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Asymptotic tail probability of weighted infinite sum of conditionally dependent and consistently varying tailed random variables

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Abstract
This paper investigates the asymptotic behavior of the tail probability of a weighted infinite sum of random variables with consistently varying tails under two conditional dependence structures. The obtained results extend and improve the existing results of Bae and Ko (J. Korean Stat. Soc. 46:321–327, 2017).

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1 Introduction

Assume that $\{X_i, i \geq 1\}$ is a sequence of random variables (r.v.s) with their respective distributions F_i, $i \geq 1$, supported on $D = [0, \infty)$ or $(-\infty, \infty)$, and that $\{\psi_i, i \geq 1\}$ is a sequence of real numbers, which represent the weights of $\{X_i, i \geq 1\}$. Denote the weighted infinite sum by $\sum_{i=1}^{\infty} \psi_i X_i$, the asymptotic tail behavior of which is the main objective of our paper.

In this paper, we consider the heavy-tailed distribution classes. Firstly, we introduce some notions and notations. All limit relationships henceforth hold as $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$, $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$. For a proper distribution V on $(-\infty, \infty)$, we denote its tail by $\overline{V}(x) = 1 - V(x)$, and its upper and lower Matuszewska indices, respectively, by

$$J_V^+ = \inf\left\{-\frac{\log \overline{V}_*(y)}{\log y} : y > 1\right\} \quad \text{and} \quad J_V^- = \sup\left\{-\frac{\log \overline{V}^*(y)}{\log y} : y > 1\right\},$$

where $\overline{V}_*(y) = \liminf \overline{V}(xy)/\overline{V}(x)$ and $\overline{V}^*(y) = \limsup \overline{V}(xy)/\overline{V}(x)$ for $y > 0$.

An important class of heavy-tailed distributions is the subexponential class. Say that a distribution V on $[0, \infty)$ belongs to the subexponential class, denoted by $V \in \mathscr{S}$, if

$$\overline{V^{*2}}(x) \sim \overline{V}(x),$$

where V^{*2} is the 2-fold convolution of V. Note that if $V \in \mathscr{S}$ then V is long-tailed, denoted by $V \in \mathscr{L}$, in the sense that

$$\overline{V}(x+y) \sim \overline{V}(x), \quad \text{for any } y > 0.$$

Besides, if $V \in \mathscr{L}$ then

$$\begin{aligned}\mathscr{H}(V) = \bigl\{&h : x \in [0,\infty), h(x) \uparrow \infty, h(x) = o(1)x \text{ and } \overline{V}(x+y) \sim \overline{V}(x) \text{ holds uniformly} \\ &\text{for all } |y| \le h(x)\bigr\} \\ \neq \varnothing&.\end{aligned}$$

Moreover, the class $\mathscr{S}$ covers the class $\mathscr{C}$ of distributions with consistently varying tails, characterized by

$$\lim_{y\downarrow 1} \overline{V}_*(y) = 1, \quad \text{or equivalently,} \quad \lim_{y\uparrow 1} \overline{V}^*(y) = 1;$$

and also the class $\mathscr{C}$ covers the class $\mathscr{R}_{-\alpha}$, $0 < \alpha < \infty$, of distributions with regularly varying tails, characterized by

$$\overline{V}(xy) \sim y^{-\alpha}\overline{V}(x).$$

Another important class of heavy-tailed distributions is the dominant variation class, denoted by $\mathscr{D}$. Say that a distribution V belongs to the class $\mathscr{D}$, if

$$\overline{V}^*(y) < \infty, \quad \text{for any } y > 0.$$

More generally, when V is supported on $(-\infty,\infty)$, we say that V belongs to a distribution class if $V(x)1_{\{x\ge 0\}}$ belongs to the class. In conclusion,

$$\mathscr{R}_{-\alpha} \subset \mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{S} \subset \mathscr{L}.$$

For more details of heavy-tailed distributions and their applications, the reader is referred to Bingham et al. [2] and Embrechts et al. [5].

By inequality (2.1.9) in Theorem 2.18 and Proposition 2.2.1 of Bingham et al. [2], we know that $V \in \mathscr{D}$ if and only if $J_V^+ < \infty$; and if $V \in \mathscr{D}$, then, for all $0 < p_1 < J_V^-$ and $p_2 > J_V^+$, there exist $C_i > 0$ and $D_i > 0$, $i = 1,2$ such that

$$\frac{\overline{V}(xy)}{\overline{V}(x)} \le C_1 y^{-p_1}, \quad xy \ge x \ge D_1; \tag{1.1}$$

and

$$\frac{\overline{V}(x)}{\overline{V}(xy)} \le C_2 y^{p_2}, \quad xy \ge x \ge D_2. \tag{1.2}$$

We now give a proposition, which will play a key role in the proofs of the main results.

Proposition 1.1 *If $V \in \mathscr{C}$, then $J_V^- > 0$.*

Proof For any fixed $x > 0$, $\overline{V}(xy)/\overline{V}(x)$ is a monotonically decreasing function of y, which leads to $\overline{V}^*(y) \le \overline{V}^*(z)$ for $y > z > 0$, and then by $V \in \mathscr{C}$, $\overline{V}^*(y) \le \lim_{z\uparrow 1} \overline{V}^*(z) = 1$. Since $\limsup_{x\to\infty} \lim_{y\to\infty} \overline{V}(xy)/\overline{V}(x) = 0$, there exists a sufficiently large number $y_0 > 1$ such that $\overline{V}^*(y) < 1$ for all $y > y_0$, and further $\log \overline{V}^*(y)/\log y < 0$, $y > y_0 > 1$. Hence by the definition of J_V^-, it follows that $J_V^- \ge \sup\{-\log \overline{V}^*(y)/\log y : y > y_0\} > 0$. □

It is well known that an increasing number of researchers introduce many dependence structures to extensively study the asymptotic tail behaviors of sums of r.v.s in the literature of applied probability. See, for example, Ko and Tang [14], Geluk and Tang [12], Chen and Yuan [4], Foss and Richards [6], Gao and Wang [10], Yi et al. [21], Liu et al. [17], Gao and Liu [9], Chen et al. [3], Li [15], Wang et al. [20], Jiang et al. [13], Gao and Yang [11], Gao and Jin [8], Liu et al. [16, 18], Bae and Ko [1], Gao et al. [7], among which Ko and Tang [14] proposed a conditional dependence structure as follows.

