Yang et al. *Journal of Inequalities and Applications* (2019) 2019:115
https://doi.org/10.1186/s13660-019-2070-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some reverse mean inequalities for operators and matrices

Chaojun Yang[1], Yaxin Gao[1] and Fangyan Lu[1*]

*Correspondence: fylu@suda.edu.cn
[1]Department of Mathematics, Soochow University, Suzhou, P.R. China

Abstract

In this paper, we present some new reverse arithmetic–geometric mean inequalities for operators and matrices due to Lin (Stud. Math. 215:187–194, 2013). Among other inequalities, we prove that if $A, B \in B(\mathcal{H})$ are accretive and $0 < mI \le \Re(A), \Re(B) \le MI$, then, for every positive unital linear map Φ,

$$\Phi^2\left(\Re\left(\frac{A+B}{2}\right)\right) \le (K(h))^2 \Phi^2(\Re(A\sharp B)),$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$. Moreover, some reverse harmonic–geometric mean inequalities are also presented.

MSC: Primary 47A63; secondary 15A15

Keywords: Positive linear maps; Arithmetic–geometric–harmonic mean; Sector matrix; Inequality

1 Introduction

Throughout this paper, $B(\mathcal{H})$ stands for all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In the finite-dimensional setting, $\mathbb{M}_n$ denotes the set of all $n \times n$ complex matrices. For $A, B \in B(\mathcal{H})$, we use $A \ge B$ ($B \le A$) to mean that $A - B$ is positive. The operator norm is denoted by $\|\cdot\|$. An operator $A \in B(\mathcal{H})$ is called accretive if in its Cartesian (or Toeptliz) decomposition, $A = \Re A + i\Im A$, $\Re A$ is positive, where $\Re A = \frac{A+A^*}{2}$, $\Im A = \frac{A-A^*}{2i}$. A linear map $\Phi: B(\mathcal{H}) \to B(\mathcal{H})$ is called positive if $\Phi(A) \ge 0$ whenever $A \ge 0$. If $\Phi(I) = I$, where I denotes the identity operator, then we say that Φ is unital. We reserve M, m for scalars. In the finite-dimensional setting, we use I_n for the identity.

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, S_α denote the sector regions in the complex plane as follows:

$$S_\alpha = \{z \in \mathbb{C} : \Re z \ge 0, |\Im z| \le (\Re z)\tan\alpha\}.$$

Clearly, A is positive semidefinite if and only if $W(A) \subset S_0$, and if $W(A), W(B) \subset S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A+B) \subset S_\alpha$. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, then A is nonsingular. Moreover, $W(A) \subset S_\alpha$ implies $W(X^*AX) \subset S_\alpha$ for any nonzero $n \times m$ matrix X, thus $W(A^{-1}) \subset S_\alpha$.

Recent studies on matrices with numerical ranges in a sector can be found in [3–5, 8–10, 14] and the references therein.

In [13], Tominaga presented an operator inequality as follows: Let A, B be positive operators on a Hilbert space with $0 < mI \le A, B \le MI$, then

$$\frac{A+B}{2} \le S(h) A \sharp B, \tag{1}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is called Specht's ratio and $h = \frac{M}{m}$.

Lin [6] found that, for a positive unital linear map Φ between C^*-algebra,

$$\Phi\left(\frac{A+B}{2}\right) \le K(h) \Phi(A \sharp B) \tag{2}$$

due to (1) and the following observation [6]:

$$S(h) \le K(h) \le S^2(h) \quad (h \ge 1),$$

where $K(h) = \frac{(h+1)^2}{4h}$.

