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Theorems of complete convergence and complete integral convergence for END random variables under sub-linear expectations

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Abstract

The goal of this paper is to build complete convergence and complete integral convergence for END sequences of random variables under sub-linear expectation space. By using the Markov inequality, we extend some complete convergence and complete integral convergence theorems for END sequences of random variables when we have a sub-linear expectation space, and we provide a way to learn this subject.

MSC: 60F15

Keywords: Sub-linear expectation; Complete convergence; Complete integral convergence; END random variables

1 Introduction

Classical probability theorems were widely used in many fields, which only hold on some occasions of model certainty. However, there are uncertainties, such as measures of risk, non-linear stochastic calculus and statistics in the process of finance. At this time, sub-linear expectation and capacity are not additive, the limit theorems of classical probability space are no longer valid. Therefore, the study of the limit theorems of sub-linear expectation becomes more complex. Peng Shige [1–3] of Shandong University constructed the basic concept of sub-linear expectation and gave a complete set of axioms of sub-linear expectation theories. The sub-linear expectation axiom system makes up for the deficiency of limit theorems of classical probability space. Since the general framework of sub-linear expectation was introduced by Peng Shige, many scholars have paid close attention to it, and lots of excellent results have been established. For example, Zhang [4–6] studied the sub-linear expectation space in depth and proved some important inequalities under the sub-linear expectations, Xu and Zhang [7] proved a three series theorem for independent random variables under sub-linear expectations with applications. Wu and Jiang [8] proved the strong law of large numbers and the law of iterated logarithm in the sub-linear expectation space. Chen [9] obtained the strong law of large numbers for independent isomorphic sequences in sub-linear expectation space, and he also obtained the central limit theorem of weighted sums in sub-linear expectation space. The complete convergence and complete integral convergence have a relatively complete development in limit theories of probability. The notion of complete convergence was raised by Hsu and Robbins [10], and Chow [11] built complete moment convergence. However, to the best of our knowledge, except for Liu [12], Chen [13], Wu and Guan [14], not many authors discussed the properties for END random variables. We know Sung [15] founded the notion of uniform integrability, we obtained complete convergence and complete integral convergence for the array of END random variables under uniform integrability, which were not considered in Sung [15]. We found the concept of uniform integrability on random variable sequences and uniform integrability is a more extensive condition than that of Cesàro [16, 17].

The complete integral convergence is a more important version of the complete convergence, and both of them are most important problems in classical probability theories. Many of the related results have already been obtained in classical probability space. Now, some corresponding results were obtained by Gut and Stadtmuller [18], Qiu and Chen [19], Wu and Jiang [20] and Feng and Wang [21], we still need to perfect the complete convergence and complete integral convergence under sub-linear expectation. We establish the complete convergence and complete integral convergence for END random variables under sub-linear expectation and generalize them [22] to the sub-linear expectation space.

2 Preliminaries

We use the framework and notions of Peng [1]. Let $(\Omega,\mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega,\mathcal{F})$ such that if $X_1,X_2,\ldots,X_n\in\mathcal{H}$ then $\varphi(X_1,\ldots,X_n)\in\mathcal{H}$ for each $\varphi\in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$\bigl|\varphi(\mathbf{x})-\varphi(\mathbf{y})\bigr|\le c\bigl(1+|\mathbf{x}|^m+|\mathbf{y}|^m\bigr)|\mathbf{x}-\mathbf{y}|,\quad \forall\mathbf{x},\mathbf{y}\in\mathbb{R}_n,$$

for some $c>0$, $m\in\mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X\in\mathcal{H}$.

Definition 2.1 ([1]) A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}}:\mathcal{H}\to\bar{\mathbb{R}}$ satisfying the following properties: for all $X,Y\in\mathcal{H}$, we have

(a) monotonicity: if $X\ge Y$ then $\hat{\mathbb{E}}X\ge\hat{\mathbb{E}}Y$;
(b) the constant preserving property: $\hat{\mathbb{E}}c=c$;
(c) sub-additivity: $\hat{\mathbb{E}}(X+Y)\le\hat{\mathbb{E}}X+\hat{\mathbb{E}}Y$; whenever $\hat{\mathbb{E}}X+\hat{\mathbb{E}}Y$ is not of the form $+\infty-\infty$ or $-\infty+\infty$;
(d) positive homogeneity: $\hat{\mathbb{E}}(\lambda X)=\lambda\hat{\mathbb{E}}X$, $\lambda\ge 0$.

Here $\bar{\mathbb{R}}=[-\infty,\infty]$. The triple $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}X:=-\hat{\mathbb{E}}(-X),\quad \forall X\in\mathcal{H}.$$

$$\hat{\mathbb{E}}f\le\mathbb{V}(A)\le\hat{\mathbb{E}}g,\qquad \hat{\varepsilon}f\le\mathcal{V}(A)\le\hat{\varepsilon}g,\quad \text{if } f\le I(A)\le g, f,g\in\mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$

$$\begin{aligned} &\hat{\varepsilon}X \le \hat{\mathbb{E}}X, \qquad \hat{\mathbb{E}}(X+c) = \hat{\mathbb{E}}X + c, \\ &\hat{\mathbb{E}}(X-Y) \ge \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y. \end{aligned} \tag{2.1}$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$. Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0,1]$ is called a capacity if

$$V(\emptyset) = 0, \qquad V(\Omega) = 1 \quad \text{and} \quad V(A) \le V(B) \quad \text{for } \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called sub-additive if $V(A \cup B) \le V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\bigl\{\hat{\mathbb{E}}\xi; I(A) \le \xi, \xi \in \mathcal{H}\bigr\}, \qquad \mathcal{V}(A) := 1 - \mathbb{V}\bigl(A^c\bigr), \quad \forall A \in \mathcal{F},$$

where A^c is the complement set of A. By definition of $\mathbb{V}$ and $\mathcal{V}$, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\mathcal{V}[A] \le \mathbb{V}[A], \quad \forall A \in \mathcal{F}.$$