Assumption A For $n \ge 2$ and $D = [0, \infty)$, there exist some large constants $x_0 = x_0(n) > 0$ and $C = C(n) > 0$ such that, for every $2 \le j \le n$,

$$\limsup \sup_{x_0 \le t \le x - x_0} \frac{P(X_1 + \cdots + X_{j-1} > x - t \mid X_j = t)}{P(X_1 + \cdots + X_{j-1} > x - t)} \le C.$$

In this paper, we extend the support of corresponding distribution in Assumption A from $[0, \infty)$ to $(-\infty, \infty)$, and we denote by Assumption A* the modified dependence structure.

Besides, Geluk and Tang [12] introduced another conditional dependence structure.

Assumption B For $n \ge 2$ and $D = (-\infty, \infty)$, there exist some large constants $x_0 = x_0(n) > 0$ and $C = C(n) > 0$ such that the inequality

$$P\bigl(|X_i| > x_i \mid X_j = x_j \text{ with } j \in J\bigr) \le C\overline{F_i}(x_i)$$

holds for all $1 \le i \le n$, $J := \{j : 1 \le j \le n\} \setminus \{i\}$, $x_i > x_0$, and $x_j > x_0$, $j \in J$.

Obviously, the relation in Assumption B is equivalent to the conjunction of the relations

$$P(X_i > x_i \mid X_j = x_j \text{ with } j \in J) \le C\overline{F_i}(x_i)$$

and

$$P(X_i < -x_i \mid X_j = x_j \text{ with } j \in J) \le C\overline{F_i}(x_i). \tag{1.3}$$

In this paper, for Assumption B, relation (1.3) is replaced by the following relation:

$$P(X_i < -x_i \mid X_j = x_j \text{ with } j \in J) \le CF_i(-x_i)$$

to cover all independent r.v.s In fact, when $\{X_i, 1 \le i \le n\}$ is a sequence of mutually independent r.v.s such that $\lim_{x_i\to\infty} \overline{F_i}(x_i)/F_i(-x_i) = 0$ for some $1 \le i \le n$, relation (1.3) is not satisfied, and then neither is Assumption B. Hence, the extended conditional dependence structure from Assumption B is labeled as Assumption B*. Note that these extended conditional dependence structures denoted by Assumptions A* and B* were firstly considered by Jiang et al. [13].

This paper will mainly focus on the asymptotic behavior of the tail probability of a weighted infinite sum of heavy-tailed r.v.s under the above two extended conditional dependence structures. In the rest of this paper, we will state our main results in Sect. 2, and prove them in Sect. 3.

2 Main results

In this section we firstly review the related results, and then present the main result of this paper. For the case when r.v.s X_i, $1 \le i \le n$, satisfy Assumption A, Bae and Ko [1] obtained the following theorem on a weighted infinite sum.

Theorem 1.A *Let $\{X_i, i \ge 1\}$ be a sequence of nonnegative r.v.s with common distribution $F \in \mathscr{R}_{-\alpha}$, and for each n, X_i, $1 \le i \le n$, satisfy Assumption A. If $\sum_{i=1}^{\infty} |\psi_i|^p < \infty$ for some $0 < p < \min\{\alpha, 1\}$, then*

$$P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) \sim \sum_{i \in \mathbb{I}_+} \overline{F}\left(\psi_i^{-1} x\right) \sim \overline{F}(x) \sum_{i \in \mathbb{I}_+} \psi_i^{\alpha},$$

where $\mathbb{I}_+$ denotes the set $\{i \mid \psi_i > 0\}$.

For the case when r.v.s X_i, $1 \le i \le n$, satisfy Assumption B, Geluk and Tang [12] presented a theorem as below.

Theorem 1.B *Assume that X_i, $1 \le i \le n$, are real-valued r.v.s with distributions F_i, $1 \le i \le n$. If $F_i \in \mathscr{S}$ for all $1 \le i \le n$ and $F_i * F_j \in \mathscr{S}$ for all $1 \le i < j \le n$, and Assumption B holds. Then, for all $n \ge 1$,*

$$P\left(\sum_{i=1}^{n} X_i > x\right) \sim \sum_{i=1}^{n} \overline{F}_i(x). \tag{2.1}$$

For the case when r.v.s X_i, $1 \le i \le n$, satisfy Assumption A* or B*, Jiang et al. [13] gave the following two results on sums of these r.v.s.

Theorem 1.C *Assume that X_i, $1 \le i \le n$, satisfy Assumption A*, and $F_i \in \mathscr{L}$ for all $1 \le i \le n$ and $F_i * F_j \in \mathscr{S}$ for all $1 \le i < j \le n$. Furthermore, when these r.v.s do not satisfy Assumption B or B*, there exists a function $h \in \bigcap_{i=1}^{n} \mathscr{H}(F_i)$ such that, for all $1 \le i \le n$,*

$$F_i\left(-h(x)\right) = o\left(\sum_{i=1}^{n} \overline{F}_i(x)\right).$$

Then, for all $n \ge 1$, Eq. (2.1) holds.

