Qi et al. *Journal of Inequalities and Applications* (2019) 2019:135
https://doi.org/10.1186/s13660-019-2079-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalized fractional integral inequalities of Hermite–Hadamard type for (α, m)-convex functions

Feng Qi[1,2,5*], Pshtiwan Othman Mohammed[3], Jen-Chih Yao[4] and Yong-Hong Yao[5]

*Correspondence:
qifeng618@hotmail.com
[1]Institute of Mathematics, Henan Polytechnic University, Jiaozuo, China
[2]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, China
Full list of author information is available at the end of the article

Abstract

In the paper, the authors establish some generalized fractional integral inequalities of the Hermite–Hadamard type for (α, m)-convex functions, show that one can find some Riemann–Liouville fractional integral inequalities and classical integral inequalities of the Hermite–Hadamard type, and generalize and extend some known results.

MSC: Primary 26A33; secondary 26D07; 26D10; 26D15

Keywords: Generalized fractional integral; Hermite–Hadamard type; Integral inequality; Convex function; Riemann–Liouville fractional integral

1 Introduction

Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b h(x)\,\mathrm{d}x \le \frac{h(a)+h(b)}{2}. \tag{1.1}$$

Inequality (1.1) is well known in the literature as the Hermite–Hadamard inequality. A number of mathematicians have devoted their efforts to generalize, refine, counterpart, and extend the Hermite–Hadamard inequality (1.1) for different classes of convex functions and mappings. For several recent results concerning inequality (1.1), we may refer the interested reader to [1, 10, 14, 27, 33, 34, 39].

Let us recall some definitions and known results concerning convexity.

Definition 1.1 ([33]) A function $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality

$$h\big(\lambda x + (1-\lambda)y\big) \le \lambda h(x) + (1-\lambda)h(y)$$

holds for all $x, y \in I$ and $\lambda \in (0, 1)$.

Definition 1.2 ([1, 33]) A function: $h : [0, b] \to \mathbb{R}$ is said to be m-convex if

$$h\big(\lambda a + m(1-\lambda)b\big) \le \lambda h(a) + m(1-\lambda)h(b)$$

holds for all $a, b \in [0, b]$ and $\lambda \in [0, 1]$ and for some $m \in (0, 1]$.

Definition 1.3 ([1, 33]) Let $(\alpha, m) \in (0,1]^2$. A function: $h : [0,b] \to \mathbb{R}$ is said to be (α, m)-convex if

$$h\bigl(\lambda a + m(1-\lambda)b\bigr) \le \lambda^\alpha h(a) + m\bigl(1-\lambda^\alpha\bigr)h(b)$$

holds for all $a, b \in [0,b]$ and $\lambda \in [0,1]$ and for some $m \in (0,1]$.

The Riemann–Liouville integrals $J_{a+}^\alpha h(t)$ and $J_{b-}^\alpha h(t)$ of order $\alpha \ge 0$ are defined in [5] respectively by $J_{a+}^0 h(t) = J_{b-}^0 h(t) = h(t)$,

$$J_{a+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} h(u)\,\mathrm{d}\,u, \quad t > a$$

and

$$J_{b-}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (u-t)^{\alpha-1} h(u)\,\mathrm{d}\,u, \quad t < b$$

for $h \in L_1([a,b])$ and $\alpha > 0$, where Γ denotes the classical Euler gamma function which can be defined [17, 22] by

$$\Gamma(w) = \lim_{n\to\infty} \frac{n! n^w}{\prod_{k=0}^n (w+k)}, \quad w \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$$

or by

$$\Gamma(w) = \int_0^\infty u^{w-1} e^{-u}\,\mathrm{d}\,u, \quad \Re(w) > 0.$$

Recently, the following integral identity and the Riemann–Liouville fractional integral inequalities of the Hermite–Hadamard type for (α, m)-convex functions were obtained.

Lemma 1.1 ([26, Lemma 2.1]) *Let $h : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on an interval (a,b) with $a < b$ such that $h' \in L_1([a,b])$. Then*

$$\begin{aligned} Q_\alpha(a,b) = \frac{b-a}{16}\biggl[&\int_0^1 \bigl(1-u^\alpha\bigr) h'\biggl(\frac{3a+b}{4}u + \frac{a+b}{2}(1-u)\biggr)\,\mathrm{d}\,u \\ &- \int_0^1 u^\alpha h'\biggl(au + \frac{3a+b}{4}(1-u)\biggr)\,\mathrm{d}\,u \\ &+ \int_0^1 \bigl(1-u^\alpha\bigr) h'\biggl(\frac{a+3b}{4}u + b(1-u)\biggr)\,\mathrm{d}\,u \\ &- \int_0^1 u^\alpha h'\biggl(\frac{a+b}{2}u + \frac{a+3b}{4}(1-u)\biggr)\,\mathrm{d}\,u \biggr] \end{aligned} \tag{1.2}$$

for $\alpha > 0$, where

$$\begin{aligned} Q_\alpha(a,b) = \frac{1}{2}\biggl[\frac{h(a)+h(b)}{2} + h\biggl(\frac{a+b}{2}\biggr)\biggr] - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\biggl[J_{a+}^\alpha h\biggl(\frac{3a+b}{4}\biggr) \\ + J_{[(3a+b)/4]+}^\alpha h\biggl(\frac{a+b}{2}\biggr) + J_{[(a+b)/2]+}^\alpha h\biggl(\frac{a+3b}{4}\biggr) + J_{[(a+3b)/4]+}^\alpha h(b)\biggr]. \end{aligned}$$

Theorem 1.1 ([26, Theorem 3.1]) *Let $h:[0,\infty)\to\mathbb{R}$ be differentiable on $[0,\infty)$ and $h'\in L_1([a,b])$ for $0\le a<b$ and $\alpha>0$. If $|h'|^q$ is (α_1,m)-convex on $[0,\frac{b}{m}]$ for some $(\alpha_1,m)\in(0,1]^2$ and $q\ge 1$, then*

$$\begin{aligned}\bigl|Q_\alpha(a,b)\bigr| \le{}& \frac{b-a}{16(\alpha+1)}\biggl[\frac{1}{(\alpha_1+1)(\alpha+\alpha_1+1)}\biggr]^{1/q}\biggl[\biggl((\alpha+1)(\alpha_1+1)\bigl|h'(a)\bigr|^q \\ &+ m\alpha_1(\alpha_1+1)\biggl|h'\biggl(\frac{3a+b}{4m}\biggr)\biggr|^q\biggr)^{1/q} + \alpha\biggl((\alpha+1)\biggl|h'\biggl(\frac{3a+b}{4}\biggr)\biggr|^q \\ &+ m\alpha_1(\alpha_1+\alpha+2)\biggl|h'\biggl(\frac{a+b}{2m}\biggr)\biggr|^q\biggr)^{1/q} \\ &+ \biggl((\alpha+1)(\alpha_1+1)\biggl|h'\biggl(\frac{a+b}{2}\biggr)\biggr|^q + m\alpha_1(\alpha_1+1)\biggl|h'\biggl(\frac{a+3b}{4m}\biggr)\biggr|^q\biggr)^{1/q} \\ &+ \alpha\biggl((\alpha+1)\biggl|h'\biggl(\frac{a+3b}{4}\biggr)\biggr|^q + m\alpha_1(\alpha_1+\alpha+2)\biggl|h'\biggl(\frac{b}{m}\biggr)\biggr|^q\biggr)^{1/q}\biggr].\end{aligned}\tag{1.3}$$