This implies the Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}\bigl(|X| \ge x\bigr) \le \hat{\mathbb{E}}\bigl(|X|^p\bigr)/x^p, \quad \forall x > 0, p > 0,$$

from $I(|X| \ge x) \le |X|^p/x^p \in \mathcal{H}$.

Definition 2.2 ([1]) A sequence of random variables $\{X_n; n \ge 1\}$ is said to be upper (resp. lower) extended negatively dependent if there is some dominating constant $K \ge 1$ such that

$$\hat{\mathbb{E}}\left(\prod_{i=1}^n \varphi_i(X_i)\right) \le K \prod_{i=1}^n \hat{\mathbb{E}}\bigl(\varphi_i(X_i)\bigr), \quad \forall n \ge 2,$$

whenever the non-negative functions $\varphi_i(x) \in C_{l,\mathrm{Lip}}(\mathbb{R})$, $i = 1, 2, \ldots$ are all non-decreasing (resp. all non-increasing). They are called extended negatively dependent (END) if they are both upper extended negatively dependent and lower extended negatively dependent.

It is obvious that, if $\{X_n; n \ge 1\}$ is a sequence of extended negatively dependent random variables and $f_1(x), f_2(x), \ldots \in C_{l,\mathrm{Lip}}(\mathbb{R})$ are non-decreasing (resp. non-increasing) functions, then $\{f_n(X_n); n \geqq 1\}$ is also a sequence of END random variables.

Definition 2.3 ([4]) The Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ are defined by

$$C_V = \int_0^\infty V(X \ge t)\,\mathrm{d}t + \int_{-\infty}^0 \bigl[V(X \ge t) - 1\bigr]\,\mathrm{d}t,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

We define C to be various positive constants at different places in this paper.

Lemma 2.1 ([4], Theorem 3.1) *Assume that* $\{X_i; u_n \le i \le m_n\}$ *is an array of row-wise END random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *with* $\hat{\mathbb{E}}X_i \le 0$, *for* $u_n \le i \le m_n$. *Let* $B_n = \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}[X_i^2]$. *Then, for any given n and for all* $x > 0$, $y > 0$, $K \ge 1$. *Then*

$$\mathbb{V}\left(\sum_{i=u_n}^{m_n} X_i \ge x\right) \le \mathbb{V}\Big(\max_{u_n \le i \le m_n} X_i \ge y\Big) + K\exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right\}.$$

Here K is the dominating constant in Definition 2.2.

For $0 < \mu < 1$, *let* $g(x) \in C_{l,\mathrm{Lip}}(\mathbb{R})$ *be a non-increasing function such that* $0 \le g(x) \le 1$ *for all x and* $g(x) = 1$ *if* $x \le \mu$, $g(x) = 0$ *if* $x > 1$. *Then*

$$I\big(|x| \le \mu\big) \le g\big(|x|\big) \le I\big(|x| \le 1\big). \tag{2.2}$$

Lemma 2.2 *Assume that* $\{X_{ni}; u_n \le i \le m_n, n \ge 1\}$ *is an array of row-wise END random variables. Let* $\{h_n; n \ge 1\}$ *and* $\{k_n; n \ge 1\}$ *be increasing sequences of positive constants with* $h_n \to \infty$, $k_n \to \infty$ *as* $n \to \infty$ *and* $\frac{h_n}{k_n} \to 0$, *for some* $r > 0$, *satisfying*

$$\sup_{n\ge 1} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r < \infty \tag{2.3}$$

and

$$\lim_{n\to\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) = 0. \tag{2.4}$$

Then the following statements hold:

$$\lim_{n\to\infty} k_n^{-\frac{\alpha}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\alpha \left(1 - g\left(\frac{|X_{ni}|^r}{k_n}\right)\right) = 0, \quad \textit{for any } 0 < \alpha \le r, \tag{2.5}$$

$$\lim_{n\to\infty} k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\beta g\left(\frac{|X_{ni}|^r}{k_n}\right) = 0, \quad \textit{for any } \beta > r. \tag{2.6}$$

