}{(1 + \frac{|x-z|}{\rho(x)})^N} \\ &\times \frac{|x-y|^\delta}{|x-z|^{n-2\beta+\delta}} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}} \, d\xi + \frac{1}{|x-z|}\right)\end{aligned}$$

whenever $|x-y| < \frac{1}{16}|x-z|$. Moreover, the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$.

Proof The proof of (i) can be found in [5], page 449. Let us prove (ii). By (6) of [5] we know

$$K(x,z) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{-\beta} \nabla_x \Gamma(x,z,\tau) \, d\tau, & \text{for } \frac{1}{2} < \beta < 1, \\ \nabla_x \Gamma(x,z,0), & \text{for } \beta = 1. \end{cases}$$

Then

$$\left|K^*(x,z) - K^*(y,z)\right| \lesssim \int_{-\infty}^{\infty} |\tau|^{-\beta} \left|\nabla_z \Gamma(z,x,\tau) - \nabla_z \Gamma(z,y,\tau)\right| d\tau$$

for $\frac{1}{2} < \beta < 1$ and

$$\left|K^*(x,z) - K^*(y,z)\right| \lesssim \left|\nabla_z \Gamma(z,x,0) - \nabla_z \Gamma(z,y,0)\right|$$

for $\beta = 1$.

Fix $x, z \in \mathbb{R}^n$ and let $R = |x-z|/8$, $1/t = 1/s - 1/n$, $\delta = 2 - n/s > 0$. For any $|x-y| < R/2$, it follows from the Morrey embedding theorem (see [16]) and Lemma 4 that

$$\begin{aligned}
&\left|\nabla_z \Gamma(z,x,\tau) - \nabla_z \Gamma(z,y,\tau)\right| \\
&\quad \lesssim |x-y|^{1-n/t}\left(\int_{B(x,R)} \left|\nabla_u \nabla_z \Gamma(z,u,\tau)\right|^t \, du\right)^{1/t} \\
&\quad \lesssim |x-y|^{1-n/t} R^{(n/s)-2}\left(1 + \frac{R}{\rho(x)}\right)^{k_0} \sup_{u \in B(x,2R)} \left|\nabla_z \Gamma(z,u,\tau)\right|.
\end{aligned}$$

It follows from [11, p. 428] that

$$\begin{aligned}
&\sup_{u \in B(x,2R)} \left|\nabla_z \Gamma(z,u,\tau)\right| \\
&\quad \lesssim \frac{C_{k_1}}{(1 + |\tau|^{1/2}|z-u|)^{k_1}(1 + \frac{|z-u|}{\rho(z)})^{k_1}} \frac{1}{|z-u|^{n-2}} \\
&\qquad \times \left(\int_{B(z,|z-u|/4)} \frac{V(\xi)}{|z-\xi|^{n-1}} \, d\xi + \frac{1}{|z-u|}\right).
\end{aligned}$$

Then, by the fact that $6R \le |z-u| \le 10R$, we get

$$\begin{aligned}
&\left|\nabla_z \Gamma(z,x,\tau) - \nabla_z \Gamma(z,y,\tau)\right| \\
&\quad \lesssim \frac{|x-y|^\delta}{|x-z|^{n-2+\delta}} \frac{C_N}{(1 + |\tau|^{1/2}|x-z|)^N (1 + \frac{|x-z|}{\rho(x)})^N} \\
&\qquad \times \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|z-\xi|^{n-1}} \, d\xi + \frac{1}{|x-z|}\right).
\end{aligned}$$

Thus, for $\beta = 1$,

$$\begin{aligned}
\left|K^*(x,z) - K^*(y,z)\right| &\lesssim \left|\nabla_z \Gamma(z,x,0) - \nabla_z \Gamma(z,y,0)\right| \\
&\lesssim \frac{|x-y|^\delta}{|x-z|^{n-2+\delta}} \frac{C_N}{(1 + \frac{|x-z|}{\rho(x)})^N} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|z-\xi|^{n-1}} \, d\xi + \frac{1}{|x-z|}\right).
\end{aligned}$$

Note that

$$\int_{-\infty}^{\infty} \frac{|\tau|^{-\beta} \, d\tau}{(1 + |\tau|^{1/2}|x-z|)^k} \lesssim |x-z|^{2\beta-2}.$$

Then, for $\frac{1}{2} < \beta < 1$, we have

$$\begin{aligned}
\left|K^*(x,z) - K^*(y,z)\right| &\lesssim \frac{|x-y|^\delta}{|x-z|^{n+\delta-2\beta}} \\
&\quad \times \frac{C_N}{(1 + \frac{|x-z|}{\rho(x)})^N} \left(\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi-z|^{n-1}} \, d\xi + \frac{1}{|x-z|}\right).
\end{aligned}$$

By Lemma 2, we know that the inequality above also holds with $\rho(x)$ replaced by $\rho(z)$. $\square$

3 Proof of main results

Before proving Theorem 2, we need to give some necessary lemmas.

Lemma 6 *Let $V \in RH_s$ with $n/2 < s < n$, $\frac{1}{p_\alpha} = \frac{\alpha+1}{s} - \frac{1}{n}$, and $b \in \mathrm{BMO}_\theta(\rho)$. Then, for any $p'_\alpha < t < \infty$, we have*

$$\frac{1}{|Q|}\int_Q \bigl|\bigl[b, T_\beta^*\bigr]f\bigr| \lesssim [b]_\theta \inf_{y\in Q} M_t f(y)$$

for all $f \in L^t_{\mathrm{loc}}(\mathbb{R}^n)$ and every ball $Q = B(x_0, \rho(x_0))$.