Theorem 1.2 ([26, Theorem 3.2]) *Let $h:[0,\infty)\to\mathbb{R}$ be differentiable on $[0,\infty)$ and $h'\in L_1([a,b])$ for $0\le a<b$ and $\alpha>0$. If $|h'|^q$ is (α_1,m)-convex on $[0,\frac{b}{m}]$ for some $(\alpha_1,m)\in(0,1]^2$ and for $q>1$ and $q\ge r\ge 0$, then*

$$\begin{aligned}\bigl|Q_\alpha(a,b)\bigr| \le{}& \frac{b-a}{16}\biggl\{\biggl(\frac{q-1}{\alpha(q-r)+q-1}\biggr)^{1-1/q}\biggl[\frac{1}{\alpha r+\alpha_1+1}\bigl|h'(a)\bigr|^q \\ &+ \frac{m\alpha_1}{(\alpha r+1)(\alpha r+\alpha_1+1)}\biggl|h'\biggl(\frac{3a+b}{4m}\biggr)\biggr|^q\biggr]^{1/q} \\ &+ \frac{1}{\alpha}B^{1-1/q}\biggl(\frac{2q-r-1}{q-1},\frac{1}{\alpha}\biggr)\biggl[B\biggl(r+1,\frac{\alpha_1+1}{\alpha}\biggr)\biggl|h'\biggl(\frac{3a+b}{4}\biggr)\biggr|^q \\ &+ m\biggl(B\biggl(r+1,\frac{1}{\alpha}\biggr)-B\biggl(r+1,\frac{\alpha_1+1}{\alpha}\biggr)\biggr)\biggl|h'\biggl(\frac{a+b}{2m}\biggr)\biggr|^q\biggr]^{1/q} \\ &+ \biggl(\frac{q-1}{\alpha(q-r)+q-1}\biggr)^{1-1/q}\biggl[\frac{1}{\alpha r+\alpha_1+1}\biggl|h'\biggl(\frac{a+b}{2}\biggr)\biggr|^q \\ &+ \frac{m\alpha_1}{(\alpha r+1)(\alpha r+\alpha_1+1)}\biggl|h'\biggl(\frac{a+3b}{4m}\biggr)\biggr|^q\biggr]^{1/q} \\ &+ \frac{1}{\alpha}B^{1-1/q}\biggl(\frac{2q-r-1}{q-1},\frac{1}{\alpha}\biggr)\biggl[B\biggl(r+1,\frac{\alpha_1+1}{\alpha}\biggr)\biggl|h'\biggl(\frac{a+3b}{4}\biggr)\biggr|^q \\ &+ m\biggl(B\biggl(r+1,\frac{1}{\alpha}\biggr)-B\biggl(r+1,\frac{\alpha_1+1}{\alpha}\biggr)\biggr)\biggl|h'\biggl(\frac{b}{m}\biggr)\biggr|^q\biggr]^{1/q}\biggr\},\end{aligned}\tag{1.4}$$

where $B(s,t)$ denotes the classical beta function which can be defined [18, 19] *by*

$$B(s,t)=\int_0^1 u^{s-1}(1-u)^{t-1}\,\mathrm{d}u,\quad s,t>0.$$

For more information about the Hermite–Hadamard type inequalities for (α,m)-convex functions, please refer to the papers [2, 3, 6, 15, 21, 26, 28–30, 32, 35, 38] and closely related references therein.

2 A review for generalized fractional integral operators

Now we recall some necessary definitions and mathematical preliminaries of the generalized fractional integrals which are defined by Sarikaya and Ertuğral in [24].

Let $\varphi : [0,\infty) \to [0,\infty)$ satisfy the condition $\int_0^1 \frac{\varphi(t)}{t}\,\mathrm{d}t < \infty$.

We now define the left-sided and right-sided generalized fractional integral operators ${}_{a^+}I_\varphi h(t)$ and ${}_{b^-}I_\varphi h(t)$ by

$$ {}_{a^+}I_\varphi h(t) = \int_a^t \frac{\varphi(t-u)}{t-u} h(u)\,\mathrm{d}u, \quad t > a \tag{2.1} $$

and

$$ {}_{b^-}I_\varphi h(t) = \int_t^b \frac{\varphi(u-t)}{u-t} h(u)\,\mathrm{d}u, \quad t < b. \tag{2.2} $$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as the Riemann–Liouville fractional integrals [25, 26, 31], the k-Riemann–Liouville fractional integrals [11, 36], the Katugampola fractional integrals [7, 8], conformable fractional integrals [23, 37], the Hadamard fractional integrals [16], and so on. These important special cases of the integral operators in (2.1) and (2.2) are mentioned below.

1. If we take $\varphi(u) = u$, the operators in (2.1) and (2.2) reduce to the Riemann integrals

$$ I_{a^+}h(t) = \int_a^t h(u)\,\mathrm{d}u, \quad t > a \quad \text{and} \quad I_{b^-}h(t) = \int_t^b h(u)\,\mathrm{d}u, \quad t < b. $$

2. If we take $\varphi(u) = \frac{u^\alpha}{\Gamma(\alpha)}$, the operators in (2.1) and (2.2) become the Riemann–Liouville fractional integrals

$$ I_{a^+}h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} h(u)\,\mathrm{d}u, \quad t > a $$

and

$$ I_{b^-}h(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (u-t)^{\alpha-1} h(u)\,\mathrm{d}u, \quad t < b. $$

3. If we take $\varphi(u) = \frac{u^{\alpha/k}}{k\Gamma_k(\alpha)}$, the operators in (2.1) and (2.2) are the k-Riemann–Liouville fractional integrals

$$ I_{a^+,k}h(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-u)^{\alpha/k-1} h(u)\,\mathrm{d}u, \quad t > a $$

and

$$ I_{b^-,k}h(t) = \frac{1}{k\Gamma_k(\alpha)} \int_t^b (u-t)^{\alpha/k-1} h(u)\,\mathrm{d}u, \quad t < b, $$

where

$$ \Gamma_k(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u^k/k}\,\mathrm{d}u, \quad \mathbb{R}(\alpha) > 0 $$

and

$$\Gamma_k(\alpha) = k^{\alpha/k-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \mathbb{R}(\alpha) > 0, k > 0$$

are given in [13, 20].

4. If we take $\varphi(u) = \frac{u}{\alpha}\exp(-\frac{1-\alpha}{\alpha}u)$, the operators in (2.1) and (2.2) reduce to the right-sided and left-sided fractional integral operators with exponential kernel for $\alpha \in (0,1)$

$$\mathcal{I}^{\alpha}_{a^+}h(t) = \frac{1}{\alpha}\int_a^t \exp\left(-\frac{1-\alpha}{\alpha}(t-u)\right)h(u)\,\mathrm{d}u, \quad t > a$$

and

$$\mathcal{I}^{\alpha}_{b^-}h(t) = \frac{1}{\alpha}\int_t^b \exp\left(-\frac{1-\alpha}{\alpha}(u-t)\right)h(u)\,\mathrm{d}u, \quad t < b$$

which are defined in [9].

Recently, Sarikaya and Ertuğral [24] established the following trapezoid inequalities for generalized fractional integrals.

Theorem 2.1 ([24]) *Let $h : [a,b] \to \mathbb{R}$ be differentiable on (a,b) with $a < b$. If $|h'|$ is convex on $[a,b]$, then*

$$\left|\frac{h(a)+h(b)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}h(b) + {}_{b^-}I_{\varphi}h(a)\right]\right| \le \frac{|h'(a)|+|h'(b)|}{2}\frac{b-a}{\Lambda(1)}\int_0^1 u\left|\Lambda(1-u)-\Lambda(u)\right|\,\mathrm{d}u,$$

where

$$\Lambda(u) = \int_0^u \frac{\varphi((b-a)t)}{t}\,\mathrm{d}t < \infty.$$

Theorem 2.2 ([24]) *Let $h : [a,b] \to \mathbb{R}$ be differentiable on (a,b) with $a < b$. If $|h'|^q$ is convex on $[a,b]$ for $p,q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\frac{h(a)+h(b)}{2} - \frac{{}_{a^+}I_{\varphi}h(b) + {}_{b^-}I_{\varphi}h(a)}{2\Lambda(1)}\right| \le \frac{b-a}{2\Lambda(1)}\left[\frac{|h'(a)|^q+|h'(b)|^q}{2}\right]^{1/q}\left[\int_0^1 \left|\Lambda(1-u)-\Lambda(u)\right|^p\,\mathrm{d}u\right]^{1/p}.$$

In [4], Ertuğral and Sarikaya established the following trapezoid inequalities for generalized fractional integrals.