Proof We have $\frac{h_n}{k_n} \to 0$ as $n \to \infty$, so there exists N such that $h_n \le k_n$ if $n > N$. We know $g(x) \in C_{l,\mathrm{Lip}}(\mathbb{R})$ is a non-increasing function and $h_n \le k_n$, by (2.2) we obtain

$$1 - g\left(\frac{|X_{ni}|^r}{k_n}\right) \le 1 - g\left(\frac{|X_{ni}|^r}{h_n}\right), \tag{2.7}$$

and $1 - g(\frac{|X_{ni}|^r}{k_n}) \le I(|X_{ni}|^r > \mu k_n)$. When $0 < \alpha \le r$ and $0 < \mu < 1$, then, for $n > N$, combining (2.4), we can get

$$\begin{aligned} k_n^{-\frac{\alpha}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\alpha \left(1 - g\left(\frac{|X_{ni}|^r}{k_n}\right)\right) &= k_n^{-1} \cdot k_n^{-\frac{\alpha}{r}+1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\alpha \left(1 - g\left(\frac{|X_{ni}|^r}{k_n}\right)\right) \\ &\le k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\alpha \left(\frac{|X_{ni}|^r}{\mu}\right)^{\frac{r-\alpha}{r}} \left(1 - g\left(\frac{|X_{ni}|^r}{k_n}\right)\right) \end{aligned}$$

$$\begin{aligned}
&\le \mu^{\frac{\alpha-r}{r}} \cdot k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

Therefore, (2.5) has been proven.

Now we prove (2.6). Assume that $A_n = k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\beta g(\frac{|X_{ni}|^r}{k_n})$, on considering (2.1), we conclude that

$$\begin{aligned}
A_n &= k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\beta g\left(\frac{|X_{ni}|^r}{k_n}\right) \\
&\le k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\beta g\left(\frac{\mu|X_{ni}|^r}{1}\right) + k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\beta \left(g\left(\frac{|X_{ni}|^r}{k_n}\right) - g\left(\frac{\mu|X_{ni}|^r}{1}\right)\right) \\
&=: A_{n1} + A_{n2}.
\end{aligned}$$

For A_{n1}, note that $\frac{\beta}{r} > 1$, $0 < \mu < 1$, $g(\frac{\mu|X_{ni}|^r}{1}) \le I(\mu|X_{ni}|^r \le 1)$ and (2.3), while $k_n \to \infty$ as $n \to \infty$, so we can obtain

$$\begin{aligned}
A_{n1} &= k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^\beta g\left(\frac{\mu|X_{ni}|^r}{1}\right) \\
&= k_n^{-\frac{\beta}{r}+1} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r |X_{ni}|^{\beta-r} g\left(\frac{\mu|X_{ni}|^r}{1}\right) \\
&\le k_n^{-\frac{\beta}{r}+1} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(\frac{1}{\mu}\right)^{\frac{\beta-r}{r}} \\
&\le (\mu k_n)^{1-\frac{\beta}{r}} \cdot \sup_{n\ge1} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

For A_{n2}, because of (2.2), we obtain $I(|X_{ni}|^r > j) \le 1 - g(\frac{|X_{ni}|^r}{j})$. So we have

$$\begin{aligned}
&|X_{ni}|^\beta \left(g\left(\frac{|X_{ni}|^r}{k_n}\right) - g\left(\frac{\mu|X_{ni}|^r}{1}\right)\right) \\
&\quad \le |X_{ni}|^\beta I\big(1 < |X_{ni}|^r \le k_n\big) \\
&\quad = |X_{ni}|^r |X_{ni}|^{\beta-r} I\big(1 < |X_{ni}|^r \le k_n\big) \\
&\quad \le |X_{ni}|^r \sum_{j=1}^{k_n-1} (j+1)^{\frac{\beta-r}{r}} I\big(j < |X_{ni}|^r \le j+1\big) \\
&\quad = \sum_{j=1}^{k_n-1} |X_{ni}|^r (j+1)^{\frac{\beta-r}{r}} I\big(|X_{ni}|^r > j\big) - \sum_{j=1}^{k_n-1} |X_{ni}|^r (j+1)^{\frac{\beta-r}{r}} I\big(|X_{ni}|^r > j+1\big) \\
&\quad \le \sum_{j=1}^{k_n-1} |X_{ni}|^r (j+1)^{\frac{\beta-r}{r}} I\big(|X_{ni}|^r > j\big) - \sum_{j=2}^{k_n} |X_{ni}|^r j^{\frac{\beta-r}{r}} I\big(|X_{ni}|^r > j\big)
\end{aligned}$$

$$\leq \sum_{j=1}^{k_n-1} |X_{ni}|^r \big((j+1)^{\frac{\beta-r}{r}} - j^{\frac{\beta-r}{r}}\big) I\big(|X_{ni}|^r > j\big) + C|X_{ni}|^r I\big(|X_{ni}|^r > 1\big)$$

$$\leq \sum_{j=1}^{k_n-1} |X_{ni}|^r \big((j+1)^{\frac{\beta-r}{r}} - j^{\frac{\beta-r}{r}}\big)\left(1 - g\left(\frac{|X_{ni}|^r}{j}\right)\right) + C|X_{ni}|^r.$$