Proof Let $f \in L^t_{\mathrm{loc}}(\mathbb{R}^n)$ and $Q = B(x_0, \rho(x_0))$. We consider

$$\bigl[b, T_\beta^*\bigr]f = (b - b_Q)T_\beta^* f - T_\beta^*\bigl(f(b - b_Q)\bigr). \tag{1}$$

By Hölder's inequality with $t > p'_\alpha$ and Lemma 3,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \bigl|(b - b_Q)T_\beta^* f\bigr| &\lesssim \biggl(\frac{1}{|Q|}\int_Q |b - b_Q|^{t'}\biggr)^{1/t'}\biggl(\frac{1}{|Q|}\int_Q \bigl|T_\beta^* f\bigr|^t\biggr)^{1/t} \\ &\lesssim [b]_\theta\biggl(\frac{1}{|Q|}\int_Q \bigl|T_\beta^* f\bigr|^t\biggr)^{1/t}.\end{aligned}$$

Write $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$. By Theorem 1, we know that T_β^* is bounded on $L^t(\mathbb{R}^n)$ with $t > p'_\alpha$, and then

$$\biggl(\frac{1}{|Q|}\int_Q \bigl|T_\beta^* f_1\bigr|^t\biggr)^{1/t} \lesssim \biggl(\frac{1}{|Q|}\int_{2Q} |f|^t\biggr)^{1/t} \lesssim \inf_{y\in Q} M_t f(y).$$

For $x \in Q$, using (i) in Lemma 5, we get

$$\bigl|T_\beta^* f_2(x)\bigr| = \biggl|\int_{(2Q)^c} V(z)^\alpha K^*(x,z) f(z)\,dz\biggr| \lesssim I_1(x) + I_2(x),$$

where

$$I_1(x) \lesssim \int_{(2Q)^c} \frac{|f(z)|}{(1 + \frac{|x-z|}{\rho(x)})^N}\frac{V(z)^\alpha}{|x-z|^{n-2\beta+1}}\,dz$$

and

$$I_2(x) \lesssim \int_{(2Q)^c} \frac{|f(z)|}{(1 + \frac{|x-z|}{\rho(x)})^N}\frac{V(z)^\alpha}{|x-z|^{n-2\beta}}\int_{B(z,|x-z|/4)} \frac{V(\xi)}{|\xi - z|^{n-1}}\,d\xi\,dz.$$

To deal with $I_2(x)$, note that $\rho(x) \sim \rho(x_0)$ and $|x - z| \sim |x_0 - z|$ for $x \in Q$. We split $(2Q)^c$ into annuli to obtain

$$I_2(x) \lesssim \sum_{k\geq 2} \frac{2^{-kN}(2^k\rho(x_0))^{2\beta}}{(2^k\rho(x_0))^n}\int_{2^kQ} \bigl|f(z)\bigr| V(z)^\alpha \mathcal{I}_1(V\chi_{2^kQ})(z)\,dz.$$

Observe that $\frac{1}{p'_\alpha} + \frac{\alpha}{s} + \frac{1}{q_1} = 1$, $\frac{1}{q_1} = \frac{1}{s} - \frac{1}{n}$, $t > p'_\alpha$, and $\beta - \alpha = 1/2$. Then by Hölder's inequality and the boundedness of fractional integral $\mathcal{I}_1 : L^s \to L^{q_1}$ with $\frac{1}{q_1} = \frac{1}{s} - \frac{1}{n}$, we get

$$\begin{aligned} I_2(x) &\lesssim \sum_{k\geq 2} 2^{-kN}\bigl(2^k\rho(x_0)\bigr)^{2\beta}\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^kQ}\bigl|f(z)\bigr|^{p'_\alpha}\,dz\biggr)^{1/p'_\alpha} \\ &\quad\times\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^kQ}V(z)^s\,dz\biggr)^{\alpha/s}\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^{k+1}Q}\bigl|\mathcal{I}_1(V\chi_{2^kQ})(z)\bigr|^{q_1}\,dz\biggr)^{1/q_1} \\ &\lesssim \sum_{k\geq 2} 2^{-kN}\bigl(2^k\rho(x_0)\bigr)^{2\beta+n/s-n/q_1}\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^kQ}V(z)^s\,dz\biggr)^{\alpha/s} \\ &\quad\times\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^kQ}V(z)^s\,dz\biggr)^{1/s}\inf_{y\in Q}M_tf(y). \end{aligned}$$

Then, since $V \in RH_s$, from Lemma 2 and $2\beta + n(1/s - 1/q_1) - 2\alpha - 2 = 0$, we get

$$\begin{aligned} I_2(x) &\lesssim \sum_{k\geq 2} 2^{-kN}\bigl(2^k\rho(x_0)\bigr)^{2\beta+n(1/s-1/q_1)-2\alpha-2}\bigl(1+2^k\bigr)^{(\alpha+1)l_0}\inf_{y\in Q}M_tf(y) \\ &\lesssim \inf_{y\in Q}M_tf(y). \end{aligned} \tag{2}$$

For $I_1(x)$, we split $(2Q)^c$ into annuli to obtain

$$I_1(x) \lesssim \sum_{k\geq 1}\frac{2^{-kN}(2^k\rho(x_0))^{2\beta-1}}{(2^k\rho(x_0))^n}\int_{2^{k+1}Q}\bigl|f(z)\bigr|V(z)^\alpha\,dz.$$