Theorem 1.D *Assume that X_i, $1 \le i \le n$, satisfy Assumption B*, and $F_i \in \mathscr{L}$ for all $1 \le i \le n$ and $F_i * F_j \in \mathscr{S}$ for all $1 \le i < j \le n$. Then, for all $n \ge 1$, Eq. (2.1) holds.*

Inspired by the above results, in this paper we further consider the asymptotic tail behavior of weighted infinite sum of consistently varying tailed r.v.s under conditional dependence structure satisfying Assumption A* or B*. The main results of this paper are given below.

Theorem 2.1 *Let $\{X_i, i \geq 1\}$ be a sequence of real-valued r.v.s. with distributions $F_i \in \mathscr{L}$, $i \geq 1$, and all weights $\{\psi_i, i \geq 1\}$ be real numbers. Assume that there exists a distribution $F \in \mathscr{C}$ such that*

$$\limsup_{i\geq 1} \frac{F_i(-x)}{\overline{F}(x)} = 0 \tag{2.2}$$

and

$$0 < S := \liminf \inf_{i\geq 1} \frac{\overline{F_i}(x)}{\overline{F}(x)} \leq \limsup \sup_{i\geq 1} \frac{\overline{F_i}(x)}{\overline{F}(x)} =: M < \infty, \tag{2.3}$$

and that $\sum_{i=1}^{\infty} |\psi_i|^p < \infty$ for some $0 < p < \min\{J_F^-, J_F^-/J_F^+\}$, then the relation

$$S \sum_{i\in\mathbb{I}_+} \overline{F}(\psi_i^{-1}x) \lesssim P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) \lesssim M \sum_{i\in\mathbb{I}_+} \overline{F}(\psi_i^{-1}x) \tag{2.4}$$

holds, if $\{X_i, i \geq 1\}$ is a sequence of r.v.s satisfy Assumption A* or B*, where $\mathbb{I}_+$ is the set given in Theorem 1.A.

Corollary 2.1 *Under the conditions of Theorem 2.1, if $F_i \in \mathscr{C}$, $i \geq 1$, then*

$$S \sum_{i\in\mathbb{I}_+} \overline{F}(\psi_i^{-1}x) \lesssim P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) \sim \sum_{i\in\mathbb{I}_+} \overline{F}_i(\psi_i^{-1}x) \lesssim M \sum_{i\in\mathbb{I}_+} \overline{F}(\psi_i^{-1}x),$$

and furthermore if $F_i \sim F$, $i \geq 1$, then

$$P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) \sim \sum_{i\in\mathbb{I}_+} \overline{F}(\psi_i^{-1}x).$$

If $F_i \sim F \in \mathscr{R}_{-\alpha}$, $i \geq 1$, then

$$P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) \sim \sum_{i\in\mathbb{I}_+} \overline{F}(\psi_i^{-1}x) \sim \overline{F}(x) \sum_{i\in\mathbb{I}_+} \psi_i^{\alpha}.$$

3 Lemmas

In order to prove Theorem 2.1 and Corollary 2.1, we now give two lemmas which are concerned with the case that weights $\{\psi_i, i \geq 1\}$ be positive.

Lemma 3.1 *Let $\{X_i, i \geq 1\}$ be a sequence of real-valued r.v.s with their respective distributions $F_i \in \mathscr{L}$, $i \geq 1$, and their weights $\{\psi_i, i \geq 1\}$ be positive. Assume that there exists a distribution $F \in \mathscr{C}$ such that (2.2) and (2.3) hold, and that $\sum_{i=1}^{\infty} \psi_i^p < \infty$ for some*

$0 < p < \min\{J_F^-, J_F^-/J_F^+\}$, *then the relation*

$$S\sum_{i=1}^{\infty}\overline{F}\bigl(\psi_i^{-1}x\bigr) \lesssim P\left(\sum_{i=1}^{\infty}\psi_i X_i > x\right) \lesssim M\sum_{i=1}^{\infty}\overline{F}\bigl(\psi_i^{-1}x\bigr) \tag{3.1}$$

holds, if $\{X_i, i \geq 1\}$ *is a sequence of r.v.s satisfy Assumption* A* *or* B*.

Proof Without loss of generality, we assume that $0 < \psi_i \leq 1$, $i \geq 1$. It is because there can be only a finite number of terms with $\psi_i > 1$ by the assumption and, if that is the case, we can divide each weight with the maximum of such ψ_is.

Take $0 < p < \min\{J_F^-, J_F^-/J_F^+\}$ such that $\sum_{i=1}^{\infty}\psi_i^p < \infty$. Then, for any $0 < \varepsilon < 1$, there exists a large positive integer n_0 such that

$$\sum_{i=n_0+1}^{\infty}\psi_i^p < \varepsilon. \tag{3.2}$$

For the above integer n_0, by $F \in \mathscr{C} \subset \mathscr{D}$, (1.1) and (3.2), there exist positive constants C_3 and D_3 such that, for all large $x \geq D_3$ and the above p,

$$\sum_{i=n_0+1}^{\infty}\overline{F}\bigl(\psi_i^{-1}x\bigr) \leq C_3\overline{F}(x)\sum_{i=n_0+1}^{\infty}\psi_i^p \leq C_3\varepsilon\overline{F}(x). \tag{3.3}$$

