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Growth of meromorphic solutions of linear difference equations without dominating coefficients

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Abstract

This paper is devoted to studying the growth of meromorphic solutions of difference equation

$$P_n(z)f(z+n) + P_{n-1}f(z+n-1) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = 0,$$

where the coefficients P_j ($j = 0, \dots, n$) are meromorphic functions. With some additional conditions on coefficients, we obtain precise estimates of the growth of meromorphic solutions of such an equation.

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1 Introduction and main results

In this article, we shall consider the linear higher order difference equation

$$P_n(z)f(z+n) + P_{n-1}(z)f(z+n-1) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = 0, \quad (1)$$

where $P_0(z), \dots, P_n(z)$ are meromorphic functions. Nevanlinna theory is an important and basic tool in our discussion; for references see [9, 12, 18]. For a meromorphic function $f(z)$, we denote by $T(r, f)$, $N(r, f)$ the characteristic function and the counting function of $f(z)$, respectively. In particular, we define the order and the lower order of a meromorphic function $f(z)$ by

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r},$$

while

$$\lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r} \quad \text{and} \quad \delta(a, f) = \liminf_{r \rightarrow \infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}$$

stand for the exponents of convergence of zero sequence of f and the deficiency of f at the point a , respectively. Let $\alpha(z)$ be a meromorphic function. We say that $\alpha(z)$ is a small function with respect to $f(z)$, if $T(r, \alpha(z)) = o(T(r, f))$, possible outside of a set E with finite linear measure.

In [5], Chiang and Feng considered the growth of transcendental entire solutions of linear higher order difference equations.

Theorem A *Let $P_j(z)$ ($j = 0, 1, \dots, n$) be polynomials. If there exists an integer l , $0 \leq l \leq n$, such that*

$$\deg(P_l) > \max_{0 \leq j \leq n, j \neq l} \{\deg(P_j)\}. \quad (2)$$

Then every meromorphic solution $f(z)$ of Eq. (1) satisfies $\sigma(f) \geq 1$.

The condition (2) shows the degree of P_l is larger than the other coefficients which means the polynomial P_l is dominating coefficient. Chen [3] weakens the condition (2), and obtained.

Theorem B *Let $P_j(z)$ ($j = 0, \dots, n$) be polynomials such that $P_n P_0 \neq 0$ and*

$$\deg\{P_n + \dots + P_0\} = \max\{\deg P_j : j = 0, \dots, n\} \geq 1. \quad (3)$$

If $f(z)$ is a meromorphic solution of Eq. (1), then $\sigma(f) \geq 1$, and $f(z)$ assumes every finite value $a \in \mathbb{C}$ except zero infinitely often and $\lambda(f - a) = \sigma(f)$.

The above theorems deal with the case that the coefficients of Eq. (1) are polynomials, and there exists a dominating coefficient. Then a natural question arises: can we estimate the growth of the Eq. (1) provided the coefficients are entire functions or meromorphic functions. To answer this question, Chiang and Feng [5] obtained the following.

Theorem C *Let $P_0(z), \dots, P_n(z)$ be entire functions such that there exists an integer l , $0 \leq l \leq n$, such that*

$$\sigma(P_l) > \max\{\sigma(P_j) : 0 \leq j \leq n, j \neq l\}. \quad (4)$$

If $f(z)$ is a nontrivial meromorphic solution of Eq. (1), then $\sigma(f) \geq \sigma(P_l) + 1$.

Clearly, since the order of P_l is larger than that of the others, P_l in Theorem C is a dominating coefficient. But if no such coefficient exists, Laine and Yang [13] also proved a similar conclusion in Theorem D below.

Definition 1.1 ([9]) The type of a meromorphic function $f(z)$ of order σ ($0 < \sigma < \infty$) is defined by

$$\tau(f) = \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^\sigma}.$$

Theorem D ([13]) *Let $P_0(z), \dots, P_n(z)$ be entire functions of finite order so that among those having the maximal order $\sigma := \max\{\sigma(P_j) : 0 \leq j \leq n\}$, exactly one has its type strictly greater than the others. Then, for any meromorphic solution of Eq. (1), we have $\sigma(f) \geq \sigma + 1$.*

The condition about type in Theorem D still means the growth of some coefficient is faster than the others, although there exist several coefficients have the same order. This implies there still exists dominating coefficient. For a linear difference equation with meromorphic coefficients, Chen [4] proved the following result.

Theorem E *Let $P_0(z), \dots, P_n(z)$ be meromorphic functions such that there exists an integer l , $0 \leq l \leq n$, such that $\sigma(P_l) > \max\{\sigma(P_j) : 0 \leq j \leq n, j \neq l\}$ and $\delta(\infty, P_l) > 0$. If $f(z)$ is a nontrivial meromorphic solution of Eq. (1), then $\sigma(f) \geq \sigma(P_l) + 1$.*

Laine and Yang [13] posed the following question, which it is natural to ask.