It is well known that, for two general positive operators (or positive definite matrices) A, B,

$$A \ge B \quad \nRightarrow \quad A^2 \ge B^2.$$

However, Lin [6] showed that (2) can be squared as follows:

$$\Phi^2\left(\frac{A+B}{2}\right) \le K^2(h) \Phi^2(A \sharp B). \tag{3}$$

Zhang [15] generalized (3) when $p \ge 2$ as follows:

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \le \frac{(K(h)(M^2+m^2))^{2p}}{16 M^{2p} m^{2p}} \Phi^{2p}(A \sharp B). \tag{4}$$

For two accretive operators $A, B \in B(\mathcal{H})$, Drury [3] defined the geometric mean of A and B as follows:

$$A \sharp B = \left(\frac{2}{\pi} \int_0^\infty \left(tA + t^{-1}B\right)^{-1} \frac{dt}{t}\right)^{-1}. \tag{5}$$

This new geometric mean defined by (5) possesses some similar properties compared to the geometric mean of two positive operators. For instance, $A \sharp B = B \sharp A$, $(A \sharp B)^{-1} = A^{-1} \sharp B^{-1}$. For more information about the geometric mean of two accretive operators, see [3]. Moreover, if $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$, then $W(A \sharp B) \subset S_\alpha$.

Following an idea of Lin [6], we shall give some new reverse arithmetic–geometric mean inequalities for operators and matrices which can be seen as a complementary of (3) and (4). Moreover, some reverse harmonic–geometric mean inequalities are also presented.

2 Main results

To reach our goal, we need the following lemmas.

Lemma 2.1 ([10]) *If $A, B \in B(\mathcal{H})$ are accretive, then*

$$\Re(A)\sharp\Re(B) \leq \Re(A\sharp B).$$

Lemma 2.2 ([10]) *If $A, B \in B(\mathcal{H})$ are accretive, then*

$$\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right) \geq \left(\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right)^{-1}.$$

Lemma 2.3 ([9]) *If $A \in \mathbb{M}_n$ has a positive definite real part, then*

$$\Re\left(A^{-1}\right) \leq \Re(A)^{-1}.$$

Lemma 2.4 ([4]) *If $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$, then*

$$\sec^2(\alpha)\Re\left(A^{-1}\right) \geq \Re(A)^{-1}.$$

It is easy to verify that $\Re((\frac{A^{-1}+B^{-1}}{2})^{-1}) \leq \Re(A\sharp B) \leq \Re(\frac{A+B}{2})$ does not persist for two accretive operators A and B. However, Lin presented the following extension of the arithmetic–geometric mean inequality.

Lemma 2.5 ([9]) *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\Re(A\sharp B) \leq \sec^2(\alpha)\Re\left(\frac{A+B}{2}\right). \tag{6}$$

Lemma 2.6 ([2]) *Let $A, B \in B(\mathcal{H})$ be positive. Then*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2.$$

Lemma 2.7 ([1]) *Let $A \in B(\mathcal{H})$ be positive. Then, for every positive unital linear map Φ,*

$$\Phi^{-1}(A) \leq \Phi\left(A^{-1}\right).$$

Lemma 2.8 ([1]) *Let $A, B \in B(\mathcal{H})$ be positive. Then, for $1 \leq r < +\infty$,*

$$\left\|A^r + B^r\right\| \leq \left\|(A+B)^r\right\|.$$

An operator Kantorovich inequality obtained by Marshall and Olkin [12] reads as follows.

Let $0 < mI \leq A \leq MI$, then, for every positive unital linear map Φ,

$$\Phi\left(A^{-1}\right) \leq K(h)\Phi(A)^{-1}, \tag{7}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Lin [7] showed that (7) can be squared as follows:

$$\Phi^2\left(A^{-1}\right) \leq \left(K(h)\right)^2 \Phi(A)^{-2}, \tag{8}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Let $A \in \mathbb{M}_n$ have a positive definite real part, $0 < mI_n \leq \Re(A) \leq MI_n$ and Φ be a unital positive linear map. By (7) and Lemma 2.3, we can obtain the following inequality:

$$\Phi\left(\Re\left(A^{-1}\right)\right) \leq K(h)\Phi\left(\Re(A)\right)^{-1}, \tag{9}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

As an analog of inequality (8), we show that inequality (9) can be squared nicely as follows.