Firstly, to prove the upper bound of Eq. (3.1), we follow the approach used in the proof of Lemma 4.24 in Resnick [19] or Theorem 2 in Bae and Ko [1]. For any $0 < \delta < 1$ and integer n_0 in (3.2), we have

$$\begin{aligned} P\left(\sum_{i=1}^{\infty}\psi_i X_i > x\right) &\leq P\left(\sum_{i=1}^{n_0}\psi_i X_i^+ > (1-\delta)x\right) + P\left(\sum_{i=n_0+1}^{\infty}\psi_i X_i^+ > \delta x\right) \\ &=: I_1(x) + I_2(x), \end{aligned} \tag{3.4}$$

where $X_i^+ = \max\{X_i, 0\}$, $i \geq 1$. For convenience's sake, we remark that $F_i \in \mathscr{L} \cap \mathscr{D}$, $i \geq 1$, can imply $F_i \in \mathscr{S}$, $1 \leq i \leq n$, and $F_i * F_j \in \mathscr{S}$ for all $1 \leq i < j \leq n$; see Jiang et al. [13]. Therefore, the distributions F_i, $i \geq 1$, in Theorem 2.1 and Lemma 3.1, can also satisfy the conditions in Theorem 1.C. For $I_1(x)$, by Theorem 1.C or 1.D, and (2.3), it follows that

$$\begin{aligned} I_1(x) &\sim \sum_{i=1}^{n_0}P\bigl(\psi_i X_i^+ > (1-\delta)x\bigr) \\ &\lesssim M\sum_{i=1}^{n_0}\overline{F}\bigl(\psi_i^{-1}(1-\delta)x\bigr) \\ &\leq M\sup_{0<\psi_i\leq 1}\frac{\overline{F}(\psi_i^{-1}(1-\delta)x)}{\overline{F}(\psi_i^{-1}x)}\sum_{i=1}^{n_0}\overline{F}\bigl(\psi_i^{-1}x\bigr). \end{aligned} \tag{3.5}$$

By $F \in \mathscr{C}$, we get

$$\lim_{\delta\downarrow 0}\limsup_{x\to\infty}\sup_{0<\psi_i\leq 1}\frac{\overline{F}(\psi_i^{-1}(1-\delta)x)}{\overline{F}(\psi_i^{-1}x)} = \lim_{\delta\downarrow 0}\limsup_{x\to\infty}\frac{\overline{F}((1-\delta)x)}{\overline{F}(x)} = 1. \tag{3.6}$$

Hence, we substitute (3.6) into (3.5) to obtain

$$I_1(x) \lesssim M \sum_{i=1}^{n_0} \overline{F}(\psi_i^{-1} x). \tag{3.7}$$

For $I_2(x)$, when $0 < J_F^+ < 1$, we have

$$\begin{aligned} I_2(x) &\le P\left(\bigcup_{i=n_0+1}^{\infty} \{\psi_i X_i^+ > \delta x\} \right) + P\left(\sum_{i=n_0+1}^{\infty} \psi_i X_i^+ \mathbf{1}_{\{\psi_i X_i^+ \le \delta x\}} > \delta x \right) \\ &=: I_{21}(x) + I_{22}(x). \end{aligned} \tag{3.8}$$

By (1.2), (2.3), (3.3) and $F \in \mathscr{C} \subset \mathscr{D}$, for any $p_2 > J_F^+$, there exist some large positive constants C_4 and D_4 such that, for all $x \ge \max\{D_3, D_4\}$,

$$\begin{aligned} I_{21}(x) &\le \sum_{i=n_0+1}^{\infty} P(\psi_i X_i^+ > \delta x) \\ &\lesssim M \sum_{i=n_0+1}^{\infty} \overline{F}(\psi_i^{-1} \delta x) \\ &\le C_3 C_4 M \delta^{-p_2} \varepsilon \overline{F}(x). \end{aligned} \tag{3.9}$$

By Markov's inequality and the monotone convergence theorem, we see that

$$\begin{aligned} I_{22}(x) &\le (\delta x)^{-1} E\left(\sum_{i=n_0+1}^{\infty} \psi_i X_i^+ \mathbf{1}_{\{\psi_i X_i^+ \le \delta x\}} \right) \\ &= (\delta x)^{-1} \sum_{i=n_0+1}^{\infty} \psi_i E\left(X_i^+ \mathbf{1}_{\{X_i^+ \le \psi_i^{-1} \delta x\}}\right). \end{aligned} \tag{3.10}$$

By $F \in \mathscr{C} \subset \mathscr{D}$, (1.2) and (2.3), for any $J_F^+ < p_2 < 1$, there exist some large positive constants C_5 and D_5 such that, for all $x \ge D_5$,

$$\begin{aligned} E\left(X_i^+ \mathbf{1}_{\{X_i^+ \le \psi_i^{-1} \delta x\}}\right) &= -\int_0^{\psi_i^{-1} \delta x} u \, d\overline{F_i}(u) \\ &= -\psi_i^{-1} \delta x \overline{F_i}(\psi_i^{-1} \delta x) + \int_0^{\psi_i^{-1} \delta x} \overline{F_i}(u) \, du \\ &\le \psi_i^{-1} \delta x \int_0^1 \overline{F_i}(t \psi_i^{-1} \delta x) \, dt \\ &\lesssim M \psi_i^{-1} \delta x \overline{F}(\psi_i^{-1} \delta x) \int_0^1 \frac{\overline{F}(t \psi_i^{-1} \delta x)}{\overline{F}(\psi_i^{-1} \delta x)} \, dt \\ &\le \frac{C_5 M}{1 - p_2} \psi_i^{-1} \delta x \overline{F}(\psi_i^{-1} \delta x). \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10) and using the last step of (3.9) can lead to

$$I_{22}(x) \lesssim \frac{C_3 C_4 C_5 M}{1 - p_2} \delta^{-p_2} \varepsilon \overline{F}(x). \tag{3.12}$$

Therefore by (3.4), (3.7)–(3.9), (3.12) and the arbitrariness of ε, we derive that

$$P\left(\sum_{i=1}^{\infty}\psi_i X_i > x\right) \lesssim M\sum_{i=1}^{n_0}\overline{F}\left(\psi_i^{-1}x\right) \le M\sum_{i=1}^{\infty}\overline{F}\left(\psi_i^{-1}x\right). \tag{3.13}$$