Question 1.1 Do all meromorphic solution f of Eq. (1) satisfy $\sigma(f) \geq 1 + \max\{\sigma(P_j) : 0 \leq j \leq n\}$, even if there is no dominating coefficient?

In [11], some examples are given to illustrate that the Laine–Yang conjecture is not true in whole. In this paper, we consider this question and shall give estimates of growth of solutions of Eq. (1), and there is no dominating coefficient.

Theorem 1.1 *Let $P_j(z)$ ($j = 0, \dots, n$) be meromorphic functions. If $P_1(z)$ has a finite deficient value a , and $\sigma(P_j) < \sigma(P_0) < \frac{1}{2}$, $j \neq 0, 1$, then every finite order meromorphic solution $f(z)$ of Eq. (1) satisfies $\sigma(f) \geq \mu(P_0) - \max\{\sigma(P_j), j \neq 0, 1\} + 1$.*

Edrei and Fuchs proved that the number of deficient values cannot be infinite when the zeros and poles of a meromorphic function distributed near some curves. We first give some definitions and results in order to relate their results with Eq. (1).

Let

$$\Omega(\theta_1, \theta_2, r) = \{z : \theta_1 < \arg z < \theta_2, |z| < r\},$$

$$\overline{\Omega}(\theta_1, \theta_2, r) = \{z : \theta_1 \leq \arg z \leq \theta_2, |z| < r\}.$$

Definition 1.2 ([6, 7]) Let $f(z)$ be a meromorphic function with finite order of growth ($0 < \sigma(f) < \infty$). A ray $\arg z = \theta$ starting from the origin is called a zero-pole accumulation ray of $f(z)$, if, for any given real number $\varepsilon > 0$, the following equality holds:

$$\lim_{r \rightarrow \infty} \frac{\log n\{\overline{\Omega}(\theta - \varepsilon, \theta + \varepsilon, r), f = 0\} + \log n\{\overline{\Omega}(\theta - \varepsilon, \theta + \varepsilon, r), f = \infty\}}{\log r} = \sigma(f).$$

Zhang [18] proved that if $f(z)$ is a meromorphic function with order $\sigma = \sigma(f)$, $0 < \sigma < \infty$ and $f(z)$ has q zero-pole accumulation rays and p deficient values other than 0 and ∞ , then $p \leq q$. We say $f(z) \in EF$ if $p = q \geq 1$, and the set of such $f(z)$ is called an Edrei–Fuchs set. This means that $f(z)$ is a positive finite order meromorphic function with p zero-pole accumulation rays and p nonzero finite deficient values.

Now we are in the position to give a result concerning an Edrei–Fuchs set.

Theorem 1.2 *Let $P_j(z)$ ($j = 0, \dots, n$) be meromorphic functions. If $P_0(z)$ has a deficient value ∞ , $P_1(z) \in EF$, and $\sigma(P_j) < \sigma(P_0)$ for $j \neq 0, 1$, then every meromorphic solution $f(z)$ of Eq. (1) satisfies $\sigma(f) \geq \sigma(P_0) - \max\{\sigma(P_j), j \neq 0, 1\} + 1$.*

Remark 1.1 In Theorems 1.1 and 1.2, we require one coefficient in the equation to have one or several finite deficient value, but we do not know which one of P_0 and P_1 has faster growth, and this means there does not exist a dominating coefficient.

Recently, some results about the connections between complex dynamics with linear differential equations have been obtained; see [10, 14]. In the following, we shall consider Question 1.1 from the dynamical system point of view. Some notations of complex dynamics are needed; see [10].

Let $f: \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ be a transcendental meromorphic function. Denote by f^n , $n \in \mathbb{N}$, the n th iterate of f , that is, $f^1 = f, \dots, f^n = f \circ (f^{n-1})$. The Fatou set of f is denoted by $F(f)$, and the Julia set of f is defined by $J(f) = \widehat{\mathbb{C}} \setminus F(f)$. Let U be a connected component of $F(f)$, then $f^n(U)$ is contained in a component of $F(f)$, denoted by U_n . We say U is a wandering domain, if $U_n \cap U_m = \emptyset$. Furthermore, if U is wandering, and all U_n are multiply-connected and surround 0 and the Euclidean distance $\text{dist}(0, U_n) \rightarrow \infty$ as $n \rightarrow \infty$, then U is called a Baker wandering domain. The reader can refer to [2, 19] for more details.