Theorem 2.9 *If $A \in \mathbb{M}_n$ has a positive definite real part and $0 < mI_n \leq \Re(A) \leq MI_n$, then, for every positive unital linear map Φ,*

$$\Phi^2\left(\Re\left(A^{-1}\right)\right) \leq \left(K(h)\right)^2 \Phi\left(\Re(A)\right)^{-2}, \tag{10}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Proof Since

$$mI_n \leq \Re(A) \leq MI_n,$$

we have

$$\left(MI_n - \Re(A)\right)\left(mI_n - \Re(A)\right)\Re(A)^{-1} \leq 0,$$

which is equivalent to

$$\Re(A) + Mm\Re(A)^{-1} \leq (M+m)I_n. \tag{11}$$

By Lemma 2.3 and (11), we get

$$\begin{aligned} &\Re(A) + Mm\Re\left(A^{-1}\right) \\ &\quad \leq \Re(A) + Mm\Re(A)^{-1} \\ &\quad \leq (M+m)I_n. \end{aligned} \tag{12}$$

Inequality (10) is equivalent to

$$\left\| \Phi\left(\Re\left(A^{-1}\right)\right)\Phi\left(\Re(A)\right)\right\| \leq K(h).$$

By computation, we have

$$\left\| Mm\Phi\left(\Re\left(A^{-1}\right)\right)\Phi\left(\Re(A)\right)\right\|$$

$$\le \frac{1}{4}\left\|Mm\Phi\left(\Re\left(A^{-1}\right)\right)+\Phi\left(\Re(A)\right)\right\|^2 \quad \text{(by Lemma 2.6)}$$

$$\le \frac{1}{4}(M+m)^2 \quad \big(\text{by (12)}\big).$$

That is,

$$\left\|\Phi\left(\Re\left(A^{-1}\right)\right)\Phi\left(\Re(A)\right)\right\| \le K(h).$$

This completes the proof. □

Let $A, B \in B(\mathcal{H})$ be accretive, $0 < mI \le \Re(A), \Re(B) \le MI$ and Φ be a unital positive linear map. By inequality (2) and Lemma 2.1, we can obtain the following inequality:

$$\Phi\left(\Re\left(\frac{A+B}{2}\right)\right) \le K(h)\Phi\left(\Re(A\sharp B)\right), \tag{13}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Following an idea of Lin [6], we give a squaring version of inequality (13) below.

Theorem 2.10 *If $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$ and $0 < mI_n \le \Re(A), \Re(B) \le MI_n$, then, for every positive unital linear map Φ,*

$$\Phi^2\left(\Re\left(\frac{A+B}{2}\right)\right) \le \left(\sec^4(\alpha)K(h)\right)^2\Phi^2\left(\Re(A\sharp B)\right), \tag{14}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Proof From Theorem 2.9 we have

$$\frac{1}{2}\Re(A) + \frac{1}{2}Mm\Re(A)^{-1} \le \frac{1}{2}(M+m)I_n \tag{15}$$

and

$$\frac{1}{2}\Re(B) + \frac{1}{2}Mm\Re(B)^{-1} \le \frac{1}{2}(M+m)I_n. \tag{16}$$

Summing up inequalities (15) and (16), we get

$$\Re\left(\frac{A+B}{2}\right) + Mm\left(\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right) \le (M+m)I_n. \tag{17}$$

Inequality (14) is equivalent to

$$\left\|\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-1}\left(\Re(A\sharp B)\right)\right\| \le \sec^4(\alpha)K(h).$$

By computation, we have

$$\left\|\sec^4(\alpha)Mm\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-1}\left(\Re(A\sharp B)\right)\right\|$$

$$
\begin{aligned}
&\le \frac{1}{4}\left\|\sec^4(\alpha)\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)+Mm\Phi^{-1}\left(\Re(A\sharp B)\right)\right\|^2 \quad \text{(by Lemma 2.6)}\\
&\le \frac{1}{4}\left\|\sec^4(\alpha)\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)+Mm\Phi\left(\left(\Re(A\sharp B)\right)^{-1}\right)\right\|^2 \quad \text{(by Lemma 2.7)}\\
&\le \frac{1}{4}\left\|\sec^4(\alpha)\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)+\sec^2(\alpha)Mm\Phi\left(\Re\left(A^{-1}\sharp B^{-1}\right)\right)\right\|^2 \quad \text{(by Lemma 2.4)}\\
&\le \frac{1}{4}\left\|\sec^4(\alpha)\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)+\sec^4(\alpha)Mm\Phi\left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)\right\|^2 \quad \text{(by (6))}\\
&= \frac{1}{4}\left\|\sec^4(\alpha)\Phi\left(\Re\left(\frac{A+B}{2}\right)+Mm\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)\right\|^2\\
&\le \frac{1}{4}\left\|\sec^4(\alpha)\Phi\left(\Re\left(\frac{A+B}{2}\right)+Mm\left(\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right)\right)\right\|^2 \quad \text{(by Lemma 2.3)}\\
&\le \frac{1}{4}\sec^8(\alpha)(M+m)^2 \quad \text{(by (17))}.
\end{aligned}
$$