By Hölder's inequality with $\frac{1}{p'_\alpha} + \frac{\alpha}{s} + \frac{1}{q_1} = 1$, $t > p'_\alpha$, $\beta - \alpha = 1/2$, and Lemma 2, we get

$$\begin{aligned} I_1(x) &\lesssim \sum_{k\geq 1} 2^{-kN}\bigl(2^k\rho(x_0)\bigr)^{2\beta-1}\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^{k+1}Q}\bigl|f(z)\bigr|^{p'_\alpha}\,dz\biggr)^{1/p'_\alpha} \\ &\quad\times\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^{k+1}Q}V(z)^s\,dz\biggr)^{\alpha/s} \\ &\lesssim \sum_{k\geq 1}\frac{2^{-kN}}{(2^k\rho(x_0))^{1-2\beta}}\biggl(\frac{1}{(2^k\rho(x_0))^n}\int_{2^{k+1}Q}V(z)\,dz\biggr)^{\alpha}\inf_{y\in Q}M_tf(y) \\ &\lesssim \sum_{k\geq 1}2^{-kN}\bigl(1+2^k\bigr)^{\alpha l_0}\inf_{y\in Q}M_tf(y) \lesssim \inf_{y\in Q}M_tf(y). \end{aligned} \tag{3}$$

To deal with the second term of (1), we write again $f = f_1 + f_2$. Choosing $p'_\alpha < \bar{t} < t$ and denoting $\nu = \frac{\bar{t}t}{t-\bar{t}}$, using the boundedness of T^*_β on $L^{\bar{t}}(\mathbb{R}^n)$ and applying Hölder's inequality,

$$\begin{aligned} \frac{1}{|Q|}\int_Q\bigl|T^*_\beta f_1(b-b_Q)\bigr| &\lesssim \biggl(\frac{1}{|Q|}\int_Q\bigl|T^*_\beta f_1(b-b_Q)\bigr|^{\bar{t}}\biggr)^{1/\bar{t}} \\ &\lesssim \biggl(\frac{1}{|Q|}\int_Q\bigl|f_1(b-b_Q)\bigr|^{\bar{t}}\biggr)^{1/\bar{t}} \end{aligned}$$

$$\lesssim \left(\frac{1}{|Q|}\int_{2Q}|f|^t\right)^{1/t}\left(\frac{1}{|Q|}\int_{2Q}|b-b_q|^\nu\right)^{1/\nu}$$
$$\lesssim [b]_\theta \inf_{y\in Q} M_t f(y).$$

For the remaining term, we have

$$I_1'(x) \lesssim \int_{(2Q)^c}\frac{|f(z)(b-b_Q)|}{(1+\frac{|x-z|}{\rho(x)})^N}\frac{V(z)^\alpha}{|x-z|^{n-2\beta+1}}\,dz$$

and

$$I_2'(x) \lesssim \int_{(2Q)^c}\frac{|f(z)(b-b_Q)|}{(1+\frac{|x-z|}{\rho(x)})^N}\frac{V(z)^\alpha}{|x-z|^{n-2\beta}}\int_{B(z,|x-z|/4)}\frac{V(\xi)}{|\xi-z|^{n-1}}\,d\xi\,dz.$$

Since $1 \le p_\alpha' < t$, we can choose $\bar{t}$ such that $p_\alpha' < \bar{t} < t$. Let $\nu = \frac{\bar{t}t}{t-\bar{t}}$, and then by Hölder's inequality and Lemma 3, we get

$$\begin{aligned}
&\left(\frac{1}{(2^k\rho(x_0))^n}\int_{2^kQ}\left|f(z)\left(b(z)-b_Q\right)\right|^{p_\alpha'}dz\right)^{1/p_\alpha'}\\
&\quad\lesssim \left(\frac{1}{(2^k\rho(x_0))^n}\int_{2^{k+1}Q}\left|f(z)\left(b(z)-b_Q\right)\right|^{\bar{t}}dz\right)^{1/\bar{t}}\\
&\quad\lesssim \left(\frac{1}{(2^k\rho(x_0))^n}\int_{2^kQ}\left|f(z)\right|^{t}dz\right)^{1/t}\\
&\qquad\times\left(\frac{1}{(2^k\rho(x_0))^n}\int_{2^kQ}\left|\left(b(z)-b_Q\right)\right|^{\nu}dz\right)^{1/\nu}\\
&\quad\lesssim k2^{k\theta'}[b]_\theta \inf_{y\in Q} M_t f(y).
\end{aligned}\tag{4}$$

Then, similar to the estimate of (3), we get

$$I_1'(x) \lesssim \sum_{k\ge 1}2^{-kN}\left(1+2^k\right)^{\alpha l_0}k2^{k\theta'}[b]_\theta \inf_{y\in Q}M_t f(y) \lesssim [b]_\theta \inf_{y\in Q} M_t f(y).$$

By (4) and similar to the estimate of (2), we can get

$$I_2'(x) \lesssim [b]_\theta \inf_{y\in Q} M_t f(y).$$

This completes the proof of Lemma 6. □

Lemma 7 *Let $V \in RH_s$ for $n/2 < s < n$, $\frac{1}{p_\alpha} = \frac{\alpha+1}{s} - \frac{1}{n}$, and $b \in \mathrm{BMO}_\theta(\rho)$. Then, for any $p_\alpha' < t < \infty$ and $\gamma \ge 1$ we have*

$$\int_{(2B)^c}\left|K^*(x,z)-K^*(y,z)\right|V(z)^\alpha\left|b(z)-b_B\right|\left|f(z)\right|dz \lesssim [b]_\theta \inf_{u\in B} M_t f(u),$$

for all f and $x,y \in B = B(x_0,r)$ with $r < \gamma\rho(x_0)$.