For the case when $J_F^+ \ge 1$, we choose some constant $\beta \in (J_F^+, J_F^- p^{-1})$ such that $p < \beta^{-1}J_F^- \le \beta^{-1}J_F^+ < 1$. Set $\psi = \sum_{i=n_0+1}^{\infty}\psi_i$, which is assumed to be less than 1 without loss of generality. Then by Jensen's inequality, it follows that

$$\begin{aligned}
I_2(x) &= P\left(\psi^{\beta}\left(\sum_{i=n_0+1}^{\infty}\frac{\psi_i}{\psi}X_i^+\right)^{\beta} > \delta^{\beta}x^{\beta}\right)\\
&\le P\left(\sum_{i=n_0+1}^{\infty}\psi_i X_i^{+\beta} > \psi^{1-\beta}\delta^{\beta}x^{\beta}\right)\\
&\le P\left(\bigcup_{i=n_0+1}^{\infty}\left\{\psi_i X_i^{+\beta} > \psi^{1-\beta}\delta^{\beta}x^{\beta}\right\}\right)\\
&\quad + P\left(\sum_{i=n_0+1}^{\infty}\psi_i X_i^{+\beta}\mathbf{1}_{\{\psi_i X_i^{+\beta}\le\psi^{1-\beta}\delta^{\beta}x^{\beta}\}} > \psi^{1-\beta}\delta^{\beta}x^{\beta}\right)\\
&=: I_{21}'(x) + I_{22}'(x).
\end{aligned} \tag{3.14}$$

For $I_{21}'(x)$, by using $F \in \mathscr{C} \subset \mathscr{D}$ and (1.1), and arguing as (3.9), for any $p_1 \in (\beta p, J_F^-)$ and $p_2 > J_F^+$, there exist some large positive constants C_6 and D_6 such that, for all $x \ge \max\{D_3, D_4, D_6\}$,

$$\begin{aligned}
I_{21}'(x) &\le \sum_{i=n_0+1}^{\infty} P\left(X_i^+ > \psi_i^{-\frac{1}{\beta}}\psi^{\frac{1-\beta}{\beta}}\delta x\right)\\
&\lesssim M\sum_{i=n_0+1}^{\infty}\overline{F}\left(\psi_i^{-\frac{1}{\beta}}\psi^{\frac{1-\beta}{\beta}}\delta x\right)\\
&\le C_6 M\psi^{\frac{p_1(\beta-1)}{\beta}}\sum_{i=n_0+1}^{\infty}\overline{F}\left(\psi_i^{-\frac{1}{\beta}}\delta x\right)\\
&\le C_3C_4C_6M\psi^{\frac{p_1(\beta-1)}{\beta}}\delta^{-p_2}\varepsilon\overline{F}(x).
\end{aligned} \tag{3.15}$$

For $I_{22}'(x)$, by going along the same lines of the derivation of $I_{22}(x)$, we conclude that, for any $J_F^+ < p_2 < \beta$, there exist some large positive constants C_7 and D_7 such that, for all $x \ge \max\{D_3, D_4, D_6, D_7\}$,

$$\begin{aligned}
I_{22}'(x) &\lesssim \frac{C_7M\beta}{\beta - p_2}\sum_{i=n_0+1}^{\infty}\overline{F}\left(\psi_i^{-\beta}\psi^{\frac{1-\beta}{\beta}}\delta x\right)\\
&\le \frac{C_3C_4C_6C_7M\beta}{\beta - p_2}\psi^{\frac{p_1(\beta-1)}{\beta}}\delta^{-p_2}\varepsilon\overline{F}(x),
\end{aligned} \tag{3.16}$$

where the last step is obtained similarly to (3.15). Then by (3.4), (3.7), (3.14)–(3.16) and the arbitrariness of ε, we prove that Eq. (3.13) holds.

Secondly, we deal with the lower bound of Eq. (3.1). Let n_0 and p be fixed as those in (3.2). For any $0 < \delta < 1$, we have

$$\begin{aligned} P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) &= P\left(\sum_{i=1}^{\infty} \psi_i X_i^+ - \sum_{i=1}^{\infty} \psi_i X_i^- > x\right) \\ &\geq P\left(\sum_{i=1}^{\infty} \psi_i X_i^+ > (1+\delta)x, \sum_{i=1}^{\infty} \psi_i X_i^- \leq \delta x\right) \\ &\geq P\left(\sum_{i=1}^{\infty} \psi_i X_i^+ > (1+\delta)x\right) - P\left(\sum_{i=1}^{\infty} \psi_i X_i^- > \delta x\right) \\ &=: I_3(x) - I_4(x), \end{aligned} \tag{3.17}$$

where $X_i^- = -\min\{X_i, 0\}$, $i \geq 1$. For $I_3(x)$, by (2.3), Theorem 1.C or 1.D, we have

$$\begin{aligned} I_3(x) &\geq P\left(\sum_{i=1}^{n_0} \psi_i X_i^+ > (1+\delta)x\right) \\ &\sim \sum_{i=1}^{n_0} P\bigl(\psi_i X_i^+ > (1+\delta)x\bigr) \\ &\gtrsim S \sum_{i=1}^{n_0} \overline{F}\bigl(\psi_i^{-1}(1+\delta)x\bigr) \\ &\geq S \inf_{0<\psi_i\leq 1} \frac{\overline{F}(\psi_i^{-1}(1+\delta)x)}{\overline{F}(\psi_i^{-1}x)} \sum_{i=1}^{n_0} \overline{F}\bigl(\psi_i^{-1}x\bigr). \end{aligned} \tag{3.18}$$

By $F \in \mathscr{C}$, it follows that

$$\lim_{\delta\downarrow 0} \liminf_{x\to\infty} \inf_{0<\psi_i\leq 1} \frac{\overline{F}(\psi_i^{-1}(1+\delta)x)}{\overline{F}(\psi_i^{-1}x)} = \lim_{\delta\downarrow 0} \liminf_{x\to\infty} \frac{\overline{F}((1+\delta)x)}{\overline{F}(x)} = 1. \tag{3.19}$$