That is,

$$
\left\|\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-1}\left(\Re(A\sharp B)\right)\right\| \le \sec^4(\alpha)K(h).
$$

This completes the proof. □

Next we give a pth ($p \ge 2$) powering of inequality (14).

Theorem 2.11 *If $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$, $0 < mI_n \le \Re(A), \Re(B) \le MI_n$, $1 < \beta \le 2$ and $p \ge 2\beta$, then, for every positive unital linear map Φ,*

$$
\Phi^p\left(\Re\left(\frac{A+B}{2}\right)\right) \le \frac{\left(\sec^{2\beta}(\alpha)K(h)^{\frac{\beta}{2}}(M^\beta+m^\beta)\right)^{\frac{2p}{\beta}}}{16M^pm^p}\Phi^p\left(\Re(A\sharp B)\right), \tag{18}
$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Proof Since

$$
mI_n \le \Phi\left(\Re\left(\frac{A+B}{2}\right)\right) \le MI_n,
$$

we have

$$
M^\beta m^\beta \Phi^{-\beta}\left(\Re\left(\frac{A+B}{2}\right)\right) + \Phi^\beta\left(\Re\left(\frac{A+B}{2}\right)\right) \le M^\beta + m^\beta. \tag{19}
$$

By (14) and the L-H inequality [1], we obtain

$$
\Phi^{-\beta}\left(\Re(A\sharp B)\right) \le \left(\sec^4(\alpha)K(h)\right)^\beta \Phi^{-\beta}\left(\Re\left(\frac{A+B}{2}\right)\right). \tag{20}
$$

Inequality (18) is equivalent to

$$\left\| \Phi^{\frac{p}{2}}\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-\frac{p}{2}}\left(\Re(A\sharp B)\right)\right\| \le \frac{(\sec^{2\beta}(\alpha)K(h)^{\frac{\beta}{2}}(M^{\beta}+m^{\beta}))^{\frac{p}{\beta}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.$$

By computation, we have

$$\begin{aligned}
&\left\| M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{\frac{p}{2}}\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-\frac{p}{2}}\left(\Re(A\sharp B)\right)\right\| \\
&\le \frac{1}{4}\left\| \left(\sec^{4}(\alpha)K(h)\right)^{\frac{p}{4}}\Phi^{\frac{p}{2}}\left(\Re\left(\frac{A+B}{2}\right)\right) + \left(\frac{M^2m^2}{\sec^4(\alpha)K(h)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}\left(\Re(A\sharp B)\right)\right\|^2 \\
&\le \frac{1}{4}\left\| \left(\sec^{4}(\alpha)K(h)\right)^{\frac{\beta}{2}}\Phi^{\beta}\left(\Re\left(\frac{A+B}{2}\right)\right) + \left(\frac{M^2m^2}{\sec^4(\alpha)K(h)}\right)^{\frac{\beta}{2}}\Phi^{-\beta}\left(\Re(A\sharp B)\right)\right\|^{\frac{p}{\beta}} \\
&\le \frac{1}{4}\left\| \left(\sec^{4}(\alpha)K(h)\right)^{\frac{\beta}{2}}\Phi^{\beta}\left(\Re\left(\frac{A+B}{2}\right)\right)\right. \\
&\qquad \left. + \left(\sec^{4}(\alpha)K(h)\right)^{\frac{\beta}{2}}M^{\beta}m^{\beta}\Phi^{-\beta}\left(\Re\left(\frac{A+B}{2}\right)\right)\right\|^{\frac{p}{\beta}} \\
&= \frac{1}{4}\left\| \left(\sec^{4}(\alpha)K(h)\right)^{\frac{\beta}{2}}\left(\Phi^{\beta}\left(\Re\left(\frac{A+B}{2}\right)\right) + M^{\beta}m^{\beta}\Phi^{-\beta}\left(\Re\left(\frac{A+B}{2}\right)\right)\right)\right\|^{\frac{p}{\beta}} \\
&\le \frac{1}{4}\left(\sec^{2\beta}(\alpha)K(h)^{\frac{\beta}{2}}\left(M^{\beta}+m^{\beta}\right)\right)^{\frac{p}{\beta}},
\end{aligned}$$