Definition 1.3 ([17]) Let $f(z)$ be a meromorphic function of order σ , where $0 < \sigma < \infty$. A ray $\arg z = \theta_0$ ($0 \leq \theta_0 < 2\pi$) is named a Borel direction of order σ of $f(z)$, if for any positive number ε , the inequality

$$\lim_{r \rightarrow \infty} \frac{\log n(r, \theta_0 - \varepsilon, \theta_0 + \varepsilon, f = a)}{\log r} \geq \mu$$

holds for any finite complex value a , with possibly one exceptional value, where $n(r, \theta_0 - \varepsilon, \theta_0 + \varepsilon, f = a)$ denotes the number of zeros of $f(z) - a$ in the region $(|z| \leq r) \cap (\theta_0 - \varepsilon \leq \arg z \leq \theta_0 + \varepsilon)$, multiple zeros being counted with their multiplicities.

Suppose that $f(z)$ is an entire function of finite lower order $\mu > 0$. Let q ($< \infty$) denote the number of Borel directions and p denote the number of finite deficient values of $f(z)$. Yang in [17] proved that $p \leq \frac{q}{2}$. We will say that an entire function $f(z)$ is extremal for Yang's inequality if $f(z)$ satisfies the assumptions of Definition 1.3 with $p = \frac{q}{2}$.

Next, we consider Eq. (1) from the dynamical system point of view, and we can establish the following result.

Theorem 1.3 *Let $P_j(z)$ ($j = 0, \dots, n$) be entire functions. Suppose that $P_1(z)$ is extremal for Yang's inequality, and $F(P_0)$ has a Baker wandering domain. If $\sigma(P_j) < \sigma(P_0)$ for $j \neq 0, 1$, then every meromorphic solution $f(z)$ of Eq. (1) satisfies $\sigma(f) \geq \sigma(P_0) - \max\{\sigma(P_j), j \neq 0, 1\} + 1$.*

2 Auxiliary results

In order to prove the theorems, we need some lemmas.

Lemma 2.1 ([1]) *Let $f(z)$ be a transcendental meromorphic function, $0 \leq \mu(f) < 1$. Then, for every $\alpha \in (\mu(f), 1)$, there exists a set $E \subset [0, \infty)$ such that $\overline{\log \text{dens}} E \geq 1 - \frac{\mu(f)}{\alpha}$,*

where

$$E = \{r \in [0, \infty) : m(r) > M(r) \cos \pi \alpha\},$$

$$m(r) = \inf_{|z|=r} \log |f(z)|, \text{ and } M(r) = \sup_{|z|=r} \log |f(z)|.$$

Lemma 2.2 ([8]) *Let $f(z)$ be a non-constant meromorphic function with finite order σ . For any given $\varepsilon > 0$ and $0 < l < \frac{1}{2}$, there exist a constant $K(\sigma, \varepsilon)$ and a set $E(\varepsilon) \subset [0, \infty)$ that satisfy the lower logarithmic density of $\log \text{dens } E(\varepsilon) \geq 1 - \varepsilon$, such that, for $r \in E(\varepsilon)$ and each interval J with a length of l , we have*

$$r \int_J \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta < K(\sigma, \varepsilon) \left(l \log \frac{1}{l} \right) T(r, f).$$

Lemma 2.3 ([5]) *Let η_1, η_2 be two arbitrary complex numbers, and let $f(z)$ be a meromorphic function of finite order σ . Let $\varepsilon > 0$ be given, then there exists a subset $E_1 \subset \mathbf{R}$ with finite logarithmic measure such that, for all $r \notin E_1 \cup [0, 1]$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left| \frac{f(z + \eta_1)}{f(z + \eta_2)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.4 ([5]) *Let $f(z)$ be a meromorphic function of order $\sigma(f) = \sigma < \infty$. Then, for any given $\varepsilon > 0$, there is a set $E \subset (1, \infty)$ that has finite linear measure mE and finite logarithmic measure $\text{lm } E$, such that, for all z satisfying $|z| = r \notin E \cup [0, 1]$,*

$$\exp\{-r^{\sigma+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\sigma+\varepsilon}\}.$$

Lemma 2.5 ([19]) *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If $f(z)$ has a Baker wandering domain, there exist a constant $0 < d < 1$ and two sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers with $r_n \rightarrow \infty$ and $\frac{R_n}{r_n} \rightarrow \infty$ ($n \rightarrow \infty$) such that*

$$\log |f(z)| \geq d \log M(r, f),$$

where $|z| = r$, $G = \bigcup_{n=1}^{\infty} \{r : r_n \leq r \leq R_n\}$.