Theorem 2.3 ([4]) *Let $h : [a,b] \to \mathbb{R}$ be absolutely continuous on I° such that $h' \in L_1([a,b])$ with $a, b \in I^\circ$ with $a < b$. If the mapping $|h'|$ is convex on $[a,b]$, then*

$$\left|\frac{\nabla(0)h(b)+\Delta(0)h(a)}{b-a} - \frac{1}{b-a}\left[{}_{a^+}I_{\varphi}h(b) + {}_{b^-}I_{\varphi}h(a)\right]\right|$$

$$\leq \frac{b-x}{b-a}\left|h'(x)\right|\int_0^1\left|\nabla(u)\right|u\,\mathrm{d}u+\frac{x-a}{b-a}\left|h'(x)\right|\int_0^1\left|\Delta(u)\right|u\,\mathrm{d}u$$
$$+\frac{b-x}{b-a}\left|h'(b)\right|\int_0^1\left|\nabla(u)\right|(1-u)\,\mathrm{d}u+\frac{x-a}{b-a}\left|h'(a)\right|\int_0^1\left|\Delta(u)\right|(1-u)\,\mathrm{d}u,$$

where

$$\Delta(u)=\int_t^1\frac{\varphi((x-a)t)}{t}\,\mathrm{d}t<\infty\quad\text{and}\quad\nabla(u)=\int_t^1\frac{\varphi((b-x)t)}{t}\,\mathrm{d}t<\infty.$$

Theorem 2.4 ([4]) *Let $h:[a,b]\to\mathbb{R}$ be differentiable on (a,b) with $a<b$. If $|h'|^q$ for $q>1$ is convex on $[a,b]$, then*

$$\left|\frac{h(a)+h(b)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi h(b)+{}_{b^-}I_\varphi h(a)\right]\right|$$
$$\leq\frac{b-x}{b-a}\left[\int_0^1\left|\nabla(u)\right|^p\,\mathrm{d}u\right]^{1/p}\left[\frac{|h'(a)|^q+|h'(b)|^q}{2}\right]^{1/q}$$
$$+\frac{b-x}{b-a}\left[\int_0^1\left|\Delta(u)\right|^p\,\mathrm{d}u\right]^{1/p}\left[\frac{|h'(a)|^q+|h'(b)|^q}{2}\right]^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Most recently, Mohammed and Sarikayain [12] established some generalized fractional integral inequalities of midpoint and trapezoid types for twice differential functions.

Theorem 2.5 ([12]) *Let $h:I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I° such that $h''\in L_1([a,b])$ with $a,b\in I^\circ$ and $a<b$. If the function $|h''|$ is convex on $[a,b]$, then*

$$\left|\left[{}_{(\frac{a+b}{2})^+}I_\varphi h(b)+{}_{(\frac{a+b}{2})^-}I_\varphi h(a)\right]-2\nabla(1)h\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{4}\left(\left|h''(a)\right|+\left|h''(b)\right|\right)\int_0^1\left|\Delta(t)\right|\,\mathrm{d}t,$$

where

$$\Delta(t)=\int_0^t\nabla(u)\,\mathrm{d}u<\infty\quad\text{and}\quad\nabla(u)=\int_0^u\frac{\varphi((\frac{b-a}{2})s)}{s}\,\mathrm{d}s<\infty.$$

Theorem 2.6 ([12]) *Let $h:I\subseteq\mathbb{R}\to\mathbb{R}$ be twice differentiable on I° such that $h''\in L_1([a,b])$ with $a,b\in I^\circ$ and $a<b$. If $|h''|^q$ for $q>1$ is convex on $[a,b]$, then*

$$\left|\left[{}_{(\frac{a+b}{2})^+}I_\varphi h(b)+{}_{(\frac{a+b}{2})^-}I_\varphi h(a)\right]-2\nabla(1)h\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{4}\left(\int_0^1\left|\Delta(u)\right|^p\,\mathrm{d}u\right)^{1/p}\left\{\left(\frac{|h''(a)|^q+3|h''(b)|^q}{4}\right)^{1/q}\right.$$
$$\left.+\left(\frac{3|h''(a)|^q+|h''(b)|^q}{4}\right)^{1/q}\right\}$$
$$\leq\frac{(b-a)^2}{2^{2/q}}\left(\int_0^1\left|\Delta(u)\right|^p\,\mathrm{d}u\right)^{1/p}\left(\left|h''(a)\right|+\left|h''(b)\right|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 2.7 ([12]) *Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on I° such that $h'' \in L_1([a,b])$ with $a, b \in I^\circ$ and $a < b$. If $|h''|$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{h(a)+h(b)}{2} - \frac{1}{2\Phi(0)}\left[{}_{(\frac{a+b}{2})^+}I_\varphi h(b) + {}_{(\frac{a+b}{2})^-}I_\varphi h(a)\right]\right| \\ &\quad \le \frac{(b-a)^2}{8\Phi(0)}\left[\int_0^1 \left|\Delta(u)\right| \,\mathrm{d}u\left(\left|h''(a)\right| + \left|h''(b)\right|\right)\right].\end{aligned}$$

Theorem 2.8 ([12]) *Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on I° such that $h'' \in L_1([a,b])$ with $a, b \in I^\circ$ and $a < b$. If $|h''|^q$ for $q > 1$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{h(a)+h(b)}{2} - \frac{1}{2\Phi(0)}\left[{}_{(\frac{a+b}{2})^+}I_\varphi h(b) + {}_{(\frac{a+b}{2})^-}I_\varphi h(a)\right]\right| \\ &\quad \le \frac{(b-a)^2}{8\Phi(0)}\left(\int_0^1 \left|\Delta(u)\right|^p \,\mathrm{d}u\right)^{1/p}\left\{\left(\frac{|h''(a)|^q + 3|h''(b)|^q}{4}\right)^{1/q} \right. \\ &\qquad \left. + \left(\frac{3|h''(a)|^q + |h''(b)|^q}{4}\right)^{1/q}\right\} \\ &\quad \le \frac{(b-a)^2}{2^{3/q}\Phi(0)}\left(\int_0^1 \left|\Delta(u)\right|^p \,\mathrm{d}u\right)^{1/p}\left(\left|h''(a)\right| + \left|h''(b)\right|\right),\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

3 A generalized fractional integral identity

Before stating and proving our main results, we formulate the following important fractional integral identity.

Lemma 3.1 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$ such that $f \in L_1([a,b])$. Then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) - \frac{2}{\Delta(1)}\left[{}_{a^+}I_\varphi f\left(\frac{3a+b}{4}\right)\right. \\ &\qquad \left. + {}_{(\frac{3a+b}{4})^+}I_\varphi f\left(\frac{a+b}{2}\right) + {}_{(\frac{a+b}{2})^+}I_\varphi f\left(\frac{a+3b}{4}\right) + {}_{(\frac{a+3b}{4})^-}I_\varphi f(b)\right] \\ &\quad = \frac{b-a}{8\Delta(1)}\left[\int_0^1 \nabla(t) f'\left(\frac{3a+b}{4}t + \frac{a+b}{2}(1-t)\right)\,\mathrm{d}t\right. \\ &\qquad - \int_0^1 \Delta(t) f'\left(at + \frac{3a+b}{4}(1-t)\right)\,\mathrm{d}t + \int_0^1 \nabla(t) f'\left(\frac{a+3b}{4}t + b(1-t)\right)\,\mathrm{d}t \\ &\qquad \left. - \int_0^1 \Delta(t) f'\left(\frac{a+b}{2}t + \frac{a+3b}{4}(1-t)\right)\,\mathrm{d}t\right], \qquad (3.1)\end{aligned}$$

where

$$\Delta(t) = \int_0^t \frac{\varphi((\frac{b-a}{4})u)}{u}\,\mathrm{d}u < \infty \quad \textit{and} \quad \nabla(t) = \int_t^1 \frac{\varphi((\frac{b-a}{4})u)}{u}\,\mathrm{d}u < \infty.$$