where the first inequality is by Lemma 2.6, the second one is by Lemma 2.8, the third one is by (20) and the last one is by (19).

That is,

$$\left\| \Phi^{\frac{p}{2}}\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-\frac{p}{2}}\left(\Re(A\sharp B)\right)\right\| \le \frac{(\sec^{2\beta}(\alpha)K(h)^{\frac{\beta}{2}}(M^{\beta}+m^{\beta}))^{\frac{p}{\beta}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.$$

This completes the proof. □

We are not satisfied with the factor $(\sec^4(\alpha)K(h))^2$ in Theorem 2.10, the ideal factor should be $(K(h))^2$. We shall prove it in the following theorem.

Theorem 2.12 *If $A, B \in B(\mathcal{H})$ are accretive and $0 < mI \le \Re(A), \Re(B) \le MI$, then, for every positive unital linear map Φ,*

$$\Phi^2\left(\Re\left(\frac{A+B}{2}\right)\right) \le \left(K(h)\right)^2\Phi^2\left(\Re(A\sharp B)\right), \tag{21}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Proof From Theorem 2.10 one can get

$$\Re\left(\frac{A+B}{2}\right) + Mm\left(\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right) \le (M+m)I. \tag{22}$$

Inequality (21) is equivalent to

$$\left\| \Phi\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-1}\left(\Re(A\sharp B)\right)\right\| \le K(h).$$

By computation, we have

$$\begin{aligned}
&\left\| Mm\Phi\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-1}\left(\Re(A\sharp B)\right)\right\| \\
&\quad\le \frac{1}{4}\left\| \Phi\left(\Re\left(\frac{A+B}{2}\right)\right) + Mm\Phi^{-1}\left(\Re(A\sharp B)\right)\right\|^2 \quad \text{(by Lemma 2.6)} \\
&\quad\le \frac{1}{4}\left\| \Phi\left(\Re\left(\frac{A+B}{2}\right)\right) + Mm\Phi\left(\left(\Re(A\sharp B)\right)^{-1}\right)\right\|^2 \quad \text{(by Lemma 2.7)} \\
&\quad\le \frac{1}{4}\left\| \Phi\left(\Re\left(\frac{A+B}{2}\right)\right) + Mm\Phi\left(\left(\Re(A)\sharp\Re(B)\right)^{-1}\right)\right\|^2 \quad \text{(by Lemma 2.1)} \\
&\quad= \frac{1}{4}\left\| \Phi\left(\Re\left(\frac{A+B}{2}\right)\right) + Mm\Phi\left(\Re(A)^{-1}\sharp\Re(B)^{-1}\right)\right\|^2 \\
&\quad\le \frac{1}{4}\left\| \Phi\left(\Re\left(\frac{A+B}{2}\right)\right) + Mm\Phi\left(\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right)\right\|^2 \quad \text{(by AM-GM inequality)} \\
&\quad\le \frac{1}{4}(M+m)^2 \quad \text{(by (22))}.
\end{aligned}$$

That is,

$$\left\| \Phi\left(\Re\left(\frac{A+B}{2}\right)\right)\Phi^{-1}\left(\Re(A\sharp B)\right)\right\| \le K(h).$$

This completes the proof. □

Remark 2.13 Letting $A, B \ge 0$ in Theorem 2.12, inequality (21) coincides with inequality (3).

Next we give a pth ($p \ge 2$) powering of inequality (21) along the same line as in Theorem 2.11.