Lemma 2.6 ([15]) *Let $f(z)$ be a meromorphic function of order $0 < \sigma < \infty$ having p finite deficient values $a_1, a_2, \dots, a_p$ ($p \geq 1$) and let $g(z)$ be a meromorphic function with finite order having a deficient value ∞ . Suppose that $\beta > 1$ and $0 < \eta < \sigma(f)$ are two constants. Then there exists a sequence $\{t_n\}$ such that*

$$\lim_{n \rightarrow \infty} \frac{t_n}{T(t_n, f)} = 0. \quad (5)$$

Moreover, for every sufficiently large n , there is a set $F_n \subset [t_n, (\beta + 1)t_n]$ with $m(F_n) \leq \frac{(\beta-1)t_n}{4}$ such that, for all $R \in [t_n, \beta t_n] \setminus F_n$, the arguments θ sets $E_\nu(R)$ ($\nu = 1, 2, \dots, p$) and $E_\infty(R)$ satisfy the following inequalities:

$$m(E_\nu(R)) = m\left(\left\{\theta \in [0, 2\pi) \mid \log \frac{1}{|f(Re^{i\theta}) - a_\nu|} \geq \frac{\delta_0}{4} T(R, f)\right\}\right) \geq M_1 > 0 \quad (6)$$

and

$$m(E_\infty(R)) = m\left(\left\{\theta \in [0, 2\pi) \mid \log|g(Re^{i\theta})| \geq \frac{\delta_1}{4} T(R, g)\right\}\right) \geq M_2 > 0, \quad (7)$$

where M_1, M_2 are two positive constants depending only on f, g , $\delta_0 = \min_{0 \leq v \leq p} \delta(a_v, f)$, $\delta_1 = \delta(\infty, g)$, β and η .

Lemma 2.7 ([15]) *Let $f(z) \in EF$, then, for any given $\varepsilon > 0$ (sufficiently small) and $\beta > 1$, when n is sufficiently large, there exists a sequence of angular regions $\overline{\Omega}(\theta_{k_v}, \theta_{k_v} + 1, t_n, \beta t_n)$, $n = 1, 2, 3, \dots$, $v = 1, 2, \dots, p$, such that, for every $1 \leq v \leq p$, the inequality*

$$\log \frac{1}{|f(z) - a_v|} > \log \frac{4}{d} \quad (8)$$

holds for $z \in \overline{\Omega}(\theta_{k_v}, \theta_{k_v} + 1, t_n, \beta t_n) \setminus \bigcup_{v=1}^p (\gamma_v)_n$, where $\bigcup_{v=1}^p (\gamma_v)_n$ is defined by Lemma 2.8 with the sum of total radius not exceeding $\frac{p}{8} \varepsilon t_n$ and $t_n, \beta t_n, d = \min_{1 \leq v \neq v'} \{|a_v - a_{v'}|\}$ and a_v are deficient values of $f(z)$.

Lemma 2.8 ([16]) *Suppose that $f(z)$ is extremal for Yang's inequality, i.e., $f(z)$ is an entire function of lower order $\mu < +\infty$ and it satisfies $p = \frac{q}{2}$ where p ($1 \leq p < +\infty$) denotes the number of finite deficient values and q denotes the number of Borel directions of order $\geq \mu$ of $f(z)$. Then, for every deficient value a_i ($i = 1, 2, \dots, p$), there exists a corresponding angular domain $\Omega(\theta_{k_i}, \theta_{k_i+1})$ such that for every $\varepsilon > 0$ the inequality*

$$\log \frac{1}{|f(z) - a_i|} > A(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, f)) T(|z|, f)$$

holds for $z \in \Omega(\theta_{k_i} + \varepsilon, \theta_{k_i+1} - \varepsilon, r_\varepsilon, +\infty)$, where $A(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, f))$ is a positive constant depending only on $\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, f)$.

3 Proof of Theorem 1.1

We divide our proof into two steps.

Step 1. In this step, we use the idea in [20] with some changes. According to Lemma 2.1, for any given constant $\varepsilon > 0$, there exists a set $E_1 = E_1\{r \in [0, \infty) : m(r) > M(r) \cos \pi \alpha_0\}$ with $\overline{\log \text{dens}} E_1 \geq 1 - \frac{\mu(P_0)}{\alpha_0}$, where $m(r) = \inf_{|z|=r} \log |P_0(z)|$, and $M(r) = \sup_{|z|=r} \log |P_0(z)|$. Therefore, there exists some constant r_0 such that, for $r \in E_1 \setminus [0, r_0]$, we have

$$|P_0(z)| > \exp\{r^{\mu(P_0)-\varepsilon}\}. \quad (9)$$

Suppose $P_1(z)$ has a finite deficient value a , and $\delta(a, P_1(z)) = 2\delta > 0$. Then by the definition of deficiency, there exists a constant r_1 such that, for each $r > r_1 > r_0$, $m(r, \frac{1}{P_1-a}) \geq \delta T(r, P_1)$ holds. Hence, for $r > r_1$, there exists z_r with $|z_r| = r$ such that

$$\log |P_1(z_r) - a| \leq -\delta T(r, P_1). \quad (10)$$

Set $0 < \varepsilon_0 < 1 - \frac{\mu(P_0)}{\alpha_0}$. Applying Lemma 2.2 to $P_1(z) - a$, we can choose sufficiently small l_0 such that $K(\sigma(P_1), \varepsilon_0) l_0 \log \frac{1}{l_0} < \frac{\delta}{4}$, then, for every interval J with a length of l_0 and all