Proof Denote $Q = B(x_0, \gamma\rho(x_0))$. By Lemma 5 and since in our situation $\rho(x) \sim \rho(x_0)$ and $|x - z| \sim |x_0 - z|$, we need to estimate the following four terms:

$$J_1 = r^\delta \int_{Q\setminus 2B} \frac{|f(z)|V(z)^\alpha |b(z) - b_B|}{|x_0 - z|^{n-2\beta+\delta+1}}\,dz,$$

$$J_2 = r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(z)|V(z)^\alpha |b(z) - b_B|}{|x_0 - z|^{n-2\beta+\delta+1+N}}\,dz,$$

$$J_3 = r^\delta \int_{Q\setminus 2B} \frac{|f(z)|V(z)^\alpha |b(z) - b_B|}{|x_0 - z|^{n-2\beta+\delta}} \int_{B(x_0,4|x_0-z|)} \frac{V(u)}{|u - z|^{n-1}}\,du\,dz,$$

and

$$J_4 = r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(z)|V(z)^\alpha |b(z) - b_B|}{|x_0 - z|^{n-2\beta+\delta+N}} \int_{B(x_0,4|x_0-z|)} \frac{V(u)}{|u - z|^{n-1}}\,du\,dz.$$

Splitting into annuli, we have

$$J_1 \lesssim \sum_{j=2}^{j_0} 2^{-j\delta}\left(2^j r\right)^{2\beta-1} \frac{1}{|2^j B|} \int_{2^j B} \left|f(z)\right|\left|b(z) - b_B\right| V(z)^\alpha\,dz,$$

where j_0 is the least integer such that $2^{j_0} \geq \gamma\rho(x_0)/r$. By Hölder's inequality with $\frac{1}{p_\alpha'} + \frac{\alpha}{s} + \frac{1}{q_1} = 1$, $t > p_\alpha'$, similar to the estimate of (4), we have

$$\begin{aligned}
&\frac{1}{|2^j B|} \int_{2^j B} \left|f(z)\right|\left|b(z) - b_B\right| V(z)^\alpha\,dz \\
&\quad\lesssim \left(\frac{1}{|2^j B|} \int_{2^j B} \left(\left|f(z)\right|\left|b(z) - b_B\right|\right)^{p_\alpha'}\,dz\right)^{1/p_\alpha'} \left(\frac{1}{|2^j B|} \int_{2^j B} V(z)^s\,dz\right)^{\alpha/s} \\
&\quad\lesssim j\left(2^j r\right)^{-2\alpha} [b]_\theta \inf_{y\in B} M_t f(y) \left(1 + \frac{2^j r}{\rho(x_0)}\right)^{\theta' + l_0\alpha} \\
&\quad\lesssim j\left(2^j r\right)^{1-2\beta} [b]_\theta \inf_{u\in B} M_t f(u).
\end{aligned}$$

Then, using $\beta - \alpha = 1/2$, we get

$$J_1 \lesssim [b]_\theta \inf_{u\in B} M_t f(u).$$

To deal with I_2, we split into annuli and get

$$J_2 \lesssim \left(\frac{\rho(x_0)}{r}\right)^N \sum_{j=j_0-1}^{\infty} 2^{-j(\delta+N)}\left(2^j r\right)^{2\beta-1} \frac{1}{|2^j B|} \int_{2^j B} \left|f(z)\right|\left|b(z) - b_B\right| V(z)^\alpha\,dz.$$

Notice that

$$
\begin{aligned}
&\frac{1}{|2^jB|}\int_{2^jB}|f(z)||b(z)-b_B|V(z)^\alpha\,dz\\
&\quad\lesssim j\bigl(2^jr\bigr)^{-2\alpha}[b]_\theta\inf_{y\in B}M_tf(y)\biggl(1+\frac{2^jr}{\rho(x_0)}\biggr)^{\theta'+l_0\alpha}\\
&\quad\lesssim j2^{j(\theta'+l_0\alpha)}\biggl(\frac{\rho(x_0)}{r}\biggr)^{-(\theta'+l_0\alpha)}\bigl(2^jr\bigr)^{1-2\beta}[b]_\theta\inf_{u\in B}M_tf(u).
\end{aligned}
$$

Then, taking $N>\theta'+l_0\alpha$, we get

$$
J_2\lesssim[b]_\theta\inf_{u\in B}M_tf(u).
$$

For J_3, splitting into annuli, we obtain

$$
J_3\lesssim\sum_{j=2}^{j_0}2^{-j\delta}\bigl(2^jr\bigr)^{2\beta}\frac{1}{|2^jB|}\int_{2^jB}|f(z)||b(z)-b_B|V(z)^\alpha\mathcal{I}_1(V\chi_{2^{j+2}B})(z)\,dz.
$$

By Hölder's inequality with $\frac{1}{p'_\alpha}+\frac{\alpha}{s}+\frac{1}{q_1}=1$, similar to the estimate of (2), we get

$$
\begin{aligned}
&\frac{1}{|2^jB|}\int_{2^jB}|f(z)||b(z)-b_B|V(z)^\alpha\mathcal{I}_1(V\chi_{2^{j+2}B})(z)\,dz\\
&\quad\lesssim\biggl(\frac{1}{|2^jB|}\int_{2^jB}\bigl(|f(z)||b(z)-b_B|\bigr)^{p'_\alpha}\,dz\biggr)^{1/p'_\alpha}\biggl(\frac{1}{|2^jB|}\int_{2^jB}V(z)^s\,dz\biggr)^{\alpha/s}\\
&\qquad\times\biggl(\frac{1}{|2^jB|}\int_{2^jB}\bigl|\mathcal{I}_1(V\chi_{2^{j+2}B})(z)\bigr|^{q_1}\,dz\biggr)^{1/q_1}\\
&\quad\lesssim j\bigl(2^jr\bigr)^{-2\alpha+n(1/s-1/q_1)}[b]_\theta\inf_{y\in B}M_tf(y)\biggl(1+\frac{2^jr}{\rho(x_0)}\biggr)^{\theta'+l_0\alpha}\\
&\qquad\times\biggl(\frac{1}{|2^jB|}\int_{2^jB}V(z)^s\,dz\biggr)^{1/s}\\
&\quad\lesssim j\bigl(2^jr\bigr)^{-2\beta}[b]_\theta\inf_{y\in B}M_tf(y)\biggl(1+\frac{2^jr}{\rho(x_0)}\biggr)^{\theta'+l_0(\alpha+1)}\\
&\quad\lesssim j\bigl(2^jr\bigr)^{-2\beta}[b]_\theta\inf_{u\in B}M_tf(u).
\end{aligned}
$$