Proof Integrating by parts gives

$$I_1 = \frac{b-a}{8\Delta(1)}\int_0^1 \nabla(t)f'\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\,\mathrm{d}t = \frac{1}{2[\Delta(1)+\nabla(0)]}$$
$$\times\left[\nabla(0)f\left(\frac{a+b}{2}\right)-\int_0^1\frac{\varphi((\frac{b-a}{4})t)}{t}f\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\,\mathrm{d}t\right].$$

Changing the variable $x=\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)$ yields

$$\begin{aligned}I_1 &= \frac{1}{2\Delta(1)}\left[\nabla(0)f\left(\frac{a+b}{2}\right)+\int_{\frac{a+b}{2}}^{\frac{3a+b}{4}}\frac{\varphi(\frac{a+b}{2}-x)}{\frac{a+b}{2}-x}f(x)\,\mathrm{d}x\right]\\ &= \frac{1}{2\Delta(1)}\left[\nabla(0)f\left(\frac{a+b}{2}\right)-{}_{(\frac{3a+b}{4})^+}I_\varphi f\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}I_2 &= -\frac{b-a}{8\Delta(1)}\int_0^1\Delta(t)f'\left(at+\frac{3a+b}{4}(1-t)\right)\,\mathrm{d}t\\ &= \frac{1}{2\Delta(1)}\left[\Delta(1)f(a)-\int_0^1\frac{\varphi((\frac{b-a}{4})t)}{t}f\left(at+\frac{3a+b}{4}(1-t)\right)\,\mathrm{d}t\right]\\ &= \frac{1}{2\Delta(1)}\left[\Delta(1)f(a)-{}_{a^+}I_\varphi f\left(\frac{3a+b}{4}\right)\right],\\ I_3 &= \frac{b-a}{8\Delta(1)}\int_0^1\nabla(t)f'\left(\frac{a+3b}{4}t+b(1-t)\right)\,\mathrm{d}t\\ &= \frac{1}{2\Delta(1)}\left[\nabla(0)f(b)-\int_0^1\frac{\varphi((\frac{b-a}{4})t)}{t}f\left(\frac{a+3b}{4}t+b(1-t)\right)\,\mathrm{d}t\right]\\ &= \frac{1}{2\Delta(1)}\left[\nabla(0)f(b)-{}_{(\frac{a+3b}{4})^+}I_\varphi f(b)\right]\end{aligned}$$

and

$$\begin{aligned}I_4 &= -\frac{b-a}{8\Delta(1)}\int_0^1\Delta(t)f'\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\,\mathrm{d}t\\ &= \frac{1}{2\Delta(1)}\left[\Delta(1)f\left(\frac{a+b}{2}\right)-\int_0^1\frac{\varphi((\frac{b-a}{4})t)}{t}f\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\,\mathrm{d}t\right]\\ &= \frac{1}{2\Delta(1)}\left[\Delta(1)f\left(\frac{a+b}{2}\right)-{}_{(\frac{a+b}{2})^+}I_\varphi f\left(\frac{a+3b}{4}\right)\right],\end{aligned}$$

where we used the fact that $\Delta(1)=\nabla(0)$. Adding I_1, I_2, I_3, and I_4 results in identity (3.1). The proof is thus completed. □

Remark 3.1 Since $\Delta(1)=\nabla(0)$, we can write identity (3.1) in Lemma 3.1 as

$$\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2}{\nabla(0)}\left[{}_{a^+}I_\varphi f\left(\frac{3a+b}{4}\right)\right.$$
$$\left.+{}_{(\frac{3a+b}{4})^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{(\frac{a+b}{2})^+}I_\varphi f\left(\frac{a+3b}{4}\right)+{}_{(\frac{a+3b}{4})^-}I_\varphi f(b)\right]$$

$$= \frac{b-a}{8\nabla(0)}\left[\int_0^1 \nabla(t)f'\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\,\mathrm{d}t - \int_0^1 \Delta(t)f'\left(at+\frac{3a+b}{4}(1-t)\right)\,\mathrm{d}t + \int_0^1 \nabla(t)f'\left(\frac{a+3b}{4}t + b(1-t)\right)\,\mathrm{d}t - \int_0^1 \Delta(t)f'\left(\frac{a+b}{2}t+\frac{a+3b}{4}(1-t)\right)\,\mathrm{d}t\right].$$

Remark 3.2 Under assumptions of Lemma 3.1, if $\varphi(t) = t$, then identity (3.1) reduces to

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{16}\left[\int_0^1 (1-t)f'\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\,\mathrm{d}t - \int_0^1 tf'\left(at+\frac{3a+b}{4}(1-t)\right)\,\mathrm{d}t + \int_0^1 (1-t)f'\left(\frac{a+3b}{4}t + b(1-t)\right)\,\mathrm{d}t - \int_0^1 tf'\left(\frac{a+b}{2}t+\frac{a+3b}{4}(1-t)\right)\,\mathrm{d}t\right],$$

which has been proved in [26].

Remark 3.3 Under assumptions of Lemma 3.1, if $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then identity (3.1) reduces to identity (1.2).

Remark 3.4 Under assumptions of Lemma 3.1, if $\varphi(t) = \frac{t^{\alpha/k}}{k\Gamma_k(\alpha)}$, then

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{4^{\alpha/k-1}\Gamma_k(\alpha+1)}{(b-a)^{\alpha/k}}\left[I^\alpha_{a^+,k}f\left(\frac{3a+b}{4}\right) + I^\alpha_{(\frac{3a+b}{4})^+,k}f\left(\frac{a+b}{2}\right) + I^\alpha_{(\frac{a+b}{2})^+,k}f\left(\frac{a+3b}{4}\right) + I^\alpha_{(\frac{a+3b}{4})^+,k}f(b)\right]$$
$$= \frac{b-a}{16}\left[\int_0^1 \left(1-t^{\alpha/k}\right)f'\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\,\mathrm{d}t - \int_0^1 t^{\alpha/k}f'\left(at+\frac{3a+b}{4}(1-t)\right)\,\mathrm{d}t + \int_0^1 \left(1-t^{\alpha/k}\right)f'\left(\frac{a+3b}{4}t + b(1-t)\right)\,\mathrm{d}t - \int_0^1 t^{\alpha/k}f'\left(\frac{a+b}{2}t+\frac{a+3b}{4}(1-t)\right)\,\mathrm{d}t\right].$$

Remark 3.5 Under assumptions of Lemma 3.1, applying $\varphi(t) = \frac{t}{\alpha}\exp(-\frac{1-\alpha}{\alpha}t)$ gives

$$\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2(1-\alpha)}{1-\exp(-A)}\left[\mathcal{I}^\alpha{}_{a^+}f\left(\frac{3a+b}{4}\right) + \mathcal{I}^\alpha_{(\frac{3a+b}{4})^+}f\left(\frac{a+b}{2}\right) + \mathcal{I}^\alpha{}_{(\frac{a+b}{2})^+}f\left(\frac{a+3b}{4}\right) + \mathcal{I}^\alpha_{(\frac{a+3b}{4})^-}f(b)\right]$$
$$= \frac{b-a}{8[1-\exp(-A)]}\left[\int_0^1 \left[\exp(-At)-\exp(-A)\right]f'\left(\frac{3a+b}{4}t\right.\right.$$

$$+\frac{a+b}{2}(1-t)\Big)\,\mathrm{d}t-\int_0^1\big[1-\exp(-At)\big](t)f'\left(at+\frac{3a+b}{4}(1-t)\right)\mathrm{d}t$$
$$+\int_0^1\big[\exp(-At)-\exp(-A)\big]f'\left(\frac{a+3b}{4}t+b(1-t)\right)\mathrm{d}t$$
$$-\int_0^1\big[1-\exp(-At)\big]f'\left(\frac{a+b}{2}t+\frac{a+3b}{4}(1-t)\right)\mathrm{d}t\Bigg]$$

for $A=\frac{1-\alpha}{\alpha}\frac{b-a}{2}$.

4 Generalized fractional integral inequalities of Hermite–Hadamard type

Now we are in a position to state and prove our main results.