$r > r_1 > r_0$, $r \in E(\varepsilon_0)$, we have

$$r \int_J \left| \frac{P_1'(re^{i\theta})}{P_1(re^{i\theta}) - a} \right| d\theta < \frac{\delta}{2} T(r, P_1), \quad (11)$$

where $E(\varepsilon_0)$ is the set of lower logarithmic density $\underline{\log \text{dens}} E(\varepsilon_0) \geq 1 - \varepsilon_0$ determined by Lemma 2.2. Let $z_r = re^{i\theta}$ and $\phi_0 = \frac{l_0}{4}$. It follows from (10)–(11) that, for all $|z_r| = r \in E(\varepsilon_0) \setminus [0, r_1]$ and $\theta \in [\theta_r - \phi_0, \theta_r + \phi_0]$,

$$\begin{aligned} \log |P_1(re^{i\theta}) - a| &= \log |P_1(re^{i\theta_r}) - a| + \int_{\theta_r}^{\theta} \frac{d \log |P_1(re^{it}) - a|}{dt} dt \\ &\leq -\delta T(r, P_1) + \int_{\theta_r}^{\theta} \left| \frac{P_1'(re^{it})}{P_1(re^{it}) - a} \right| dt \\ &\leq -\frac{\delta}{2} T(r, P_1) \leq 0. \end{aligned}$$

Therefore, for $|z_r| = r \in E(\varepsilon_0) \setminus [0, r_1]$ and $\theta \in [\theta_r - \phi_0, \theta_r + \phi_0]$, we get

$$|P_1(re^{i\theta}) - a| \leq 1. \quad (12)$$

Set $E = E(\varepsilon_0) \cap E_1$. Clearly,

$$\overline{\log \text{dens}} E + \underline{\log \text{dens}} (E(\varepsilon_0) \setminus E_1) \geq \underline{\log \text{dens}} E(\varepsilon_0).$$

Hence,

$$\overline{\log \text{dens}} E \geq 1 - \varepsilon_0 - \underline{\log \text{dens}} E_1^c.$$

Since $\overline{\log \text{dens}} E_1 + \underline{\log \text{dens}} E_1^c = 1$ and $\overline{\log \text{dens}} E_1 \geq 1 - \frac{\mu(P_0)}{\alpha_0}$, we have $\underline{\log \text{dens}} E_1^c \leq \frac{\mu(P_0)}{\alpha_0}$. Thus,

$$\overline{\log \text{dens}} E \geq 1 - \varepsilon_0 - \frac{\mu(P_0)}{\alpha_0} > 0.$$

Therefore, from (9) and (12), for any sequence $\{r_n\} \subset E$ and $\theta_n \in [\theta_r - \phi_0, \theta_r + \phi_0]$,

$$\begin{aligned} |P_0(z_n)| &> \exp\{r_n^{\mu(P_0)-\varepsilon}\}, \\ |P_1(r_n e^{i\theta_n}) - a| &\leq 1, \end{aligned} \quad (13)$$

where $z_n = r_n e^{i\theta_n}$.

Step 2. Now suppose that $f(z)$ is a non-trivial meromorphic solution of Eq. (1) with $\sigma(f) < \mu(P_0) - \max\{\sigma(P_j), j \neq 0, 1\} + 1$. We shall seek a contradiction. Set $\sigma = \sigma(f)$, $\alpha = \max\{\sigma(P_j), j \neq 0, 1\}$. From Eq. (1),

$$|P_0(z)| \leq |P_n(z)| \left| \frac{f(z+n)}{f(z)} \right| + \cdots + |P_2| \left| \frac{f(z+2)}{f(z)} \right| + (|P_1(z) - a| + |a|) \left| \frac{f(z+1)}{f(z)} \right|. \quad (14)$$

We may choose ε so that $0 < 4\varepsilon < \mu(P_0) - \alpha + 1 - \sigma$. By using Lemma 2.3, for ε there exists a set $E_2 \subset (1, \infty)$ with finite logarithmic measure $lm E_2 < \infty$ such that, for all $|z| = r \notin E_2 \cup [0, r_1]$, we have

$$\left| \frac{f(z+j)}{f(z)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\}, \quad j = 1, \dots, n. \quad (15)$$