Theorem 2.14 *If $A, B \in B(\mathcal{H})$ are accretive and $0 < mI \le \Re(A), \Re(B) \le MI$ $1 < \beta \le 2$ and $p \ge 2\beta$, then, for every positive unital linear map Φ,*

$$\Phi^p\left(\Re\left(\frac{A+B}{2}\right)\right) \le \frac{(K(h)^{\frac{\beta}{2}}(M^\beta + m^\beta))^{\frac{2p}{\beta}}}{16M^p m^p}\Phi^p\left(\Re(A\sharp B)\right), \tag{23}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Remark 2.15 Letting $A, B \ge 0$ and $\beta = 2$ in Theorem 2.14, inequality (23) coincides with inequality (4).

The following theorem corrects Theorem 1.2 of Liu et al. [11].

Theorem 2.16 *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$, then*

$$\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right) \le \sec^4(\alpha)\Re(A\sharp B). \tag{24}$$

Proof We can get

$$\left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)^{-1} \le \sec^2(\alpha)\bigl(\Re\bigl(A^{-1}\bigr)\sharp\Re\bigl(B^{-1}\bigr)\bigr)^{-1} \tag{25}$$

along the same line as Liu et al. did in [11] by Lemma 2.1 and Lemma 2.5.

Thus we have

$$\begin{aligned}
\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right) &\le \left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)^{-1} \quad \text{(by Lemma 2.3)}\\
&\le \sec^2(\alpha)\bigl(\Re\bigl(A^{-1}\bigr)\sharp\Re\bigl(B^{-1}\bigr)\bigr)^{-1} \quad \bigl(\text{by (25)}\bigr)\\
&= \sec^2(\alpha)\bigl(\Re\bigl(A^{-1}\bigr)\bigr)^{-1}\sharp\bigl(\Re\bigl(B^{-1}\bigr)\bigr)^{-1}\\
&\le \sec^4(\alpha)\bigl(\Re(A)\sharp\Re(B)\bigr) \quad \text{(by Lemma 2.4)}\\
&\le \sec^4(\alpha)\Re(A\sharp B) \quad \text{(by Lemma 2.1)}.
\end{aligned}$$

This completes the proof. □

Remark 2.17 Maybe it is just a clerical error in Theorem 1.2 of their work [11]. However, the authors present the following inequalities in their proof:

$$\begin{aligned}
\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right) &\le \sec^2(\alpha)\bigl(\Re\bigl(A^{-1}\bigr)\bigr)^{-1}\sharp\bigl(\Re\bigl(B^{-1}\bigr)\bigr)^{-1}\\
&\le \sec^2(\alpha)\bigl(\Re(A)\sharp\Re(B)\bigr).
\end{aligned}$$

Obviously, such a deduction in their proof collapses given the property of geometric mean for positive definite matrices. Thus we give Theorem 2.16 and the proof.

Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$, $0 < mI_n \le \Re(A^{-1}), \Re(B^{-1}) \le MI_n$ and Φ be a unital positive linear map. As a complement of inequalities (13) and (24), we have the following reverse harmonic–geometric mean inequality:

$$\Phi\bigl(\Re(A\sharp B)\bigr) \le \sec^2(\alpha)K(h)\Phi\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right), \tag{26}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Proof Compute

$$\begin{aligned}
\Re(A\sharp B) &= \Re\bigl(\bigl(A^{-1}\sharp B^{-1}\bigr)^{-1}\bigr)\\
&\le \Re\bigl(A^{-1}\sharp B^{-1}\bigr)^{-1}\\
&\le K(h)\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}
\end{aligned}$$

$$\le \sec^2(\alpha)K(h)\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right),$$

in which the first inequality is by Lemma 2.3, the second one is by inequality (13) and the last one is by Lemma 2.4.

Imposing Φ on both sides of the inequalities above, we thus obtain inequality (26). □

As an analog of Theorem 2.12, we shall present a squaring version of inequality (26).

Theorem 2.18 *If $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$ and $0 < mI_n \le \Re(A^{-1}), \Re(B^{-1}) \le MI_n$, then, for every positive unital linear map Φ,*

$$\Phi^2\big(\Re(A\sharp B)\big) \le \big(\sec^4(\alpha)K(h)\big)^2\Phi^2\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right), \tag{27}$$

where $K(h) = \frac{(h+1)^2}{4h}$ and $h = \frac{M}{m}$.