Theorem 4.1 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $f'\in L_1([a,b])$ for $0\le a<b$ and $\alpha>0$. If the mapping $|f'|^q$ for $q\ge 1$ is (α_1,m)-convex on $[0,\frac{b}{m}]$ for some $(\alpha_1,m)\in(0,1]^2$, then*

$$\begin{aligned}&\Bigg|\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2}{\Delta(1)}\Bigg[{}_{a^+}I_\varphi f\left(\frac{3a+b}{4}\right)\\&\quad+{}_{(\frac{3a+b}{4})^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{(\frac{a+b}{2})^+}I_\varphi f\left(\frac{a+3b}{4}\right)+{}_{(\frac{a+3b}{4})^-}I_\varphi f(b)\Bigg]\Bigg|\\&\le\frac{b-a}{8\Delta(1)}\Bigg[\left(\int_0^1\big|\Delta(t)\big|\,\mathrm{d}t\right)^{1-1/q}\left(A_1\big|f'(a)\big|^q+A_2\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q}\\&\quad+\left(\int_0^1\big|\nabla(t)\big|\,\mathrm{d}t\right)^{1-1/q}\left(B_1\left|f'\left(\frac{3a+b}{4}\right)\right|^q+B_2\left|f'\left(\frac{a+b}{4m}\right)\right|^q\right)^{1/q}\\&\quad+\left(\int_0^1\big|\Delta(t)\big|\,\mathrm{d}t\right)^{1-1/q}\left(A_1\left|f'\left(\frac{a+b}{4}\right)\right|^q+A_2\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q}\\&\quad+\left(\int_0^1\big|\nabla(t)\big|\,\mathrm{d}t\right)^{1-1/q}\left(B_1\left|f'\left(\frac{a+3b}{4}\right)\right|^q+B_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\Bigg],\end{aligned}$$

where the constants A_1, A_2, B_1, and B_2 are defined by

$$A_1=\int_0^1 t^{\alpha_1}\big|\Delta(t)\big|\,\mathrm{d}t,\qquad A_2=\int_0^1 m\big(1-t^{\alpha_1}\big)\big|\Delta(t)\big|\,\mathrm{d}t,$$
$$B_1=\int_0^1 t^{\alpha_1}\big|\nabla(t)\big|\,\mathrm{d}t,\qquad B_2=\int_0^1 m\big(1-t^{\alpha_1}\big)\big|\nabla(t)\big|\,\mathrm{d}t.$$

Proof Using Lemma 3.1, the well-known power mean inequality, and the (α_1,m)-convexity of $|f'|^q$ on $[0,\frac{b}{m}]$ gives

$$\begin{aligned}&\Bigg|\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2}{\Delta(1)}\Bigg[{}_{a^+}I_\varphi f\left(\frac{3a+b}{4}\right)\\&\quad+{}_{(\frac{3a+b}{4})^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{(\frac{a+b}{2})^+}I_\varphi f\left(\frac{a+3b}{4}\right)+{}_{(\frac{a+3b}{4})^-}I_\varphi f(b)\Bigg]\Bigg|\\&\le\frac{b-a}{8\Delta(1)}\Bigg[\int_0^1\big|\Delta(t)\big|\left|f'\left(at+\frac{3a+b}{4}(1-t)\right)\right|\mathrm{d}t\\&\quad+\int_0^1\big|\nabla(t)\big|\left|f'\left(\frac{3a+b}{4}t+\frac{a+b}{2}(1-t)\right)\right|\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 |\Delta(t)| \left| f'\left(\frac{a+b}{2}t+\frac{a+3b}{4}(1-t)\right)\right| \,\mathrm{d}t \\
&+\int_0^1 |\nabla(t)| \left| f'\left(\frac{a+3b}{4}t+b(1-t)\right)\right| \,\mathrm{d}t\Bigg] \\
\le{}& \frac{b-a}{8\Delta(1)}\Bigg\{\left(\int_0^1 |\Delta(t)|\,\mathrm{d}t\right)^{1-1/q}\Bigg[\int_0^1 |\Delta(t)|\bigg(t^{\alpha_1}|f'(a)|^q \\
&+m\big(1-t^{\alpha_1}\big)\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\bigg)\,\mathrm{d}t\Bigg]^{1/q} \\
&+\left(\int_0^1 |\nabla(t)|\,\mathrm{d}t\right)^{1-1/q}\Bigg[\int_0^1 |\nabla(t)|\bigg(t^{\alpha_1}\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&+m\big(1-t^{\alpha_1}\big)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\bigg)\,\mathrm{d}t\Bigg]^{1/q} \\
&+\left(\int_0^1 |\Delta(t)|\,\mathrm{d}t\right)^{1-1/q}\Bigg[\int_0^1 |\Delta(t)|\bigg(t^{\alpha_1}\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&+m\big(1-t^{\alpha_1}\big)\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\bigg)\,\mathrm{d}t\Bigg]^{1/q} \\
&+\left(\int_0^1 |\nabla(t)|\,\mathrm{d}t\right)^{1-1/q}\Bigg[\int_0^1 |\nabla(t)|\bigg(t^{\alpha_1}\left|f'\left(\frac{a+3b}{4}\right)\right|^q \\
&+m\big(1-t^{\alpha_1}\big)\left|f'\left(\frac{b}{m}\right)\right|^q\bigg)\,\mathrm{d}t\Bigg]^{1/q}\Bigg\} \\
={}& \frac{b-a}{8\Delta(1)}\Bigg[\left(\int_0^1 |\Delta(t)|\,\mathrm{d}t\right)^{1-1/q}\left(A_1|f'(a)|^q+A_2\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q} \\
&+\left(\int_0^1 |\nabla(t)|\,\mathrm{d}t\right)^{1-1/q}\left(B_1\left|f'\left(\frac{3a+b}{4}\right)\right|^q+B_2\left|f'\left(\frac{a+b}{4m}\right)\right|^q\right)^{1/q} \\
&+\left(\int_0^1 |\Delta(t)|\,\mathrm{d}t\right)^{1-1/q}\left(A_1\left|f'\left(\frac{a+b}{4}\right)\right|^q+A_2\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q} \\
&+\left(\int_0^1 |\nabla(t)|\,\mathrm{d}t\right)^{1-1/q}\left(B_1\left|f'\left(\frac{a+3b}{4}\right)\right|^q+B_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\Bigg].
\end{aligned}
$$

This completes the proof. □

Remark 4.1 Under assumptions of Theorem 4.1, if $\varphi(t)=t$ and $m=\alpha_1=1$, then

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{32}\left(\frac{1}{3}\right)^{1/q}\Bigg[\left(2|f'(a)|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q} \\
&+\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q} \\
&+\left(2\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q}+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+2|f'(b)|^q\right)^{1/q}\Bigg],
\end{aligned}
$$

which was proved in [26].

Remark 4.2 Under assumptions of Theorem 4.1, if $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then the inequality in Theorem 4.1 reduces to inequality (1.3).

Corollary 4.1 *Under assumptions of Theorem 4.1, if $\varphi(t) = \frac{t^{\alpha/k}}{k\Gamma_k(\alpha)}$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{4^{\alpha/k-1}\Gamma_k(\alpha+1)}{(b-a)^{\alpha/k}}\left[I^{\alpha}_{a^+,k}f\left(\frac{3a+b}{4}\right)\right.\right.\\
&\quad\left.\left.+I^{\alpha}_{(\frac{3a+b}{4})^+,k}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{(\frac{a+b}{2})^+,k}f\left(\frac{a+3b}{4}\right)+I^{\alpha}_{(\frac{a+3b}{4})^+,k}f(b)\right]\right|\\
&\quad\le\frac{b-a}{16(\frac{\alpha}{k}+1)}\left[\frac{1}{(\alpha_1+1)(\frac{\alpha}{k}+\alpha_1+1)}\right]^{1/q}\left\{\left[\left(\frac{\alpha}{k}+1\right)(\alpha_1+1)\left|f'(a)\right|^q\right.\right.\\
&\quad\left.+m\alpha_1(\alpha_1+1)\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right]^{1/q}+\frac{\alpha}{k}\left[\left(\frac{\alpha}{k}+1\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right.\\
&\quad\left.+m\alpha_1\left(\alpha_1+\frac{\alpha}{k}+2\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}+\left[\left(\frac{\alpha}{k}+1\right)(\alpha_1+1)\right.\\
&\quad\left.\times\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\alpha_1(\alpha_1+1)\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right]^{1/q}\\
&\quad\left.+\alpha\left[\left(\frac{\alpha}{k}+1\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\alpha_1\left(\alpha_1+\frac{\alpha}{k}+2\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}.
\end{aligned}$$