Again by Lemma 2.4, there exists a set $E_3 \subset (1, \infty)$ with logarithmic measure $lm E_3 < \infty$ such that, for all $|z| = r \notin E_3 \cup [0, r_1]$, we have

$$|P_j(z)| \leq \exp\{r^{\alpha+\varepsilon}\}, \quad j = 2, \dots, n. \quad (16)$$

It follows from (9), (12), (14), (15) and (16) that there exist $r_n \in E \setminus E_2 \cup E_3$ and $\theta_n \in [\theta_{r_n} - \phi_0, \theta_{r_n} + \phi_0]$ such that, for $z_n = r_n e^{i\theta_n}$, we have

$$\begin{aligned} \exp\{r_n^{\mu(P_0)-\varepsilon}\} &\leq (|P_n(z)| + \dots + |P_2(z)| + |P_1(z) - a| + |a|) \exp\{r_n^{\sigma-1+\varepsilon}\} \\ &\leq ((n-1) \exp\{r_n^{\alpha+\varepsilon}\} + 1 + |\alpha|) \exp\{r_n^{\sigma-1+\varepsilon}\}. \end{aligned}$$

Clearly $\mu(P_0) - \varepsilon \leq \sigma + \alpha - 1 + 3\varepsilon$, and hence $\mu(P_0) - \sigma - \alpha + 1 \leq 4\varepsilon$. We have a contradiction.

4 Proof of Theorem 1.2

Suppose that $f(z) \neq 0$ is a meromorphic solution of Eq. (1) with $\sigma(f) < \sigma(P_0) - \max\{\sigma(P_j), j \neq 0, 1\} + 1$. We shall seek a contradiction. Set $\sigma = \sigma(f) < \infty$, $\alpha = \max\{\sigma(P_j), j \neq 0, 1\} < \infty$. From Eq. (1), we have

$$|P_0(z)| \leq |P_n(z)| \left| \frac{f(z+n)}{f(z)} \right| + \dots + |P_2(z)| \left| \frac{f(z+2)}{f(z)} \right| + |P_1(z)| \left| \frac{f(z+1)}{f(z)} \right|. \quad (17)$$

First, we prove $\sigma(P_0) < \infty$. If $\sigma(P_0) = \infty$, by the difference logarithmic derivative lemma, we get

$$\begin{aligned} T(r, P_0(z)) &= T\left(r, P_n(z) \frac{f(z+n)}{f(z)} + \dots + P_2(z) \frac{f(z+2)}{f(z)} + P_1(z) \frac{f(z+1)}{f(z)}\right) \\ &\leq \sum_{j \neq 0} T(r, P_j(z)) + \sum_{j \neq 0} N(r, f(z+j)) + N\left(r, \frac{1}{f(z)}\right) + O(T(r, f)), \end{aligned}$$

possible outside of a set H satisfying $\limsup \frac{\int_{H \cap [1, r]} \frac{dt}{t}}{\log r} = 0$. Clearly, this is a contradiction since $\sigma(P_j) < \infty$ for $j \neq 0$ and $\sigma(f) < \infty$. Hence, $\sigma(P_0) < \infty$.

Choose ε such that $0 < 2\varepsilon < \sigma(P_0) - \alpha - \sigma + 1$. Applying Lemma 2.3 to $f(z)$, there exist a positive constant r_1 and a set $E_2 \subset [0, \infty)$ with $m(E_2) < \infty$ such that, for all z satisfying $|z| = r \notin E_2 \cup [0, r_1]$, we have

$$\left| \frac{f(z+j)}{f(z)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\}, \quad j = 1, \dots, n. \quad (18)$$

It follows from Lemma 2.4 that there exists a set $E_3 \subset (0, \infty)$ with logarithmic measure $lm E_3 < \infty$ such that, for all $|z| = r \notin E_3 \cup [0, 1]$, we have

$$|P_j(z)| \leq \exp\{r^{\alpha+\varepsilon}\}, \quad j \neq 0, 1. \quad (19)$$

Since $P_1(z) \in EF$, we can assume that $P_1(z)$ has p finite deficient values $a_1, a_2, \dots, a_p$ with $\delta(a_\nu, P_1) > 0$ for $1 \leq \nu \leq p$, and P_1 has p zero-pole accumulation rays $\arg z = \theta_k$ ($0 \leq \theta_1 < \theta_2 < \dots < \theta_p, \theta_{p+1} = \theta_1 + 2\pi$). Set $w = \min_{1 \leq k \leq p-1} (\theta_{k+1} - \theta_k)$.