By (3.3), (3.18) and (3.19), we obtain

$$\begin{aligned} I_3(x) &\gtrsim S \sum_{i=1}^{n_0} \overline{F}\bigl(\psi_i^{-1}x\bigr) \\ &= S \sum_{i=1}^{\infty} \overline{F}\bigl(\psi_i^{-1}x\bigr) - S \sum_{i=n_0+1}^{\infty} \overline{F}\bigl(\psi_i^{-1}x\bigr) \\ &\geq S \sum_{i=1}^{\infty} \overline{F}\bigl(\psi_i^{-1}x\bigr) - C_3 S \varepsilon \overline{F}(x), \end{aligned}$$

which, along with the arbitrariness of $0 < \varepsilon < 1$, implies that

$$I_3(x) \gtrsim S \sum_{i=1}^{\infty} \overline{F}\bigl(\psi_i^{-1}x\bigr). \tag{3.20}$$

By (2.2), for any $0 < \varepsilon < 1$, there exists a large positive constant D' such that, for all $x \geq D'$,

$$\sup_{i\geq 1} \frac{F_i(-x)}{\overline{F}(x)} < \varepsilon. \tag{3.21}$$

For $I_4(x)$, we only consider the case $0 < J_F^+ < 1$. In fact, the case of $J_F^+ \geq 1$ follows from similar derivations to (3.14)–(3.16) with slight modifications. Clearly,

$$\begin{aligned} I_4(x) &\leq \sum_{i=1}^{\infty} P\big(\psi_i X_i^- > \delta x\big) + P\left(\sum_{i=1}^{\infty} \psi_i X_i^- \mathbf{1}_{\{\psi_i X_i^- \leq \delta x\}} > \delta x\right) \\ &= \sum_{i=1}^{\infty} P(\psi_i X_i < -\delta x) + P\left(\sum_{i=1}^{\infty} \psi_i X_i^- \mathbf{1}_{\{\psi_i X_i^- \leq \delta x\}} > \delta x\right) \\ &=: I_{41}(x) + I_{42}(x). \end{aligned} \tag{3.22}$$

For $I_{41}(x)$, by (3.21) and the last step of (3.9), for all $x \geq \max\{D', D_4\}$,

$$I_{41}(x) < \varepsilon \sum_{i=1}^{\infty} \overline{F}\big(\psi_i^{-1}\delta x\big) \leq C_4 \delta^{-p_2} \varepsilon \sum_{i=1}^{\infty} \overline{F}\big(\psi_i^{-1} x\big). \tag{3.23}$$

For $I_{42}(x)$, similarly to (3.10), we have

$$I_{42}(x) \leq (\delta x)^{-1} \sum_{i=1}^{\infty} \psi_i E\big(X_i^- \mathbf{1}_{\{X_i^- \leq \psi_i^{-1}\delta x\}}\big). \tag{3.24}$$

Similarly to (3.11), by $F \in \mathscr{C} \subset \mathscr{D}$, (1.2), (2.2) and (2.3), for any $J_F^+ < p_2 < 1$, there exist some large positive constants C_8 and D_8 such that, for all $x \geq \max\{D', D_8\}$,

$$\begin{aligned} &E\big(X_i^- \mathbf{1}_{\{X_i^- \leq \psi_i^{-1}\delta x\}}\big) \\ &\quad = -\psi_i^{-1}\delta x P\big(X_i^- > \psi_i^{-1}\delta x\big) + \psi_i^{-1}\delta x \int_0^1 P\big(X_i^- > t\psi_i^{-1}\delta x\big)\, dt \\ &\quad \leq \psi_i^{-1}\delta x \overline{F}\big(\psi_i^{-1}\delta x\big) \int_0^1 \frac{F_i(-t\psi_i^{-1}\delta x)}{\overline{F}(t\psi_i^{-1}\delta x)} \frac{\overline{F}(t\psi_i^{-1}\delta x)}{\overline{F}(\psi_i^{-1}\delta x)}\, dt \\ &\quad < \varepsilon \psi_i^{-1}\delta x \overline{F}\big(\psi_i^{-1}\delta x\big) \int_0^1 \frac{\overline{F}(t\psi_i^{-1}\delta x)}{\overline{F}(\psi_i^{-1}\delta x)}\, dt \\ &\quad \leq \frac{C_8 \varepsilon}{1-p_2} \psi_i^{-1}\delta x \overline{F}\big(\psi_i^{-1}\delta x\big). \end{aligned} \tag{3.25}$$

Then, by substituting (3.25) into (3.24) and arguing similarly to (3.9), we prove that, for all $x \geq \max\{D', D_4, D_8\}$,

$$I_{42}(x) < \frac{C_8 \varepsilon}{1-p_2} \sum_{i=1}^{\infty} \overline{F}\big(\psi_i^{-1}\delta x\big) \leq \frac{C_4 C_8}{1-p_2} \delta^{-p_2} \varepsilon \sum_{i=1}^{\infty} \overline{F}\big(\psi_i^{-1} x\big); \tag{3.26}$$

and further we substitute (3.23) and (3.26) into (3.22) to obtain, for all $x \geq \max\{D', D_4, D_8\}$,

$$I_4(x) < \left(\frac{C_8}{1-p_2} + 1\right) C_4 \delta^{-p_2} \varepsilon \sum_{i=1}^{\infty} \overline{F}\big(\psi_i^{-1} x\big), \tag{3.27}$$

which, along with (3.17), (3.20) and the arbitrariness of $0 < \varepsilon < 1$, can show the lower bound of Eq. (3.1). □