Corollary 4.2 *Under assumptions of Theorem 4.1, if $\alpha_1 = 1$ and $\varphi(t) = \frac{t}{\alpha}\exp(-\frac{1-\alpha}{\alpha}t)$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2(1-\alpha)}{1-\exp(-A)}\left[\mathcal{I}^{\alpha}_{a^+}f\left(\frac{3a+b}{4}\right)\right.\right.\\
&\quad\left.\left.+\mathcal{I}^{\alpha}_{(\frac{3a+b}{4})^+}f\left(\frac{a+b}{2}\right)+\mathcal{I}^{\alpha}_{(\frac{a+b}{2})^+}f\left(\frac{a+3b}{4}\right)+\mathcal{I}^{\alpha}_{(\frac{a+3b}{4})^-}f(b)\right]\right|\\
&\quad\le\frac{b-a}{8[1-\exp(-A)]}\left[\left(\frac{A+\exp(-A)-1}{A}\right)^{1-1/q}\left(A_3\left|f'(a)\right|^q\right.\right.\\
&\quad\left.+A_4\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q}+\left(\frac{A\exp(-A)+\exp(-A)-1}{A}\right)^{1-1/q}\\
&\quad\times\left(B_3\left|f'\left(\frac{3a+b}{4}\right)\right|^q+B_4\left|f'\left(\frac{a+b}{4m}\right)\right|^q\right)^{1/q}\\
&\quad+\left(\frac{A+\exp(-A)-1}{A}\right)^{1-1/q}\left(A_3\left|f'\left(\frac{a+b}{4}\right)\right|^q+A_4\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q}\\
&\quad+\left(\frac{A\exp(-A)+\exp(-A)-1}{A}\right)^{1-1/q}\\
&\quad\left.\times\left(B_3\left|f'\left(\frac{a+3b}{4}\right)\right|^q+B_4\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\right],
\end{aligned}$$

where

$$\begin{aligned}
&A=\frac{1-\alpha}{\alpha}\frac{b-a}{2},\qquad A_3=\frac{A^2+2A\exp(-A)+2\exp(-A)-2}{2A^2},\\
&A_4=\frac{m(A+\exp(-A)-1)}{A^2},
\end{aligned}$$

$$B_3=\frac{A^2\exp(-A)+2A\exp(-A)+2\exp(-A)-2}{2A^2},$$
$$B_4=\frac{m(A^2+2A+2\exp(-A)-A^2\exp(-A)-2)}{2A^2}.$$

Theorem 4.2 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $f'\in L_1([a,b])$ for $0\le a<b$. If the mapping $|f'|^q$ is (α_1,m)-convex on $[0,\frac{b}{m}]$ for some $(\alpha_1,m)\in(0,1]^2$, $q\ge 1$, and $q\ge r\ge 0$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2}{\Delta(1)}\left[{}_{a^+}I_\varphi f\left(\frac{3a+b}{4}\right)\right.\right.\\
&\quad\left.\left.+{}_{(\frac{3a+b}{4})^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{(\frac{a+b}{2})^+}I_\varphi f\left(\frac{a+3b}{4}\right)+{}_{(\frac{a+3b}{4})^-}I_\varphi f(b)\right]\right|\\
&\quad\le\frac{b-a}{8\Delta(1)}\left[\left(\int_0^1\left|\Delta(t)\right|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(C_1\left|f'(a)\right|^q+C_2\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q}\right.\\
&\quad+\left(\int_0^1\left|\nabla(t)\right|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(D_1\left|f'\left(\frac{3a+b}{4}\right)\right|^q+D_2\left|f'\left(\frac{a+b}{4m}\right)\right|^q\right)^{1/q}\\
&\quad+\left(\int_0^1\left|\Delta(t)\right|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(C_1\left|f'\left(\frac{a+b}{4}\right)\right|^q+C_2\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q}\\
&\quad\left.+\left(\int_0^1\left|\nabla(t)\right|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(D_1\left|f'\left(\frac{a+3b}{4}\right)\right|^q+D_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\right],
\end{aligned}\tag{4.1}$$

where the constants C_1, C_2, D_1, and D_2 are defined by

$$C_1=\int_0^1 t^{\alpha_1}\left|\Delta(t)\right|^r\,\mathrm{d}t,\qquad C_2=\int_0^1 m\left(1-t^{\alpha_1}\right)\left|\Delta(t)\right|^r\,\mathrm{d}t,$$
$$D_1=\int_0^1 t^{\alpha_1}\left|\nabla(t)\right|^r\,\mathrm{d}t,\qquad D_2=\int_0^1 m\left(1-t^{\alpha_1}\right)\left|\nabla(t)\right|^r\,\mathrm{d}t.$$

Proof By Lemma 3.1, the well-known Hölder inequality, and the (α_1,m)-convexity of $|f'|^q$ on $[0,\frac{b}{m}]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2}{\Delta(1)}\left[{}_{a^+}I_\varphi f\left(\frac{3a+b}{4}\right)\right.\right.\\
&\quad\left.\left.+{}_{(\frac{3a+b}{4})^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{(\frac{a+b}{2})^+}I_\varphi f\left(\frac{a+3b}{4}\right)+{}_{(\frac{a+3b}{4})^-}I_\varphi f(b)\right]\right|\\
&\quad\le\frac{b-a}{8\Delta(1)}\left[\int_0^1\left|\Delta(t)\right|\left|f'\left(at+\frac{3a+b}{4}(1-t)\right)\right|\,\mathrm{d}t+\int_0^1\left|\nabla(t)\right|\left|f'\left(\frac{3a+b}{4}t\right.\right.\right.\\
&\quad\left.\left.+\frac{a+b}{2}(1-t)\right)\right|\,\mathrm{d}t+\int_0^1\left|\Delta(t)\right|\left|f'\left(\frac{a+b}{2}t+\frac{a+3b}{4}(1-t)\right)\right|\,\mathrm{d}t\\
&\quad\left.+\int_0^1\left|\nabla(t)\right|\left|f'\left(\frac{a+3b}{4}t+b(1-t)\right)\right|\,\mathrm{d}t\right]\\
&\quad\le\frac{b-a}{8\Delta(1)}\left\{\left(\int_0^1\left|\Delta(t)\right|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1\left|\Delta(t)\right|^r\left(t^{\alpha_1}\left|f'(a)\right|^q\right.\right.\right.\\
&\quad\left.\left.+m\left(1-t^{\alpha_1}\right)\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)\,\mathrm{d}t\right]^{1/q}+\left(\int_0^1\left|\nabla(t)\right|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\int_0^1|\nabla(t)|^r\left(t^{\alpha_1}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+m\left(1-t^{\alpha_1}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\\
&+\left(\int_0^1|\Delta(t)|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1|\Delta(t)|^r\left(t^{\alpha_1}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.\right.\\
&\left.\left.+m\left(1-t^{\alpha_1}\right)\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}+\left(\int_0^1|\nabla(t)|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\\
&\left.\times\left[\int_0^1|\nabla(t)|^r\left(t^{\alpha_1}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+m\left(1-t^{\alpha_1}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}\\
&=\frac{b-a}{8\Delta(1)}\left[\left(\int_0^1|\Delta(t)|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(C_1\left|f'(a)\right|^q+C_2\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q}\right.\\
&+\left(\int_0^1|\nabla(t)|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(D_1\left|f'\left(\frac{3a+b}{4}\right)\right|^q+D_2\left|f'\left(\frac{a+b}{4m}\right)\right|^q\right)^{1/q}\\
&+\left(\int_0^1|\Delta(t)|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(C_1\left|f'\left(\frac{a+b}{4}\right)\right|^q+C_2\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q}\\
&\left.+\left(\int_0^1|\nabla(t)|^{\frac{q-r}{q-1}}\,\mathrm{d}t\right)^{1-1/q}\left(D_1\left|f'\left(\frac{a+3b}{4}\right)\right|^q+D_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\right].
\end{aligned}
$$