Applying Lemma 2.6 to $P_0(z), P_1(z)$, there exists a sequence of closed intervals $\{[t_n, \beta t_n]\}$ with $t_n \rightarrow \infty$, $t_{n+1} > \beta t_n$ and a set $F_n \subset [t_n, (\beta + 1)t_n]$ with $m(F_n) \leq \frac{(\beta-1)t_n}{4}$ and a sequence $R_n \in [t_n, \beta t_n] \setminus F_n$ such that $P_0(z), P_1(z)$ satisfy

$$m(E_\infty(R_n)) = m\left(\left\{\theta \in [0, 2\pi) \mid \log|P_0(R_n e^{i\theta})| \geq \frac{\delta_1}{4} T(R_n, P_0)\right\}\right) \geq M_2 > 0 \quad (20)$$

and

$$m(E_\nu(R_n)) = m\left(\left\{\theta \in [0, 2\pi) \mid \log \frac{1}{|P_1(R_n e^{i\theta}) - a_\nu|} \geq \frac{\delta_0}{4} T(R_n, P_1)\right\}\right) \geq M_1 > 0, \quad (21)$$

where $\theta \in [0, 2\pi)$ and M_1, M_2 are two positive constants depending only on $P_0, P_1, \delta_1 = \delta(\infty, P_0), \delta_0 = \min_{0 \leq \nu \leq p} \delta(a_\nu, P_1), \beta$ and η .

Now we choose a sufficiently small ε_0 so that $0 < \varepsilon_0 < \min\{\frac{M_1}{8p}, \frac{w}{2}, \frac{M_2}{8p}, \frac{\beta-1}{2p}\}$. It follows from Lemma 2.8 that the following inequalities:

$$\log \frac{1}{|P_1(r_n e^{i\varphi}) - a_\nu|} > \log \frac{4}{d}, \quad \nu = 1, 2, \dots, p, \quad (22)$$

hold for $n \geq n_1 > n_0$ and $r_n e^{i\varphi} \in \bigcup_{\nu=1}^p \overline{\mathcal{D}}(\theta_{k_{\nu+1}} + 2\varepsilon_0, \theta_{k_\nu} - 2\varepsilon_0, t_n, \beta t_n) \setminus \bigcup_{\nu=1}^p (\gamma_\nu)_n$, where $\bigcup_{\nu=1}^p (\gamma_\nu)_n$ are some disks with the sum of total radius not exceeding $\frac{p}{8}\varepsilon_0 t_n < \frac{\beta-1}{16}t_n$, $d = \min_{1 \leq \nu, \nu' \leq p, \nu \neq \nu'} \{|a_\nu - a_{\nu'}|\}$ and $a_\nu, a_{\nu'}$ are deficient values of $P_1(z)$.

It is easy to choose $R_n^* \in [t_n, \beta t_n] \setminus (F_n \cup E_2 \cup E_3 \cup [0, 1])$ such that, for $n \geq n_0$,

$$\{z : |z| = R_n^*\} \cap \left\{\bigcup_{\nu=1}^p (\gamma_\nu)_n\right\} = \Phi.$$

On the other hand, from Lemma 2.6, for such a sequence $\{R_n^*\}$ and sufficiently large n , we also have

$$m(E_\infty(R_n^*)) = m\left(\left\{\theta \in [0, 2\pi) \mid \log|P_0(R_n^* e^{i\theta})| \geq \frac{\delta_1}{4} T(R_n^*, P_0)\right\}\right) \geq M_2 > 0. \quad (23)$$

Clearly, by the choice of ε_0 , there exists some k_ν , $1 \leq k_\nu \leq p$, such that

$$m(E_\infty(R_n^*)) \cap [\theta_{k_\nu} + 2\varepsilon_0, \theta_{k_{\nu+1}} - 2\varepsilon_0] \geq \frac{M_2}{2p} > 0. \quad (24)$$

Therefore, from (20), (22) and (24), we can choose $\theta_n \in E_\infty(R_n^*) \cap [\theta_{k_\nu} + 2\varepsilon_0, \theta_{k_{\nu+1}} - 2\varepsilon_0]$ such that

$$\log|P_0(R_n^* e^{i\theta_n})| \geq \frac{\delta_1}{4} T(R_n^*, P_0) \quad (25)$$

and

$$\log \frac{1}{|P_1(R_n^* e^{i\theta_n}) - a_v|} > \log \frac{4}{d}, \quad v = 1, 2, \dots, p, \quad (26)$$

hold simultaneously, where $z_n = r_n e^{i\theta_n}$, $\beta > 1$, $d = \min_{1 \leq v \neq v'} \{|a_v - a_{v'}|\}$ and $a_v, a_{v'}$ are deficient values of $P_1(z)$.