Then

$$
J_3\lesssim[b]_\theta\inf_{u\in B}M_tf(u).
$$

Finally, for J_4 we have

$$
\begin{aligned}
J_4&\lesssim\biggl(\frac{\rho(x_0)}{r}\biggr)^N\sum_{j_0-1}^{\infty}2^{-j(\delta+N)}\bigl(2^jr\bigr)^{2\beta}\\
&\quad\times\frac{1}{|2^jB|}\int_{2^jB}|f(z)||b(z)-b_B|V(z)^\alpha\mathcal{I}_1(V\chi_{2^{j+2}B})(z)\,dz.
\end{aligned}
$$

Notice that

$$\begin{aligned}
&\frac{1}{|2^jB|}\int_{2^jB}|f(z)||b(z)-b_B|V(z)^\alpha \mathcal{I}_1(V\chi_{2^{j+2}B})(z)\,dz \\
&\quad\lesssim j(2^jr)^{-2\beta}[b]_\theta \inf_{y\in B} M_tf(y)\left(1+\frac{2^jr}{\rho(x_0)}\right)^{\theta'+l_0(\alpha+1)} \\
&\quad\lesssim j2^{j(\theta'+l_0(\alpha+1))}\left(\frac{\rho(x_0)}{r}\right)^{-\theta'-l_0(\alpha+1)}(2^jr)^{-2\beta}[b]_\theta \inf_{u\in B} M_tf(u).
\end{aligned}$$

We choose N large enough such that $N>\theta'+l_0(\alpha+1)$, and then

$$J_4\lesssim [b]_\theta \inf_{u\in B} M_tf(u),$$

which finishes the proof of Lemma 7. □

Now we are in a position to give the proof of Theorem 2.

Proof of Theorem 2 We will prove part (i), and (ii) follows by duality. We start with a function $f\in L^p(\mathbb{R}^n)$ with $p'_\alpha<p<\infty$, and by Lemma 6 we have $[b,T^*_\beta]f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

By Proposition 3 and Lemma 6 with $p'_\alpha<t<p<\infty$, we have

$$\begin{aligned}
\big\|[b,T^*_\beta]f\big\|^p_{L^p(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n}\big|M_{\rho,\delta}[b,T^*_\beta]f\big|^p\,dx \\
&\lesssim \int_{\mathbb{R}^n}\big|M^\sharp_{\rho,\gamma}[b,T^*_\beta]f\big|^p\,dx+\sum_k|Q_k|\left(\frac{1}{|Q_k|}\int_{2Q_k}\big|[b,T^*_\beta]f\big|\right)^p \\
&\lesssim \int_{\mathbb{R}^n}\big|M^\sharp_{\rho,\gamma}[b,T^*_\beta]f\big|^p\,dx+[b]^p_\theta\sum_k\int_{2Q_k}\big|M_t(f)\big|^p\,dx.
\end{aligned}$$

By Proposition 2 and the boundedness of M_t on $L^p(\mathbb{R}^n)$, the second term is controlled by $[b]^p_\theta\|f\|^p_{L^p(\mathbb{R}^n)}$. Then, we only need to consider the first term.

Our goal is to find a point-wise estimate of $M_{\rho,\gamma}[b,T^*_\beta]f$. Let $x\in\mathbb{R}^n$ and $B=B(x_0,r)$ with $r<\gamma\rho(x_0)$ such that $x\in B$. Write $f=f_1+f_2$ with $f_1=f\chi_{2B}$, then

$$[b,T^*_\beta]f=(b-b_B)T^*_\beta f-T^*_\beta\big(f_1(b-b_B)\big)-T^*_\beta\big(f_2(b-b_B)\big).$$

Then, we need to control the mean oscillation on B of each term that we call $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

Let $t>p'_\alpha$, then, by Hölder's inequality and Lemma 3, we get

$$\begin{aligned}
\mathcal{O}_1 &\lesssim \frac{1}{|B|}\int_B\big|(b-b_B)T^*_\beta f\big| \\
&\lesssim \left(\frac{1}{|B|}\int_B|b-b_B|^{t'}\right)^{1/t'}\left(\frac{1}{|B|}\int_B\big|T^*_\beta f\big|^t\right)^{1/t} \\
&\lesssim [b]_\theta M_tT^*_\beta f(x_0),
\end{aligned}$$

since $r<\gamma\rho(x_0)$.