The required proof is complete. □

Remark 4.3 Under assumptions of Theorem 4.2, if $\varphi(t)=t$, then inequality (4.1) reduces to

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\leq\frac{b-a}{16}\left(\frac{q-1}{2q-r-1}\right)^{1-1/q}\left[\left(\frac{1}{r+\alpha_1+1}\left|f'(a)\right|^q+\frac{m\alpha_1}{(r+1)(r+\alpha_1+1)}\right.\right.\\
&\left.\times\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{1/q}+\left(B(r+1,\alpha_1+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q+m\left(\frac{1}{r+1}\right.\right.\\
&\left.\left.-B(r+1,\alpha_1+1)\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}+\left(\frac{1}{r+\alpha_1+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.\\
&\left.+\frac{m\alpha_1}{(r+1)(r+\alpha_1+1)}\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{1/q}+B(r+1,\alpha_1+1)\\
&\left.\times\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+m\left(\frac{1}{r+1}-B(r+1,\alpha_1+1)\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\right],
\end{aligned}
$$

which was proved in [26].

Remark 4.4 Under assumptions of Theorem 4.2, if $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, then inequality (4.1) reduces to inequality (1.4).

Corollary 4.3 *Under assumptions of Theorem 4.2, if $\varphi(t)=\frac{t^{\alpha/k}}{k\Gamma_k(\alpha)}$, then*

$$
\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{4^{\alpha/k-1}\Gamma_k(\alpha+1)}{(b-a)^{\alpha/k}}\left[I^{\alpha}_{a^+,k}f\left(\frac{3a+b}{4}\right)\right.\right.
$$

$$
\begin{aligned}
&+I^{\alpha}_{(\frac{3a+b}{4})^+,k}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{(\frac{a+b}{2})^+,k}f\left(\frac{a+3b}{4}\right)+I^{\alpha}_{(\frac{a+3b}{4})^+,k}f(b)\Bigg]\Bigg|\\
&\leq\frac{b-a}{16}\Bigg\{\left[\frac{q-1}{\frac{\alpha}{k}(q-r)+q-1}\right]^{1-1/q}\Bigg[\frac{1}{\frac{\alpha r}{k}+\alpha_1+1}\left|f'(a)\right|^q\\
&+\frac{m\alpha_1}{(\frac{\alpha r}{k}+1)(\frac{\alpha r}{k}+\alpha_1+1)}\left|f\left(\frac{3a+b}{4m}\right)\right|^q\Bigg]^{1/q}+\frac{k}{\alpha}B^{1-1/q}\left(\frac{2q-r-1}{q-1},\frac{k}{\alpha}\right)\\
&\times\Bigg[\left(r+1,\frac{k(\alpha_1+1)}{\alpha}\right)\left|f\left(\frac{3a+b}{4}\right)\right|^q+m\Bigg[B\left(r+1,\frac{k}{\alpha}\right)-B\Bigg(r+1,\\
&\frac{k(\alpha_1+1)}{\alpha}\Bigg)\Bigg]\left|f\left(\frac{a+b}{2m}\right)\right|^q\Bigg]^{1/q}+\left[\frac{q-1}{\frac{\alpha}{k}(q-r)+q-1}\right]^{1-1/q}\Bigg[\frac{1}{\frac{\alpha r}{k}+\alpha_1+1}\\
&\times\left|f'(a)\right|^q+\frac{m\alpha_1}{(\frac{\alpha r}{k}+1)(\frac{\alpha r}{k}+\alpha_1+1)}\left|f\left(\frac{a+3b}{4m}\right)\right|^q\Bigg]^{1/q}\\
&+\frac{k}{\alpha}B^{1-1/q}\left(\frac{2q-r-1}{q-1},\frac{k}{\alpha}\right)\Bigg[\left(r+1,\frac{k(\alpha_1+1)}{\alpha}\right)\left|f\left(\frac{a+3b}{4}\right)\right|^q\\
&+m\Bigg[B\left(r+1,\frac{k}{\alpha}\right)-B\left(r+1,\frac{k(\alpha_1+1)}{\alpha}\right)\Bigg]\left|f\left(\frac{b}{m}\right)\right|^q\Bigg]^{1/q}\\
&+\left[\frac{q-1}{\frac{\alpha}{k}(q-r)+q-1}\right]^{1-1/q}\Bigg\}.
\end{aligned}
$$

Corollary 4.4 *Under assumptions of Theorem 4.2, if $r=0$ and $\varphi(t)=\frac{t}{\alpha}\exp(-\frac{1-\alpha}{\alpha}t)$, then*

$$
\begin{aligned}
&\Bigg|\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)-\frac{2(1-\alpha)}{1-\exp(-A)}\Bigg[\mathcal{I}^{\alpha}_{a^+}f\left(\frac{3a+b}{4}\right)\\
&+\mathcal{I}^{\alpha}_{(\frac{3a+b}{4})^+}f\left(\frac{a+b}{2}\right)+\mathcal{I}^{\alpha}_{(\frac{a+b}{2})^+}f\left(\frac{a+3b}{4}\right)+\mathcal{I}^{\alpha}_{(\frac{a+3b}{4})^-}f(b)\Bigg]\Bigg|\\
&\leq\frac{b-a}{8[1-\exp(-A)]}\Bigg[\left(\left(\int_0^1[1-\exp(-At)]^p\,dt\right)^{1/p}\left(\left|f'(a)\right|^q\right.\right.\\
&\left.+\left|f'\left(\frac{3a+b}{4m}\right)\right|^q\right)^{1/q}+\left(\int_0^1[\exp(-At)-\exp(-A)]^p\,dt\right)^{1/p}\\
&\times\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{4m}\right)\right|^q\right)^{1/q}+\left(\int_0^1[1-\exp(-At)]^p\,dt\right)^{1/p}\\
&\times\left(\left|f'\left(\frac{a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+3b}{4m}\right)\right|^q\right)^{1/q}+\Bigg(\int_0^1[\exp(-At)\\
&-\exp(-A)]^p\,dt\Bigg)^{1/p}\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\Bigg],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $A=\frac{1-\alpha}{\alpha}\frac{b-a}{2}$.

Remark 4.5 Under assumptions of Theorem 4.2, if $A=\frac{1-\alpha}{\alpha}\frac{b-a}{2}$, $\alpha_1=r=1$, and $\varphi(t)=\frac{t}{\alpha}\exp(-\frac{1-\alpha}{\alpha}t)$, then Theorem 4.2 reduces to Corollary 4.2.

5 Conclusions

In this work, we establish generalized fractional integral inequalities, the Riemann–Liouville fractional integral inequalities, and some classical integral inequalities of the Hermite–Hadamard type for (α, m)-convex functions. The results presented in this paper would provide generalizations and extensions of those given in earlier works.

Acknowledgements
The authors appreciate the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Funding
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]Institute of Mathematics, Henan Polytechnic University, Jiaozuo, China. [2]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, China. [3]Department of Mathematics, College of Education, University of Sulaimani, Sulaimani, Iraq. [4]Center for General Education, China Medical University, Taichung, Taiwan. [5]School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin, China.