Rewrite (17) as

$$|P_0(z)| \leq |P_n(z)| \left| \frac{f(z+n)}{f(z)} \right| + \dots + |P_2(z)| \left| \frac{f(z+2)}{f(z)} \right| + (|P_1 - a_v| + |a_v|) \left| \frac{f(z+1)}{f(z)} \right|. \quad (27)$$

It follows from (18), (19), (25) and (26) that

$$\begin{aligned} \exp \left\{ \frac{\delta_1}{4} T(R_n^*, P_0) \right\} &\leq (n-1) \exp \{ (R_n^*)^{\alpha+\varepsilon} \} \exp \{ (R_n^*)^{\sigma-1+\varepsilon} \} \\ &\quad + \left(\frac{d}{4} + |a_v| \right) \exp \{ (R_n^*)^{\sigma-1+\varepsilon} \}. \end{aligned}$$

Clearly $\sigma(P_0) \leq \alpha + \sigma - 1 + 2\varepsilon$, and hence $\sigma(f) \geq \sigma(P_0) - \max\{\sigma(P_j), j \neq 0, 1\} + 1 + 2\varepsilon$. We have a contradiction.

5 Proof of Theorem 1.3

Suppose that $f \not\equiv 0$ is a meromorphic solution of Eq. (1) with $\sigma(f) < \sigma(P_0) - \max\{\sigma(P_j), j \neq 0, 1\} + 1$.

Since $P_1(z)$ is extremal for Yang's inequality, we can assume that a_i ($i = 1, 2, \dots, 2p$) are all the finite deficient values of $P_1(z)$. By Lemma 2.8, for every deficient value a_i , there exists a corresponding angular domain $\Omega(\theta_j, \theta_{j+1})$ such that for every $\varepsilon > 0$ the inequality

$$\log \frac{1}{|P_1(z) - a_i|} > A(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, P_1)) T(|z|, P_1)$$

holds for $z \in \Omega(\theta_j + \varepsilon, \theta_{j+1} - \varepsilon, r_\varepsilon, +\infty)$, where $\varepsilon > 0$ and $A(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, P_1))$ is a positive constant depending only on $\theta_j, \theta_{j+1}, \varepsilon$ and $\delta(a_i, f)$. Set $C = A(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, P_1))$ for brevity. Thus

$$|P_1(z) - a_1| < \exp \{ -CT(|z|, P_1) \}. \quad (28)$$

Note that $P_0(z)$ is a transcendental entire function with a Baker wandering domain. According to Lemma 2.5, there exists one integer $d < 1$ and two positive sequences $\{r_n\}, \{R_n\}$, such that $\frac{R_n}{r_n} \rightarrow \infty$ as $r_n \rightarrow \infty, n \rightarrow \infty$ and $|z| = r, r \in G$, and we obtain

$$\log |P_0(z)| \geq d \log M(r, P_0),$$

where $G = \bigcup_{n=1} \{r : r_n \leq r \leq R_n\}$. Thus

$$|P_0(z)| \geq M(r, P_0)^d. \quad (29)$$

Let $0 < 2\varepsilon < \sigma(P_0) - \sigma(f) - \alpha + 1$. By Lemma 2.3, there exists a set $E_2 \subset [0, \infty)$ with finite linear measure and a constant $\eta > 0$ such that, for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$, (18) holds. From Lemma 2.4, there still exists a set $E_3 \subset (0, \infty)$ with finite logarithmic measure such that, for all $|z| = r \notin E_3 \cup [0, 1]$, (19) holds.

Hence, for $z = re^{i\theta}$, $r \in G \setminus (E_2 \cup E_3 \cup [0, 1])$ and $\theta \in (\theta_j, \theta_{j+1})$, we get

$$|P_0(z)| \geq M(r, P_0)^d$$

and

$$|P_1(z) - a_1| < \exp\{-CT(|z|, P_1)\}. \quad (30)$$

Rewriting Eq. (1), we have

$$|P_0(z)| \leq |P_n(z)| \left| \frac{f(z+n)}{f(z)} \right| + \cdots + |P_2(z)| \left| \frac{f(z+2)}{f(z)} \right| + (|P_1(z) - a| + |a|) \left| \frac{f(z+1)}{f(z)} \right|. \quad (31)$$

Therefore, we deduce from (23), (24) and (28) that

$$M(r, P_0)^d \leq (n-1) \exp\{r^{\alpha+\varepsilon}\} \exp\{r^{\sigma(f)-1+\varepsilon}\} + (\exp\{-CT(|z|, P_1)\} + |a|) \exp\{r^{\sigma(f)-1+\varepsilon}\}.$$

Clearly $\sigma(P_0) \leq \sigma + \alpha - 1 + 2\varepsilon$, that is, $2\varepsilon \geq \sigma(P_0) - \sigma(f) - \alpha + 1$. We have a contradiction.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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