Lemma 3.2 *Under the conditions of Lemma 3.1, if $F_i \in \mathscr{C}$, $i \geq 1$, then*

$$S\sum_{i=1}^{\infty}\overline{F}(\psi_i^{-1}x) \lesssim P\left(\sum_{i=1}^{\infty}\psi_i X_i > x\right) \sim \sum_{i=1}^{\infty}\overline{F}_i(\psi_i^{-1}x) \lesssim M\sum_{i=1}^{\infty}\overline{F}(\psi_i^{-1}x),$$

and further if $F_i \sim F$, $i \geq 1$, then

$$P\left(\sum_{i=1}^{\infty}\psi_i X_i > x\right) \sim \sum_{i=1}^{\infty}\overline{F}(\psi_i^{-1}x).$$

If $F_i \sim F \in \mathscr{R}_{-\alpha}$, $i \geq 1$, then

$$P\left(\sum_{i=1}^{\infty}\psi_i X_i > x\right) \sim \sum_{i=1}^{\infty}\overline{F}(\psi_i^{-1}x) \sim \overline{F}(x)\sum_{i=1}^{\infty}\psi_i^{\alpha}.$$

Proof By Lemma 3.1, it suffices to prove that

$$P\left(\sum_{i=1}^{\infty}\psi_i X_i > x\right) \sim \sum_{i=1}^{\infty}\overline{F_i}(\psi_i^{-1}x) \tag{3.28}$$

and, when $F \in \mathscr{R}_{-\alpha}$,

$$\sum_{i=1}^{\infty}\overline{F}(\psi_i^{-1}x) \sim \overline{F}(x)\sum_{i=1}^{\infty}\psi_i^{\alpha}. \tag{3.29}$$

Firstly, we prove (3.28). By the proof of Lemma 3.1, we only need to prove

$$I_1(x) \lesssim \sum_{i=1}^{\infty}\overline{F_i}(\psi_i^{-1}x) \tag{3.30}$$

and

$$I_3(x) \gtrsim \sum_{i=1}^{\infty}\overline{F_i}(\psi_i^{-1}x). \tag{3.31}$$

Since $F_i \in \mathscr{C}$, $i \geq 1$, we know that

$$\lim_{\delta\downarrow 0}\limsup_{x\to\infty}\frac{\overline{F_i}(\psi_i^{-1}(1-\delta)x)}{\overline{F_i}(\psi_i^{-1}x)} = \lim_{\delta\downarrow 0}\limsup_{x\to\infty}\frac{\overline{F_i}((1-\delta)x)}{\overline{F_i}(x)} = 1 \tag{3.32}$$

and

$$\lim_{\delta\downarrow 0}\liminf_{x\to\infty}\frac{\overline{F_i}(\psi_i^{-1}(1+\delta)x)}{\overline{F_i}(\psi_i^{-1}x)} = \lim_{\delta\downarrow 0}\liminf_{x\to\infty}\frac{\overline{F_i}((1+\delta)x)}{\overline{F_i}(x)} = 1. \tag{3.33}$$

By (3.32) and Theorem 1.C or 1.D, it follows that

$$I_1(x) \sim \sum_{i=1}^{n_0} \overline{F}_i\big(\psi_i^{-1}(1-\delta)x\big) \sim \sum_{i=1}^{n_0} \overline{F}_i\big(\psi_i^{-1}x\big) \leq \sum_{i=1}^{\infty} \overline{F}_i\big(\psi_i^{-1}x\big),$$

which leads to (3.30). By Theorem 1.C or 1.D, (2.3), (3.3) and (3.33), we have

$$\begin{aligned} I_3(x) &\gtrsim \sum_{i=1}^{n_0} \overline{F}_i\big(\psi_i^{-1}(1+\delta)x\big) \sim \sum_{i=1}^{n_0} \overline{F}_i\big(\psi_i^{-1}x\big) \\ &\gtrsim \sum_{i=1}^{\infty} \overline{F}_i\big(\psi_i^{-1}x\big) - M \sum_{i=n_0+1}^{\infty} \overline{F}\big(\psi_i^{-1}x\big) \\ &\geq \sum_{i=1}^{\infty} \overline{F}_i\big(\psi_i^{-1}x\big) - C_3 M\varepsilon \overline{F}(x), \end{aligned}$$

which, along with the arbitrariness of $0 < \varepsilon < 1$, implies that Eq. (3.31) holds.

Secondly, we prove (3.29). By $F \in \mathscr{R}_{-\alpha}$ and the control convergence theorem, we have

$$\sum_{i=1}^{\infty} \overline{F}\big(\psi_i^{-1}x\big) = \overline{F}(x) \sum_{i=1}^{\infty} \frac{\overline{F}(\psi_i^{-1}x)}{\overline{F}(x)} \sim \overline{F}(x) \sum_{i=1}^{\infty} \psi_i^{\alpha}. \qquad \square$$

4 Proof of main result

In this section, we will prove the main result of this paper.

Proof of Theorem 2.1 Without loss of generality, we may assume that $-1 \leq \psi_i \leq 1$. Firstly, we consider the upper bound of E (2.4). For any $0 < \delta < 1$, we have

$$\begin{aligned} P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) &= P\left(\sum_{i=\mathbb{I}_+} \psi_i X_i + \sum_{i=\mathbb{I}_-} \psi_i X_i > x\right) \\ &\leq P\left(\sum_{i\in\mathbb{I}_+} \psi_i X_i > (1-\delta)x\right) + P\left(\sum_{i\in\mathbb{I}_-} \psi_i X_i > \delta x\right) \\ &=: I_5(x) + I_6(x), \end{aligned} \tag{4.1}$$

where $\mathbb{I}_-$ denotes the set $\{i \mid \psi_i < 0\}$. For $I_5(x)$, by Lemma 3.1 and (3.6), we have

$$\begin{aligned} I_5(x) &\lesssim M \sum_{i\in\mathbb{I}_+} \overline{F}\big(\psi_i^{-1}(1-\delta)x\big) \\ &\leq M \sup_{0<\psi_i\leq 1} \frac{\overline{F}(\psi_i^{-1}(1-\delta)x)}{\overline{F}(\psi_i^{-1}x)} \sum_{i\in\mathbb{I}_+} \overline{F}\big(\psi_i^{-1}x\big) \\ &\sim M \sum_{i\in\mathbb{I}_+} \overline{F}\big(\psi_i^{-1}x\big). \end{aligned} \tag{4.2}$$