Because of $\frac{h_n}{k_n} \to 0$ as $n \to \infty$, $\frac{\beta}{r} > 1$, $g(x) \in C_{l,Lip}(\mathbb{R})$ is a non-increasing function, and we have (2.3) and (2.4), then

$$\begin{aligned}
A_{n2} &\leq k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \sum_{j=1}^{h_n} \big((j+1)^{\frac{\beta-r}{r}} - j^{\frac{\beta-r}{r}}\big) \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{j}\right)\right) \\
&\quad + k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \sum_{j=h_n+1}^{k_n-1} \big((j+1)^{\frac{\beta-r}{r}} - j^{\frac{\beta-r}{r}}\big) \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{j}\right)\right) + Ck_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \\
&\leq k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{1}\right)\right)\big((h_n+1)^{\frac{\beta-r}{r}} - 1\big) \\
&\quad + k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n+1}\right)\right)\big(k_n^{\frac{\beta-r}{r}} - (h_n+1)^{\frac{\beta-r}{r}}\big) + Ck_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \\
&\leq k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \big((h_n+1)^{\frac{\beta-r}{r}} - 1\big) + k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\quad + Ck_n^{-\frac{\beta}{r}+1} \cdot k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \\
&\leq k_n^{-\frac{\beta}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r (h_n+1)^{\frac{\beta-r}{r}} + k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\quad + Ck_n^{-\frac{\beta}{r}+1} \cdot k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \\
&= \left(\frac{h_n+1}{k_n}\right)^{\frac{\beta-r}{r}} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r + k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\quad + Ck_n^{-\frac{\beta}{r}+1} \cdot k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \\
&\leq \left(\frac{h_n+1}{k_n}\right)^{\frac{\beta-r}{r}} \cdot \sup_{n\geq 1} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r + k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\quad + Ck_n^{-\frac{\beta}{r}+1} \cdot \sup_{n\geq 1} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \\
&\to 0 \quad (n \to \infty).
\end{aligned}$$

Now (2.6) has been proven. The proof is completed. □

3 Main results

Theorem 3.1 *Assume that* $\{X_{ni}; u_n \le i \le m_n, n \ge 1\}$ *is an array of row-wise END random variables;* $\{h_n; n \ge 1\}$ *and* $\{k_n; n \ge 1\}$ *are two increasing sequences of positive constants with* $h_n \to \infty$, $k_n \to \infty$ *as* $n \to \infty$. *For some* $1 \le r < 2$, *satisfying* (2.3),

$$\sum_{n=1}^{\infty}(m_n - u_n)\frac{h_n}{k_n} < \infty, \tag{3.1}$$

$$\sum_{n=1}^{\infty}\left(\frac{h_n}{k_n}\right)^{\frac{2-r}{r}} < \infty, \tag{3.2}$$

$$\sum_{n=1}^{\infty}k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) < \infty. \tag{3.3}$$

Then, for all $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=u_n}^{m_n}(X_{ni} - \hat{\mathbb{E}}X_{ni}) > \varepsilon k_n^{\frac{1}{r}}\right) < \infty \tag{3.4}$$

and

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=u_n}^{m_n}(X_{ni} - \hat{\mathcal{E}}X_{ni}) < -\varepsilon k_n^{\frac{1}{r}}\right) < \infty. \tag{3.5}$$

In particular, if $\hat{\mathbb{E}}X_{ni} = \hat{\mathcal{E}}X_{ni}$, *then*

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni} - \hat{\mathbb{E}}X_{ni})\right| > \varepsilon k_n^{\frac{1}{r}}\right) < \infty. \tag{3.6}$$

Theorem 3.2 *Assume that* $\{X_{ni}; u_n \le i \le m_n, n \ge 1\}$ *is an array of row-wise END random variables;* $\{h_n; n \ge 1\}$ *and* $\{k_n; n \ge 1\}$ *are two increasing sequences of positive constants with* $h_n \to \infty$, $k_n \to \infty$ *as* $n \to \infty$. *For some* $1 \le r < 2$, *satisfying* (2.3), (3.1), (3.2),

$$\hat{\mathbb{E}}|X_{ni}|^r\left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \le C_{\mathbb{V}}|X_{ni}|^r\left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right), \tag{3.7}$$

$$\sum_{n=1}^{\infty}k_n^{-1}\sum_{i=u_n}^{m_n}C_{\mathbb{V}}\left\{|X_{ni}|^r\left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right)\right\} < \infty. \tag{3.8}$$

Then, for all $\varepsilon > 0$ *and* $\hat{\mathbb{E}}X_{ni} = \hat{\mathcal{E}}X_{ni}$, *we have*

$$\sum_{n=1}^{\infty}k_n^{-1}C_{\mathbb{V}}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni} - \hat{\mathbb{E}}X_{ni})\right| - \varepsilon k_n^{\frac{1}{r}}\right)_+^r < \infty. \tag{3.9}$$

Proof of Theorem 3.1 For an array of row-wise END random variables $\{X_{ni}; u_n \le i \le m_n, n \ge 1\}$, to ensure the truncated random variables are also END, we demand that truncated functions belong to $C_{l,\mathrm{Lip}}$. For all $u_n \le i \le m_n$, $n \ge 1$, $\lambda \ge 0$, for all $\varepsilon > 0$, we define

$$Y_{ni} = -\lambda k_n^{\frac{1}{r}} I\{X_{ni} < -\lambda k_n^{\frac{1}{r}}\} + X_{ni} I\{|X_{ni}| \le \lambda k_n^{\frac{1}{r}}\} + \lambda k_n^{\frac{1}{r}} I\{X_{ni} > \lambda k_n^{\frac{1}{r}}\},$$
$$Z_{ni} = X_{ni} - Y_{ni} = (X_{ni} + \lambda k_n^{\frac{1}{r}}) I\{X_{ni} < -\lambda k_n^{\frac{1}{r}}\} + (X_{ni} - \lambda k_n^{\frac{1}{r}}) I\{X_{ni} > \lambda k_n^{\frac{1}{r}}\}.$$