To estimate $\mathcal{O}_2$, let $p'_\alpha < \bar{t} < t$ and $\nu = \frac{\bar{t}t}{t-\bar{t}}$. Then

$$\begin{aligned}
\mathcal{O}_2 &\lesssim \frac{1}{|B|}\int_B \bigl|T^*_\beta\bigl((b-b_B)f_1\bigr)\bigr| \\
&\lesssim \left(\frac{1}{|B|}\int_B \bigl|T^*_\beta\bigl((b-b_B)f_1\bigr)\bigr|^{\bar{t}}\right)^{1/\bar{t}} \\
&\lesssim \left(\frac{1}{|B|}\int_B \bigl|(b-b_B)f_1\bigr|^{\bar{t}}\right)^{1/\bar{t}} \\
&\lesssim \left(\frac{1}{|B|}\int_B |b-b_B|^{\nu}\right)^{1/\nu}\left(\frac{1}{|B|}\int_{2B}|f|^t\right)^{1/t} \\
&\lesssim [b]_\theta M_t f(x_0).
\end{aligned}$$

For $\mathcal{O}_3$, note that $\inf_{y\in B} M_t f(y) \le M_t f(x_0)$, and so by Lemma 7 we get

$$\begin{aligned}
\mathcal{O}_3 &\lesssim \frac{1}{|B|^2}\int_B\int_B \bigl|T^*_\beta\bigl((b-b_B)f_2\bigr)(x) - T^*_\beta\bigl((b-b_B)f_2\bigr)(y)\bigr|\,dx\,dy \\
&\lesssim [b]_\theta M_t f(x_0).
\end{aligned}$$

Thus, we have showed that

$$\bigl|M^\sharp_{\rho,\gamma}\bigl[b,T^*_\beta\bigr]f\bigr| \lesssim [b]_\theta\bigl(M_t T^*_\beta f(x) + M_t f(x)\bigr).$$

Since $t < p$, we obtain the desired result. □

Proof of Theorem 3 Let $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$. By Proposition 1, we can write $f = \sum_{j=-\infty}^{\infty}\lambda_j a_j$, where each a_j is a $(1,q)_\rho$-atom with $1 < q < p_\alpha$, $\frac{1}{p_\alpha} = \frac{\alpha+1}{q_0} - \frac{1}{n}$ and $\sum_{j=-\infty}^{\infty}|\lambda_j| \le 2\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}$. Suppose $\operatorname{supp} a_j \subset B_j = B(x_j, r_j)$ with $r_j < \rho(x_j)$. Write

$$\begin{aligned}
[b,T_\beta]f(x) &= \sum_{j=-\infty}^{\infty}\lambda_j[b,T_\beta]a_j(x)\chi_{8B_j}(x) \\
&\quad + \sum_{j:r_j\ge\rho(x_j)/4}\lambda_j\bigl(b(x)-b_{B_j}\bigr)T_\beta a_j(x)\chi_{(8B_j)^c}(x) \\
&\quad + \sum_{j:r_j<\rho(x_j)/4}\lambda_j\bigl(b(x)-b_{B_j}\bigr)T_\beta a_j(x)\chi_{(8B_j)^c}(x) \\
&\quad - \sum_{j=-\infty}^{\infty}\lambda_j T_\beta\bigl((b-b_{B_j})a_j\bigr)(x)\chi_{(8B_j)^c}(x) \\
&= \sum_{i=1}^{4}\sum_{j=-\infty}^{\infty}\lambda_j A_{ij}(x).
\end{aligned}$$

Note that

$$\left(\int_{B_j}\bigl|a_j(x)\bigr|^q\,dx\right)^{1/q} \lesssim |B_j|^{\frac{1}{q}-1}.$$

By Hölder's inequality, for $1 < q < p_\alpha$, and using Theorem 2 we get

$$\begin{aligned}\|A_{1,j}\|_{L^1(\mathbb{R}^n)} &\lesssim \left(\int_{8B_j} \left|[b, T_\beta]a_j(x)\right|^q dx\right)^{\frac{1}{q}} r_j^{\frac{n}{q'}} \\ &\lesssim [b]_\theta r_j^{\frac{n}{q'}} \left(\int_{B_j} \left|a_j(x)\right|^q dx\right)^{1/q} \\ &\lesssim [b]_\theta |B_j|^{\frac{1}{q'}+\frac{1}{q}-1} \lesssim [b]_\theta.\end{aligned}$$

Thus

$$\begin{aligned}\left\|\sum_{j=-\infty}^{\infty} \lambda_j A_{1j}\right\|_{L^1(\mathbb{R}^n)} &\lesssim \sum_{j=-\infty}^{\infty} |\lambda_j| \|A_{1j}\|_{L^1(\mathbb{R}^n)} \\ &\lesssim [b]_\theta \sum_{j=-\infty}^{\infty} |\lambda_j| \lesssim [b]_\theta \|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}.\end{aligned}$$

And so

$$\left|\left\{x \in \mathbb{R}^n : \left|\sum_{j=-\infty}^{\infty} \lambda_j A_{1j}\right| > \frac{\lambda}{4}\right\}\right| \lesssim \frac{[b]_\theta}{\lambda} \|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}.$$

Since $z \in B_j$, $x \in 2^k B_j \setminus 2^{k-1}B_j$, we have $|x - z| \sim |x - x_j| \sim 2^k r_j$, and by Lemma 1 we get

$$\frac{1}{(1 + \frac{|x-z|}{\rho(x)})^N} \lesssim \frac{1}{(1 + \frac{2^k r_j}{\rho(x_j)})^{\frac{N}{k_0+1}}}.$$

By Hölder's inequality, Lemmas 2 and 3, we get

$$\begin{aligned}&\frac{1}{|2^k B_j|} \int_{2^k B_j} \left|b(x) - b_{B_j}\right| V(x)^\alpha dx \\ &\quad\lesssim \left(\frac{1}{|2^k B_j|} \int_{2^k B_j} \left|b(x) - b_{B_j}\right|^{(\frac{s}{\alpha})'} dx\right)^{1/(\frac{s}{\alpha})'} \left(\frac{1}{|2^k B_j|} \int_{2^k B_j} V(x)^s dx\right)^{\alpha/s} \\ &\quad\lesssim k[b]_\theta \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\theta'} \left(\frac{1}{|2^k B_j|} \int_{2^k B_j} V(x) dx\right)^\alpha \\ &\quad\lesssim k[b]_\theta \left(2^k r_j\right)^{-2\alpha} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\theta' + l_0\alpha}.\end{aligned} \tag{5}$$