Proof From Theorem 2.10 we have

$$\frac{1}{2}\Re\big(A^{-1}\big) + \frac{1}{2}Mm\Re\big(A^{-1}\big)^{-1} \le \frac{1}{2}(M+m)I_n \tag{28}$$

and

$$\frac{1}{2}\Re\big(B^{-1}\big) + \frac{1}{2}Mm\Re\big(B^{-1}\big)^{-1} \le \frac{1}{2}(M+m)I_n. \tag{29}$$

Summing up inequalities (28) and (29), we get

$$\begin{aligned}
&\Re\left(\frac{A^{-1}+B^{-1}}{2}\right) + Mm\Re\left(\frac{A+B}{2}\right)\\
&\quad\le \Re\left(\frac{A^{-1}+B^{-1}}{2}\right) + Mm\left(\frac{\Re(A^{-1})^{-1}+\Re(B^{-1})^{-1}}{2}\right)\\
&\quad\le (M+m)I_n.
\end{aligned}$$

Inequality (27) is equivalent to

$$\left\|\Phi\big(\Re(A\sharp B)\big)\Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\| \le \sec^4(\alpha)K(h).$$

By computation, we have

$$\begin{aligned}
&\left\|Mm\Phi\big(\Re(A\sharp B)\big)\Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\|\\
&\quad\le \frac{1}{4}\left\|Mm\Phi\big(\Re(A\sharp B)\big) + \Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\|^2 \quad \text{(by Lemma 2.6)}\\
&\quad\le \frac{1}{4}\left\|Mm\Phi\big(\Re(A\sharp B)\big) + \Phi\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)^{-1}\right)\right\|^2 \quad \text{(by Lemma 2.7)}\\
&\quad\le \frac{1}{4}\left\|Mm\Phi\big(\Re(A\sharp B)\big) + \sec^2(\alpha)\Phi\left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)\right\|^2 \quad \text{(by Lemma 2.4)}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{4}\left\| \sec^2(\alpha) Mm\Phi\left(\Re\left(\frac{A+B}{2}\right)\right) + \sec^2(\alpha)\Phi\left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)\right\|^2 \quad (\text{by } (6))\\
&= \frac{1}{4}\left\| \sec^2(\alpha)\Phi\left(Mm\Re\left(\frac{A+B}{2}\right) + \Re\left(\frac{A^{-1}+B^{-1}}{2}\right)\right)\right\|^2\\
&\le \frac{1}{4}\sec^4(\alpha)(M+m)^2.
\end{aligned}
$$

That is,

$$
\left\| \Phi\left(\Re(A\sharp B)\right)\Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\| \le \sec^4(\alpha)K(h).
$$

This completes the proof. □

Obviously, the optimal factor in Theorem 2.18 should be $(\sec^2(\alpha)K(h))^2$. We note that it is affirmative under the condition $mI_n \le \Re(A^{-1}) \le \Re(A)^{-1} \le MI_n$ and $mI_n \le \Re(B^{-1}) \le \Re(B)^{-1} \le MI_n$ by presenting the following theorem.

Theorem 2.19 *If* $A, B \in \mathbb{M}_n$ *with* $W(A), W(B) \subset S_\alpha$ *and* $0 < mI_n \le \Re(A)^{-1}, \Re(B)^{-1} \le MI_n$, *then, for every positive unital linear map* Φ,

$$
\Phi^2\left(\Re(A\sharp B)\right) \le \left(\sec^2(\alpha)K(h)\right)^2\Phi^2\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right), \tag{30}
$$

where $K(h) = \frac{(h+1)^2}{4h}$ *and* $h = \frac{M}{m}$.