Through this, it is easy to see that $Y_{ni} \le |Y_{ni}| \le \lambda k_n^{\frac{1}{r}}$, $|Y_{ni}| \le |X_{ni}|$ and

$$|Z_{ni}| \le |X_{ni}| I\{|X_{ni}| > \lambda k_n^{\frac{1}{r}}\} \le |X_{ni}| \left(1 - g\left(\frac{|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\right)\right).$$

Now we prove (3.4), for all $\varepsilon > 0$, it suffices to verify that

$$I_1 = \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} Z_{ni} > \varepsilon k_n^{\frac{1}{r}}\right) < \infty,$$
$$I_2 = \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} (Y_{ni} - \hat{\mathbb{E}} Y_{ni}) > \varepsilon k_n^{\frac{1}{r}}\right) < \infty,$$
$$I_3 = \lim_{n\to\infty} k_n^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} (\hat{\mathbb{E}} Y_{ni} - \hat{\mathbb{E}} X_{ni}) = 0.$$

By the Markov inequality, (3.1) and (3.3), we may draw the conclusion that

$$\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{u_n} Z_{ni} > \varepsilon k_n^{\frac{1}{r}}\right) \\
&= \sum_{n=1}^{\infty} \mathbb{V}\left(\exists i; u_n \le i \le m_n, \text{ such that } |X_{ni}| > \lambda k_n^{\frac{1}{r}}\right) \\
&\le \sum_{n=1}^{\infty} \sum_{i=u_n}^{m_n} \mathbb{V}\left(|X_{ni}| > \lambda k_n^{\frac{1}{r}}\right) \le \sum_{n=1}^{\infty} \sum_{i=u_n}^{m_n} \frac{\hat{\mathbb{E}}|X_{ni}|^r}{\lambda^r k_n} \\
&\le C \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) + C \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r g\left(\frac{|X_{ni}|^r}{h_n}\right) \\
&\le C \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) + C \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} h_n g\left(\frac{|X_{ni}|^r}{h_n}\right) \\
&\le C \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) + C \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} h_n \\
&\le C \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) + C \sum_{n=1}^{\infty} (m_n - u_n) \frac{h_n}{k_n} \\
&< \infty.
\end{aligned}$$

Next we consider I_2. Let $B_n = \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}(Y_{ni} - \hat{\mathbb{E}}Y_{ni})^2$, $x = \varepsilon k_n^{\frac{1}{r}}$, $y = 2\lambda k_n^{\frac{1}{r}}$. Assume $\lambda = \frac{\varepsilon}{2}$, so $y = 2\lambda k_n^{\frac{1}{r}} = \varepsilon k_n^{\frac{1}{r}}$ in Lemma 2.1. For all $u_n \le i \le m_n$, $n \ge 1$, $\varepsilon > 0$, we have

$$\hat{\mathbb{E}}(Y_{ni} - \hat{\mathbb{E}}Y_{ni})^2 \le 2\hat{\mathbb{E}}\bigl(Y_{ni}^2 + (\hat{\mathbb{E}}Y_{ni})^2\bigr) \le 4\hat{\mathbb{E}}Y_{ni}^2$$

and

$$Y_{ni} - \hat{\mathbb{E}}Y_{ni} \le \lambda k_n^{\frac{1}{r}} + \hat{\mathbb{E}}|Y_{ni}| \le \varepsilon k_n^{\frac{1}{r}}.$$

Because of $Y_{ni} - \hat{\mathbb{E}}Y_{ni} \le \lambda k_n^{\frac{1}{r}} + \hat{\mathbb{E}}|Y_{ni}| \le \varepsilon k_n^{\frac{1}{r}}$, we have $\mathbb{V}(\max_{u_n \le i \le m_n}(Y_{ni} - \hat{\mathbb{E}}Y_{ni}) > \varepsilon k_n^{\frac{1}{r}}) = 0$. So we can get

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} \mathbb{V}\Biggl(\sum_{i=u_n}^{m_n}(Y_{ni} - \hat{\mathbb{E}}Y_{ni}) > \varepsilon k_n^{\frac{1}{r}}\Biggr) \\
&\le \sum_{n=1}^{\infty}\biggl\{\mathbb{V}\Bigl(\max_{u_n \le i \le m_n}(Y_{ni} - \hat{\mathbb{E}}Y_{ni}) > \varepsilon k_n^{\frac{1}{r}}\Bigr) + K\exp\biggl(1 - \ln\biggl(1 + \frac{\varepsilon^2 k_n^{\frac{2}{r}}}{B_n}\biggr)\biggr)\biggr\} \\
&= \sum_{n=1}^{\infty} K\exp\biggl(1 - \ln\biggl(1 + \frac{\varepsilon^2 k_n^{\frac{2}{r}}}{B_n}\biggr)\biggr) \le C\sum_{n=1}^{\infty}\exp\ln\biggl(\frac{B_n}{B_n + \varepsilon^2 k_n^{\frac{2}{r}}}\biggr) \\
&\le C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} B_n = C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n}\hat{\mathbb{E}}(Y_{ni} - \hat{\mathbb{E}}Y_{ni})^2 \le C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}Y_{ni}^2 \\
&\le C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}X_{ni}^2 g\biggl(\frac{\mu|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\biggr) + C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}\bigl(\lambda k_n^{\frac{1}{r}}\bigr)^2\biggl(1 - g\biggl(\frac{|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\biggr)\biggr) \\
&\le C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}X_{ni}^2 g\biggl(\frac{\mu|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\biggr) + C\sum_{n=1}^{\infty}\sum_{i=u_n}^{m_n}\mathbb{V}\bigl(|X_{ni}| > \mu\lambda k_n^{\frac{1}{r}}\bigr) \\
&:= I_{21} + I_{22}.
\end{aligned}$$