For $I_6(x)$, it follows from (3.27) that, for all $x \geq \max\{D', D_4, D_8\}$,

$$I_6(x) \leq P\left(\sum_{i\in\mathbb{I}_-} |\psi_i| X_i^- > \delta x\right) < \left(\frac{C_8}{1-p_2} + 1\right) C_4 \delta^{-p_2} \varepsilon \sum_{i=1}^{\infty} \overline{F}\left(|\psi_i|^{-1} x\right). \tag{4.3}$$

Thus, substituting (4.2) and (4.3) into (4.1) and considering the arbitrariness of $0 < \varepsilon < 1$, we show that

$$P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) \lesssim M \sum_{i\in\mathbb{I}_+} \overline{F}\left(\psi_i^{-1} x\right). \tag{4.4}$$

Secondly, we consider the lower bound of Eq. (2.4). By Lemma 3.1 and (3.19), we derive that

$$\begin{aligned}
P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) &= P\left(\sum_{i\in\mathbb{I}_+} \psi_i X_i + \sum_{i\in\mathbb{I}_-} \psi_i X_i > x\right) \\
&\geq P\left(\sum_{i\in\mathbb{I}_+} \psi_i X_i > (1+\delta)x, \sum_{i\in\mathbb{I}_-} \psi_i X_i \geq -\delta x\right) \\
&\sim P\left(\sum_{i\in\mathbb{I}_+} \psi_i X_i > (1+\delta)x\right) \\
&\sim \sum_{i\in\mathbb{I}_+} P\left(\psi_i X_i > (1+\delta)x\right) \\
&\gtrsim S \sum_{i\in\mathbb{I}_+} \overline{F}\left(\psi_i^{-1}(1+\delta)x\right) \\
&\geq S \inf_{0<\psi_i<1} \frac{\overline{F}(\psi_i^{-1}(1+\delta)x)}{\overline{F}(\psi_i^{-1}x)} \sum_{i\in\mathbb{I}_+} \overline{F}\left(\psi_i^{-1}x\right) \\
&\sim S \sum_{i\in\mathbb{I}_+} \overline{F}\left(\psi_i^{-1}x\right),
\end{aligned} \tag{4.5}$$

where in the third step we used the fact that the event $\{\omega : \sum_{i\in\mathbb{I}_-} \psi_i X_i \geq -\delta x\}$ increases to a certain event as x tends to infinity. Therefore, we combine (4.4) and (4.5) to conclude that Eq. (2.4) holds. □

Proof of Corollary 2.1 By (3.29) and the proof of Theorem 2.1, we only need to prove

$$I_5(x) \lesssim \sum_{i\in\mathbb{I}_+} \overline{F_i}\left(\psi_i^{-1} x\right) \tag{4.6}$$

and

$$P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) \gtrsim \sum_{i\in\mathbb{I}_+} \overline{F_i}\left(\psi_i^{-1} x\right). \tag{4.7}$$

Firstly, we consider (4.6). By Lemma 3.2, (3.3) and (3.32), we conclude that, for any $p_2 > J_F^+$, there exist some large positive constants C_9 and D_9 such that, for all $x \ge \max\{D_3, D_9\}$,

$$\begin{aligned} I_5(x) &\sim \sum_{i\in\mathbb{I}_+} \overline{F_i}\big(\psi_i^{-1}(1-\delta)x\big) \\ &\lesssim \sum_{i\in\mathbb{I}_+, i\le n_0} \overline{F_i}\big(\psi_i^{-1}(1-\delta)x\big) + M \sum_{i\in\mathbb{I}_+, i\ge n_0+1} \overline{F}\big(\psi_i^{-1}(1-\delta)x\big) \\ &\lesssim \sum_{i\in\mathbb{I}_+, i\le n_0} \overline{F_i}\big(\psi_i^{-1}x\big) + C_3C_9M(1-\delta)^{-p_2}\varepsilon\overline{F}(x), \end{aligned}$$

which, along with the arbitrariness of $0 < \varepsilon < 1$, proves (4.6).

Secondly, we consider (4.7). Similarly to (4.5), by Theorem 3.1, (3.3) and (3.33), we conclude that

$$\begin{aligned} P\left(\sum_{i=1}^{\infty} \psi_i X_i > x\right) &\gtrsim \sum_{i\in\mathbb{I}_+} \overline{F_i}\big(\psi_i^{-1}(1+\delta)x\big) \\ &\ge \sum_{i\in\mathbb{I}_+, i\le n_0} \overline{F_i}\big(\psi_i^{-1}(1+\delta)x\big) \\ &\sim \sum_{i\in\mathbb{I}_+, i\le n_0} \overline{F_i}\big(\psi_i^{-1}x\big) \\ &\gtrsim \sum_{i\in\mathbb{I}_+} \overline{F}\big(\psi_i^{-1}x\big) - M \sum_{i\in\mathbb{I}_+, i\ge n_0+1} \overline{F}\big(\psi_i^{-1}x\big) \\ &\ge \sum_{i\in\mathbb{I}_+} \overline{F}\big(\psi_i^{-1}x\big) - C_3M\varepsilon\overline{F}(x), \end{aligned}$$

which, along with the arbitrariness of $0 < \varepsilon < 1$, proves (4.7). □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The author QG found the main reference Bae and Ko [1] in the literature study, and proposed the ideas and methods of the main results in our paper. All authors implemented concretely the above ideas and methods, and accomplished this paper. All the authors read and approved the final manuscript.

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