Note that $\frac{1}{p_\alpha} + \frac{\alpha}{s} + \frac{1}{q_1} = 1$, $\frac{1}{q_1} = \frac{1}{s} - \frac{1}{n}$, so by Hölder's and Hardy–Littlewood–Sobolev's inequalities and using the fact that $V \in RH_s$, we obtain

$$\begin{aligned}&\frac{1}{|2^k B_j|} \int_{2^k B_j} \left|b(x) - b_{B_j}\right| V(x)^\alpha \left(\mathcal{I}_1(V\chi_{2^k B})(x)\right) dx \\ &\quad\lesssim \left(\frac{1}{|2^k B_j|} \int_{2^k B_j} \left|b(x) - b_{B_j}\right|^{p'_\alpha} dx\right)^{1/p'_\alpha} \left(\frac{1}{|2^k B_j|} \int_{2^k B_j} V(x)^s dx\right)^{\alpha/s}\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{|2^k B_j|} \int_{2^k B_j} \bigl(\mathcal{I}_1(V\chi_{2^k B_j})(x)\bigr)^{q_1}\, dx \right)^{1/q_1} \\
&\lesssim [b]_\theta k \bigl|2^k B_j\bigr|^{1/s-1/q_1} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\theta'} \left(\frac{1}{|2^k B_j|} \int_{2^k B_j} V(x)^s\, dx \right)^{(\alpha+1)/s} \\
&\lesssim [b]_\theta k \bigl(2^k r_j\bigr)^{-2\alpha-1} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\theta'+(\alpha+1)l_0}. \qquad (6)
\end{aligned}
$$

Recall $\int_{B_j} |a_j(y)|\, dy \lesssim 1$, $\beta - \alpha = \frac{1}{2}$ and $r_j/\rho(x_j) \geq 1/4$. Then, taking N large enough such that $\frac{N}{k_0+1} > \theta' + l_0(\alpha + 1)$, we get

$$
\begin{aligned}
&\bigl\|A_{2,j}(x)\bigr\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \sum_{k\geq 4} \frac{1}{(1+\frac{2^k r_j}{\rho(x)})^N} \frac{1}{(2^k r_j)^{n-2\beta+1}} \int_{2^k B_j \setminus 2^{k-1} B_j} \bigl|b(x) - b_{B_j}\bigr| V(x)^\alpha\, dx \int_{B_j} \bigl|a_j(z)\bigr|\, dz \\
&\quad + \sum_{k\geq 4} \frac{1}{(1+\frac{2^k r_j}{\rho(x)})^N} \frac{1}{(2^k r_j)^{n-2\beta}} \\
&\quad \times \int_{2^k B_j \setminus 2^{k-1} B_j} \bigl|b(x) - b_{B_j}\bigr| V(x)^\alpha \bigl(\mathcal{I}_1(V\chi_{2^k B})(x)\bigr)\, dx \int_{B_j} \bigl|a_j(z)\bigr|\, dz \\
&\lesssim [b]_\theta \sum_{k\geq 4} \frac{k(2^k r_j)^{2\beta-1}}{(1+\frac{2^k r_j}{\rho(x_j)})^{\frac{N}{k_0+1}}} \bigl(2^k r_j\bigr)^{-2\alpha} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\theta'+l_0\alpha} \\
&\quad + [b]_\theta \sum_{k\geq 4} \frac{(2^k r_j)^{2\beta}}{(1+\frac{2^k r_j}{\rho(x_j)})^{\frac{N}{k_0+1}}} \bigl(2^k r_j\bigr)^{-2\alpha-1} \left(1 + \frac{2^k r_j}{\rho(x_j)}\right)^{\theta'+(\alpha+1)l_0} \\
&\lesssim [b]_\theta \sum_{k\geq 3} \frac{k}{(2^k)^{\frac{N}{k_0+1}-\theta'-l_0\alpha}} + [b]_\theta \sum_{k\geq 3} \frac{k}{(2^k)^{\frac{N}{k_0+1}-\theta'-l_0(\alpha+1)}} \\
&\lesssim [b]_\theta.
\end{aligned}
$$

Thus

$$
\left\| \sum_{j=-\infty}^{\infty} \lambda_j A_{2j} \right\|_{L^1(\mathbb{R}^n)} \lesssim [b]_\theta \|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}.
$$

Therefore

$$
\left| \left\{ x \in \mathbb{R}^n : \left| \sum_{j=-\infty}^{\infty} \lambda_j A_{2j} \right| > \frac{\lambda}{4} \right\} \right| \lesssim \frac{[b]_\theta}{\lambda} \|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}.
$$

When $x \in 2^k B_j \setminus 2^{k-1} B_j$, and $z \in B_j$, by Lemmas 5 and 1, we have

$$
\begin{aligned}
\bigl|K(x,z) - K(x,x_j)\bigr| &\lesssim \frac{1}{(1+\frac{2^k r_j}{\rho(x_j)})^{N/(k_0+1)}} \frac{r_j^\delta}{(2^k r_j)^{n+\delta-2\beta+1}} \\
&\quad + \frac{1}{(1+\frac{2^k r_j}{\rho(x_j)})^{N/(k_0+1)}} \frac{r_j^\delta}{(2^k r_j)^{n+\delta-2\beta}} \mathcal{I}_1(V\chi_{2^k B_j})(z),
\end{aligned}
$$

where $\delta = 2 - n/s > 0$. Thus, by the vanishing condition of a_j, together with (5) and (6), we have