From the proof of I_1, it is easy to prove that

$$\begin{aligned}
I_{22} &= C\sum_{n=1}^{\infty}\sum_{i=u_n}^{m_n}\mathbb{V}\bigl(|X_{ni}| > \mu\lambda k_n^{\frac{1}{r}}\bigr) \\
&\le C\sum_{n=1}^{\infty}\sum_{i=u_n}^{m_n}\frac{\hat{\mathbb{E}}|X_{ni}|^r}{(\mu\lambda)^r k_n} \\
&< \infty.
\end{aligned}$$

Noting that $g(\frac{\mu|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}) \le 1$ and $g(\frac{\mu^r|X_{ni}|^r}{\lambda^r k_n}) - g(\frac{|X_{ni}|^r}{h_n}) \le I\{\mu h_n < |X_{ni}|^r \le \frac{\lambda^r k_n}{\mu^r}\}$, combining (2.1), (2.3), (3.2) and (3.3) we get

$$\begin{aligned}
I_{21} &= C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} X_{ni}^2 g\left(\frac{|X_{ni}|^r}{h_n}\right) + C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} X_{ni}^2 g\left(\frac{\mu|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\right) \\
&\quad - C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} X_{ni}^2 g\left(\frac{|X_{ni}|^r}{h_n}\right) \\
&\le C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} X_{ni}^2 g\left(\frac{|X_{ni}|^r}{h_n}\right) + C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} X_{ni}^2 \left(g\left(\frac{\mu|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\right) - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&= C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r |X_{ni}|^{2-r} g\left(\frac{|X_{ni}|^r}{h_n}\right) \\
&\quad + C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r |X_{ni}|^{2-r} \left(g\left(\frac{\mu|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\right) - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\le C\sum_{n=1}^{\infty} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r h_n^{\frac{2-r}{r}} g\left(\frac{|X_{ni}|^r}{h_n}\right) \\
&\quad + C\sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{2-r} k_n^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r k_n^{\frac{2-r}{r}} \left(g\left(\frac{\mu|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\right) - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\le C\sum_{n=1}^{\infty} \left(\frac{h_n}{k_n}\right)^{\frac{2-r}{r}} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r + C\sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&\le C\sum_{n=1}^{\infty} \left(\frac{h_n}{k_n}\right)^{\frac{2-r}{r}} \cdot \sup_{n\ge 1} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r + C\sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\
&< \infty.
\end{aligned}$$

So we have $I_2 < \infty$.

Finally, we prove $I_3 \to 0$. We only need to prove $\lim_{n\to\infty} k_n^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} |\hat{\mathbb{E}} Y_{ni} - \hat{\mathbb{E}} X_{ni}| = 0$. There exists n such $h_n \le \lambda^r k_n$, thus $1 - g(\frac{|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}) \le 1 - g(\frac{|X_{ni}|^r}{\lambda^r k_n}) \le 1 - g(\frac{|X_{ni}|^r}{h_n})$. By combining (2.4) and $|\hat{\mathbb{E}} Y_{ni} - \hat{\mathbb{E}} X_{ni}| \le \hat{\mathbb{E}} |Y_{ni} - X_{ni}|$,

$$\begin{aligned}
&\lim_{n\to\infty} k_n^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} |\hat{\mathbb{E}} Y_{ni} - \hat{\mathbb{E}} X_{ni}| \\
&\quad \le \lim_{n\to\infty} k_n^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |Y_{ni} - X_{ni}| = \lim_{n\to\infty} k_n^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |Z_{ni}| \\
&\quad \le \lim_{n\to\infty} k_n^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}| \left(1 - g\left(\frac{|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\right)\right) \\
&\quad \le \lim_{n\to\infty} k_n^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}} |X_{ni}| \left(\frac{|X_{ni}|}{\mu\lambda k_n^{\frac{1}{r}}}\right)^{r-1} \left(1 - g\left(\frac{|X_{ni}|}{\lambda k_n^{\frac{1}{r}}}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\le C\cdot \lim_{n\to\infty} k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{\lambda^r k_n}\right)\right)\\
&\le C\cdot \lim_{n\to\infty} k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{h_n}\right)\right)\\
&= 0.
\end{aligned}
\tag{3.10}
$$

So we obtain $I_3 \to 0$. So (3.4) to be established.