Publisher's Note

Received: 2 January 2019 Accepted: 23 April 2019 Published online: 14 May 2019

References

1. Alomari, M., Darus, M., Kirmaci, U.S.: Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means. Comput. Math. Appl. **59**, 225–232 (2010). https://doi.org/10.1016/j.camwa.2009.08.002
2. Bai, R.-F., Qi, F., Xi, B.-Y.: Hermite–Hadamard type inequalities for the *m*- and (α, m)-logarithmically convex functions. Filomat **27**, 1–7 (2013). https://doi.org/10.2298/FIL1301001B
3. Bai, S.-P., Wang, S.-H., Qi, F.: Some Hermite–Hadamard type inequalities for *n*-time differentiable (α, m)-convex functions. J. Inequal. Appl. **2012**, 267 (2012). https://doi.org/10.1186/1029-242X-2012-267
4. Ertuğral, F., Sarikaya, M.Z.: Some trapezoid type inequalities for generalized fractional integral. ResearchGate Article (2018). https://www.researchgate.net/publication/326463500
5. Gorenflo, R., Mainardi, F.: Fractional calculus: integral and differential equations of fractional order. In: Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996). CISM Courses and Lectures, vol. 378, pp. 223–276. Springer, Vienna (1997)
6. Huang, C.-J., Rahman, G., Nisar, K.S., Ghaffar, A., Qi, F.: Some inequalities of the Hermite–Hadamard type for *k*-fractional conformable integrals. Aust. J. Math. Anal. Appl. **16**(1), Article ID 7 (2019). http://ajmaa.org/cgi-bin/paper.pl?string=v16n1/V16I1P7.tex
7. Katugampola, U.N.: New approach to a generalized fractional integral. Appl. Math. Comput. **218**, 860–865 (2011). https://doi.org/10.1016/j.amc.2011.03.062
8. Katugampola, U.N.: Mellin transforms of generalized fractional integrals and derivatives. Appl. Math. Comput. **257**, 566–580 (2015). https://doi.org/10.1016/j.amc.2014.12.067
9. Kirane, M., Torebek, B.T.: Hermite–Hadamard, Hermite–Hadamard–Fejér, Dragomir–Agarwal and Pachpatte type inequalities for convex functions via fractional integrals. ArXiv preprint (2017). https://arxiv.org/abs/1701.00092
10. Mohammed, P.O.: Inequalities of type Hermite–Hadamard for fractional integrals via differentiable convex functions. Turk. J. Anal. Number Theory **4**, 135–139 (2016). https://doi.org/10.12691/tjant-4-5-3
11. Mohammed, P.O.: Inequalities of (k, s), (k, h)-type for Riemann–Liouville fractional integrals. Appl. Math. E-Notes **17**, 199–206 (2017)
12. Mohammed, P.O., Sarıkaya, M.Z.: On some integral inequalities for twice differentiable convex functions via generalized fractional integral. ResearchGate Article (2018). https://www.researchgate.net/publication/329102147
13. Mubeen, S., Habibullah, G.M.: *k*-Fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(1–4), 89–94 (2012)
14. Noor, M.A., Noor, K.I., Awan, M.U.: Some quantum estimates for Hermite–Hadamard inequalities. Appl. Math. Comput. **251**, 675–679 (2015). https://doi.org/10.1016/j.amc.2014.11.090
15. Özdemir, M.E., Avci, M., Kavurmaci, H.: Hermite–Hadamard-type inequalities via (α, m)-convexity. Comput. Math. Appl. **61**, 2614–2620 (2011). https://doi.org/10.1016/j.camwa.2011.02.053
16. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
17. Qi, F.: Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities. Filomat **27**, 601–604 (2013). https://doi.org/10.2298/FIL1304601Q

18. Qi, F.: Parametric integrals, the Catalan numbers, and the beta function. Elem. Math. **72**, 103–110 (2017). https://doi.org/10.4171/EM/332
19. Qi, F.: An improper integral, the beta function, the Wallis ratio, and the Catalan numbers. Probl. Anal. Issues Anal. **7**(25), 104–115 (2018). https://doi.org/10.15393/j3.art.2018.4370
20. Qi, F., Akkurt, A., Yildirim, H.: Catalan numbers, k-gamma and k-beta functions, and parametric integrals. J. Comput. Anal. Appl. **25**, 1036–1042 (2018)
21. Qi, F., Habib, S., Mubeen, S., Naeem, M.N.: Generalized k-fractional conformable integrals and related inequalities. AIMS Math. **4**, 343–358 (2019). https://doi.org/10.3934/Math.2019.3.343
22. Qi, F., Li, W.-H.: Integral representations and properties of some functions involving the logarithmic function. Filomat **30**, 1659–1674 (2016). https://doi.org/10.2298/FIL1607659Q
23. Rahman, G., Nisar, K.S., Qi, F.: Some new inequalities of the Grüss type for conformable fractional integrals. AIMS Math. **3**, 575–583 (2018). https://doi.org/10.3934/Math.2018.4.575
24. Sarıkaya, M.Z., Ertuğral, F.: On the generalized Hermite–Hadamard inequalities. ResearchGate Article (2017). https://www.researchgate.net/publication/321760443
25. Shi, D.-P., Xi, B.-Y., Qi, F.: Hermite–Hadamard type inequalities for (m,h_1,h_2)-convex functions via Riemann–Liouville fractional integrals. Turk. J. Anal. Number Theory **2**, 23–28 (2014). https://doi.org/10.12691/tjant-2-1-6
26. Shi, D.-P., Xi, B.-Y., Qi, F.: Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of (α,m)-convex functions. Fract. Differ. Calc. **4**, 31–43 (2014). https://doi.org/10.7153/fdc-04-02
27. Shuang, Y., Qi, F.: Integral inequalities of Hermite–Hadamard type for extended s-convex functions and applications. Mathematics **6**(11), Article ID 223 (2018). https://doi.org/10.3390/math6110223
28. Shuang, Y., Qi, F., Wang, Y.: Some inequalities of Hermite–Hadamard type for functions whose second derivatives are (α,m)-convex. J. Nonlinear Sci. Appl. **9**, 139–148 (2016). https://doi.org/10.22436/jnsa.009.01.13
29. Shuang, Y., Wang, Y., Qi, F.: Some inequalities of Hermite–Hadamard type for functions whose third derivatives are (α,m)-convex. J. Comput. Anal. Appl. **17**, 272–279 (2014)
30. Shuang, Y., Wang, Y., Qi, F.: Integral inequalities of Simpson's type for (α,m)-convex functions. J. Nonlinear Sci. Appl. **9**, 6364–6370 (2016). https://doi.org/10.22436/jnsa.009.12.36
31. Wang, S.-H., Qi, F.: Hermite–Hadamard type inequalities for s-convex functions via Riemann–Liouville fractional integrals. J. Comput. Anal. Appl. **22**, 1124–1134 (2017)
32. Wang, S.-H., Xi, B.-Y., Qi, F.: On Hermite–Hadamard type inequalities for (α,m)-convex functions. Int. J. Open Probl. Comput. Sci. Math. **5**, 47–56 (2012). https://doi.org/10.12816/0006138
33. Wu, S.: On the weighted generalization of the Hermite–Hadamard inequality and its applications. Rocky Mt. J. Math. **39**, 1741–1749 (2009). https://doi.org/10.1216/RMJ-2009-39-5-1741
34. Wu, Y., Qi, F.: On some Hermite–Hadamard type inequalities for (s,QC)-convex functions. SpringerPlus **5**, 49 (2016). https://doi.org/10.1186/s40064-016-1676-9
35. Xi, B.-Y., Gao, D.-D., Zhang, T., Guo, B.-N., Qi, F.: Shannon type inequalities for Kapur's entropy. Mathematics **7**(1), Article ID 22 (2019). https://doi.org/10.3390/math7010022
36. Yao, Y., Qin, X., Yao, J.-C.: Projection methods for firmly type nonexpansive operators. J. Nonlinear Convex Anal. **19**, 407–415 (2018)
37. Yao, Y., Yao, J.-C., Liou, Y.-C., Postolache, M.: Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms. Carpath. J. Math. **34**, 459–466 (2018)
38. Yin, H.-P., Qi, F.: Hermite–Hadamard type inequalities for the product of (α,m)-convex functions. J. Nonlinear Sci. Appl. **8**, 231–236 (2015). https://doi.org/10.22436/jnsa.008.03.07
39. Yin, H.-P., Wang, J.-Y., Qi, F.: Some integral inequalities of Hermite–Hadamard type for s-geometrically convex functions. Miskolc Math. Notes **19**, 699–705 (2018). https://doi.org/10.18514/MMN.2018.2451