$$
\begin{aligned}
&\left\|A_{3,j}(x)\right\|_{L^1(\mathbb{R}^n)} \\
&\quad\lesssim \sum_{k\geq 4} \int_{2^k B_j \setminus 2^{k-1}B_j} \left|b(x) - b_{B_j}\right| V(x)^\alpha \int_{B_j} \left|K_\alpha(x,z) - K_\alpha(x,x_j)\right| \left|a_j(z)\right| \,dz\,dx \\
&\quad\lesssim \sum_{k\geq 3} \frac{1}{(1+\frac{2^k r_j}{\rho(x_j)})^{\frac{N}{k_0+1}}} \frac{r_j^\delta}{(2^k r_j)^{n+\delta-2\beta+1}} \int_{2^{k+1}B_j} \left|b(x) - b_{B_j}\right| V(x)^\alpha \,dx \int_{B_j} \left|a_j(z)\right| dz \\
&\qquad + \sum_{k\geq 3} \frac{1}{(1+\frac{2^k r_j}{\rho(x_j)})^{\frac{N}{k_0+1}}} \frac{r_j^\delta}{(2^k r_j)^{(n+\delta-2\beta)}} \\
&\qquad \times \int_{2^{k+1}B_j} \left|b(x) - b_{B_j}\right| V(x)^\alpha \mathcal{I}_1(V\chi_{2^k B_j})(x)\,dx \int_{B_j} \left|a_j(z)\right| dz \\
&\quad\lesssim [b]_\theta \sum_{k\geq 3} \frac{1}{(1+\frac{2^k r_j}{\rho(x_j)})^{\frac{N}{k_0+1}-\theta'-l_0\alpha}} \frac{k}{2^{k\delta}} + [b]_\theta \sum_{k\geq 3} \frac{1}{(1+\frac{2^k r_j}{\rho(x_j)})^{\frac{N}{k_0+1}-\theta'-l_0(\alpha+1)}} \frac{k}{2^{k\delta}} \lesssim [b]_\theta.
\end{aligned}
$$

So that

$$
\left|\left\{x \in \mathbb{R}^n : \left|\sum_{j=-\infty}^{\infty} \lambda_j A_{3j}\right| > \frac{\lambda}{4}\right\}\right| \lesssim \frac{[b]_\theta}{\lambda} \|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}.
$$

Now let us deal with the last part. Since $r_j \leq \rho(x_j)$, we get

$$
\begin{aligned}
\left\|(b - b_{B_j})a_j\right\|_{L^1(\mathbb{R}^n)} &\leq \left(\int_{B_j} \left|b(x) - b_{B_j}\right|^{q'} dx\right)^{1/q'} \left(\int_{B_j} \left|a_j(x)\right|^q dx\right)^{1/q} \\
&\lesssim [b]_\theta \left(1 + \frac{r_j}{\rho(x_j)}\right)^{\theta'} \lesssim [b]_\theta.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left|A_{4j}(x)\right| &\leq \sum_{j=-\infty}^{\infty} |\lambda_j| T_\beta\left(\left|(b - b_{B_j})a_j\right|\right)(x) \chi_{(8B_j)^c}(x) \\
&\leq T_\beta\left(\sum_{j=-\infty}^{\infty} \left|\lambda_j (b - b_{B_j}) a_j\right|\right)(x).
\end{aligned}
$$

By Theorem 1, we know T_β is bounded from $L^1(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$. Then

$$
\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : \left|\sum_{j=-\infty}^{\infty} \lambda_j A_{4j}\right| > \frac{\lambda}{4}\right\}\right| \\
&\quad\leq \left|\left\{x \in \mathbb{R}^n : \left|T_\beta\left(\sum_{j=-\infty}^{\infty} \left|\lambda_j (b - b_{B_j}) a_j\right|\right)(x)\right| > \frac{\lambda}{4}\right\}\right| \\
&\quad\lesssim \frac{1}{\lambda} \left\|\sum_{j=-\infty}^{\infty} \left|\lambda_j (b - b_{B_j}) a_j\right|\right\|_{L^1(\mathbb{R}^n)}
\end{aligned}
$$

$$\begin{aligned} &\lesssim \frac{1}{\lambda}\sum_{j=-\infty}^{\infty}|\lambda_j|\big\|(b-b_{B_j})a_j\big\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \frac{[b]_\theta}{\lambda}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\right) \lesssim \frac{[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left|\left\{x\in\mathbb{R}^n : \left|\sum_{i=1}^{4}\sum_{j=-\infty}^{\infty}\lambda_j A_{ij}\right| > \lambda\right\}\right| \\ &\quad\lesssim \sum_{i=1}^{4}\left|\left\{x\in\mathbb{R}^n : \left|\sum_{j=-\infty}^{\infty}\lambda_j A_{ij}\right| > \frac{\lambda}{4}\right\}\right| \\ &\quad\lesssim \frac{[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}. \end{aligned}$$

□

4 Conclusion

In this paper, we established the L^p-boundedness of commutator operators $[b, T_\beta]$ and $[b, T^*_\beta]$, where $T_\beta = V^\alpha\nabla\mathcal{L}^{-\beta}$, $\frac{1}{2} < \beta \leq 1$, $\beta - \alpha = \frac{1}{2}$, and $b \in \mathrm{BMO}_\theta(\rho)$, which is larger than the space $\mathrm{BMO}(\mathbb{R}^n)$. At the endpoint, we show that the operator $[b, T_\beta]$ is bounded from Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ continuously into weak $L^1(\mathbb{R}^n)$. These results enrich the theory of Schrödinger operator.

Acknowledgements
The authors are very grateful to the anonymous referees and the editor for their insightful comments and suggestions.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 19 July 2018 Accepted: 26 April 2019 Published online: 08 May 2019

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