Now we should prove (3.5), because $\{-X_{ni}; u_n \le i \le m_n, n \ge 1\}$ is also an array of rowwise END random variables. We use $\{-X_{ni}; u_n \le i \le m_n, n \ge 1\}$ instead of $\{X_{ni}; u_n \le i \le m_n, n \ge 1\}$ in (3.4), and by $\hat{\mathbb{E}}X_{ni} = -\hat{\mathcal{E}}[-X_{ni}]$, then we get (3.5). Finally, we need to prove (3.6). We have $\hat{\mathbb{E}}X_{ni} = \hat{\mathcal{E}}X_{ni}$, so we obtain

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{V}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni})\right| > \varepsilon k_n^{\frac{1}{r}}\right)\\
&\quad\le \sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni}) > \varepsilon k_n^{\frac{1}{r}}\right) + \sum_{n=1}^{\infty}\mathbb{V}\left(-\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathcal{E}}X_{ni}) > \varepsilon k_n^{\frac{1}{r}}\right)\\
&\quad= \sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni}) > \varepsilon k_n^{\frac{1}{r}}\right) + \sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathcal{E}}X_{ni}) < -\varepsilon k_n^{\frac{1}{r}}\right)\\
&\quad< \infty.
\end{aligned}
$$

The proof is completed. □

Proof of Theorem 3.2 We know $\hat{\mathbb{E}}|X_{ni}|^r(1-g(\frac{|X_{ni}|^r}{h_n})) \le C_{\mathbb{V}}|X_{ni}|^r(1-g(\frac{|X_{ni}|^r}{h_n}))$, hence (3.8) implies (3.3). We have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}k_n^{-1}C_{\mathbb{V}}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni})\right| - \varepsilon k_n^{\frac{1}{r}}\right)_+^r\\
&\quad= \sum_{n=1}^{\infty}k_n^{-1}\int_0^{\infty}\mathbb{V}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni})\right| - \varepsilon k_n^{\frac{1}{r}} > t^{\frac{1}{r}}\right)\mathrm{d}t\\
&\quad\le \sum_{n=1}^{\infty}k_n^{-1}\int_0^{k_n}\mathbb{V}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni})\right| > \varepsilon k_n^{\frac{1}{r}}\right)\mathrm{d}t\\
&\qquad+ \sum_{n=1}^{\infty}k_n^{-1}\int_{k_n}^{\infty}\mathbb{V}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni})\right| > t^{\frac{1}{r}}\right)\mathrm{d}t\\
&\quad\le \sum_{n=1}^{\infty}\mathbb{V}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni})\right| > \varepsilon k_n^{\frac{1}{r}}\right)\\
&\qquad+ \sum_{n=1}^{\infty}k_n^{-1}\int_{k_n}^{\infty}\mathbb{V}\left(\left|\sum_{i=u_n}^{m_n}(X_{ni}-\hat{\mathbb{E}}X_{ni})\right| > t^{\frac{1}{r}}\right)\mathrm{d}t\\
&\quad=: I_4 + I_5.
\end{aligned}
$$

If we want to prove (3.9), it suffices to prove $I_4 < \infty$ and $I_5 < \infty$. Because of Theorem 3.1, we obtain $I_4 < \infty$. For all $u_n \le i \le m_n$, $n \ge 1$, $t \ge k_n$, $\delta > 0$, we define

$$\begin{aligned} Y_{ni} &= -\delta t^{\frac{1}{r}} I\{X_{ni} < -\delta t^{\frac{1}{r}}\} + X_{ni} I\{|X_{ni}| \le \delta t^{\frac{1}{r}}\} + \delta t^{\frac{1}{r}} I\{X_{ni} > \delta t^{\frac{1}{r}}\}, \\ Z_{ni} &= X_{ni} - Y_{ni} = (X_{ni} + \delta t^{\frac{1}{r}}) I\{X_{ni} < -\delta t^{\frac{1}{r}}\} + (X_{ni} - \delta t^{\frac{1}{r}}) I\{X_{ni} > \delta t^{\frac{1}{r}}\}. \end{aligned}$$

Through this, we can get

$$|Z_{ni}| \le |X_{ni}| I\{|X_{ni}| > \delta t^{\frac{1}{r}}\} \le |X_{ni}| \left(1 - g\left(\frac{|X_{ni}|}{\delta t^{\frac{1}{r}}}\right)\right).$$

Next we need to prove $I_5 < \infty$. Let $I_6 = \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}(\sum_{i=u_n}^{m_n} (X_{ni} - \hat{\mathbb{E}} X_{ni}) > t^{\frac{1}{r}})\,\mathrm{d}t$, noting that

$$\begin{aligned} I_6 &= \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} (X_{ni} - \hat{\mathbb{E}} X_{ni}) > t^{\frac{1}{r}}\right) \mathrm{d}t \\ &\le \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} Z_{ni} > \frac{t^{\frac{1}{r}}}{3}\right) \mathrm{d}t + \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} (Y_{ni} - \hat{\mathbb{E}} Y_{ni}) > \frac{t^{\frac{1}{r}}}{3}\right) \mathrm{d}t \\ &\quad + \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(t^{-\frac{1}{r}} \sum_{i=u_n}^{m_n} (\hat{\mathbb{E}} Y_{ni} - \hat{\mathbb{E}} X_{ni}) > \frac{1}{3}\right) \mathrm{d}t \\ &=: I_{61} + I_{62} + I_{63}. \end{aligned}$$