Proof From Theorem 2.10 we obtain

$$
\frac{\Re(A)^{-1} + \Re(B)^{-1}}{2} + Mm\Re\left(\frac{A+B}{2}\right) \le (M+m)I_n. \tag{31}
$$

Inequality (30) is equivalent to

$$
\left\| \Phi\left(\Re(A\sharp B)\right)\Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\| \le \sec^2(\alpha)K(h).
$$

By computation, we have

$$
\begin{aligned}
&\left\| \sec^2(\alpha)Mm\Phi\left(\Re(A\sharp B)\right)\Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\|\\
&\le \frac{1}{4}\left\| Mm\Phi\left(\Re(A\sharp B)\right) + \sec^2(\alpha)\Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\|^2 \quad (\text{by Lemma 2.6})\\
&\le \frac{1}{4}\left\| Mm\Phi\left(\Re(A\sharp B)\right) + \sec^2(\alpha)\Phi\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)^{-1}\right)\right\|^2 \quad (\text{by Lemma 2.7})\\
&\le \frac{1}{4}\left\| Mm\Phi\left(\Re(A\sharp B)\right) + \sec^2(\alpha)\Phi\left(\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right)\right\|^2 \quad (\text{by Lemma 2.2})\\
&\le \frac{1}{4}\left\| \sec^2(\alpha)Mm\Phi\left(\Re\left(\frac{A+B}{2}\right)\right) + \sec^2(\alpha)\Phi\left(\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right)\right\|^2 \quad (\text{by } (6))
\end{aligned}
$$

$$= \frac{1}{4}\left\|\sec^2(\alpha)\Phi\left(Mm\Re\left(\frac{A+B}{2}\right)+\frac{\Re(A)^{-1}+\Re(B)^{-1}}{2}\right)\right\|^2$$
$$\le \frac{1}{4}\sec^4(\alpha)(M+m)^2 \quad (\text{by (31)}).$$

That is,

$$\left\|\Phi\left(\Re(A\sharp B)\right)\Phi^{-1}\left(\Re\left(\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\right)\right\| \le \sec^2(\alpha)K(h).$$

This completes the proof. □

Funding
This research is supported by the National Natural Science Foundation of P.R. China (No. 11571247).

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
All authors contributed almost the same amount of work to manuscript. All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 7 January 2019 Accepted: 15 April 2019 Published online: 27 April 2019

References
1. Bhatia, R.: Positive Definite Matrices. Princeton University Press, Princeton (2007)
2. Bhatia, R., Kittaneh, F.: Notes on matrix arithmetic–geometric mean inequalities. Linear Algebra Appl. **308**, 203–211 (2000)
3. Drury, S.: Principal powers of matrices with positive definite real part. Linear Multilinear Algebra **63**, 296–301 (2015)
4. Drury, S., Lin, M.: Singular value inequalities for matrices with numerical ranges in a sector. Oper. Matrices **8**, 1143–1148 (2014)
5. Fu, X., Liu, Y., Liu, S.: Extension of determinantal inequalities of positive definite matrices. J. Math. Inequal. **11**, 355–358 (2017)
6. Lin, M.: Squaring a reverse AM-GM inequality. Stud. Math. **215**, 187–194 (2013)
7. Lin, M.: On an operator Kantorovich inequality for positive linear maps. J. Math. Anal. Appl. **402**, 127–132 (2013)
8. Lin, M.: Extension of a result of Haynsworth and Hartfiel. Arch. Math. **104**, 93–100 (2015)
9. Lin, M.: Some inequalities for sector matrices. Oper. Matrices **10**, 915–921 (2016)
10. Lin, M., Sun, F.: A property of the geometric mean of accretive operators. Linear Multilinear Algebra **65**, 433–437 (2017)
11. Liu, J., Wang, Q.: More inequalities for sector matrices. Bull. Iran. Math. Soc. **44**, 1059–1066 (2018)
12. Marshall, A.W., Olkin, I.: Matrix versions of Cauchy and Kantorovich inequalities. Aequ. Math. **40**, 89–93 (1990)
13. Tominaga, M.: Specht's ratio in the Young inequality. Sci. Math. Jpn. **55**, 583–588 (2002)
14. Yang, C., Lu, F.: Some generalizations of inequalities for sector matrices. J. Inequal. Appl. **2018**, 183 (2018)
15. Zhang, P.: More operator inequalities for positive linear maps. Banach J. Math. Anal. **9**, 166–172 (2015)