Because of (2.7) and (3.8), we get

$$\begin{aligned} I_{61} &= \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} Z_{ni} > \frac{t^{\frac{1}{r}}}{3}\right) \mathrm{d}t \\ &\le \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}(\exists\, i; u_n \le i \le m_n, \text{such that } |X_{ni}| > \delta t^{\frac{1}{r}})\,\mathrm{d}t \\ &\le \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \sum_{i=u_n}^{m_n} \mathbb{V}(|X_{ni}| > \delta t^{\frac{1}{r}})\,\mathrm{d}t \\ &\le \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} C_{\mathbb{V}}(|X_{ni}|^r I(|X_{ni}|^r > k_n)) \\ &\le \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} C_{\mathbb{V}}\left(|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)\right) \\ &\le \sum_{n=1}^{\infty} k_n^{-1} \sum_{i=u_n}^{m_n} C_{\mathbb{V}}\left(|X_{ni}|^r \left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right)\right) \\ &< \infty. \end{aligned}$$

So we have $I_{61} < \infty$.

Next we consider I_{62}, Let $B_n = \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}(Y_{ni} - \hat{\mathbb{E}}Y_{ni})^2$, $x = \frac{t^{\frac{1}{r}}}{3}$, $y = \frac{t^{\frac{1}{r}}}{6}$ in Lemma 2.1. For all $u_n \le i \le m_n$, $n \ge 1$, $t \ge k_n$, $\delta > 0$, suppose that $\delta = \frac{1}{12}$, there are $Y_{ni} - \hat{\mathbb{E}}Y_{ni} \le 2\delta t^{\frac{1}{r}} \le \frac{t^{\frac{1}{r}}}{6}$ and $\hat{\mathbb{E}}(Y_{ni} - \hat{\mathbb{E}}Y_{ni})^2 \le 4\hat{\mathbb{E}}Y_{ni}^2$. By Lemma 2.1, we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n}(Y_{ni} - \hat{\mathbb{E}}Y_{ni}) > \frac{t^{\frac{1}{r}}}{3}\right) \mathrm{d}t \\
&\quad \le \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\max_{u_n \le i \le m_n} (Y_{ni} - \hat{\mathbb{E}}Y_{ni}) > \frac{t^{\frac{1}{r}}}{6}\right) \mathrm{d}t \\
&\qquad + \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} K \exp\left\{2 - 2\ln\left(1 + \frac{2\delta t^{\frac{2}{r}}}{3B_n}\right)\right\} \mathrm{d}t \\
&\quad \le C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left(t^{-\frac{2}{r}} B_n\right)^2 \mathrm{d}t = C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left(t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}\left(Y_{ni} - \hat{\mathbb{E}}Y_{ni}^2\right)\right)^2 \mathrm{d}t \\
&\quad \le C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left(t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}Y_{ni}^2\right)^2 \mathrm{d}t \\
&\quad \le C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left\{t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}\left[|X_{ni}|^2 g\left(\frac{\mu|X_{ni}|}{\delta t^{\frac{1}{r}}}\right) + \delta^2 t^{\frac{2}{r}}\left(1 - g\left(\frac{|X_{ni}|}{\delta t^{\frac{1}{r}}}\right)\right)\right]\right\}^2 \mathrm{d}t \\
&\quad \le C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left\{t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{\mu|X_{ni}|}{\delta t^{\frac{1}{r}}}\right)\right\}^2 \mathrm{d}t \\
&\qquad + C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left\{\sum_{i=u_n}^{m_n} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X_{ni}|}{\delta t^{\frac{1}{r}}}\right)\right)\right\}^2 \mathrm{d}t \\
&\quad \le C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left\{t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{|X_{ni}|^r}{k_n}\right)\right. \\
&\qquad \left. + t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r |X_{ni}|^{2-r}\left(g\left(\frac{\mu|X_{ni}|}{\delta t^{\frac{1}{r}}}\right) - g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)\right\}^2 \mathrm{d}t \\
&\qquad + C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left(\sum_{i=u_n}^{m_n} \mathbb{V}\left(|X_{ni}| > \mu\delta t^{\frac{1}{r}}\right)\right)^2 \mathrm{d}t \\
&\quad \le C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left\{t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{|X_{ni}|^r}{k_n}\right)\right. \\
&\qquad \left. + t^{-1} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^r \left(g\left(\frac{\mu|X_{ni}|}{\delta t^{\frac{1}{r}}}\right) - g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)\right\}^2 \mathrm{d}t \\
&\qquad + C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left(\sum_{i=u_n}^{m_n} \mathbb{V}\left(|X_{ni}| > \mu\delta t^{\frac{1}{r}}\right)\right)^2 \mathrm{d}t \\
&\quad \le C\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \left\{t^{-\frac{2}{r}} \sum_{i=u_n}^{m_n} \hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{|X_{ni}|^r}{k_n}\right)\right\}^2 \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty}k_n^{-1}\int_{k_n}^{\infty}\left\{t^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)\right\}^2\mathrm{d}t\\
&+C\sum_{n=1}^{\infty}k_n^{-1}\int_{k_n}^{\infty}\left(\sum_{i=u_n}^{m_n}\mathbb{V}\left(|X_{ni}|>\mu\delta t^{\frac{1}{r}}\right)\right)^2\mathrm{d}t\\
&=: I'_{62}+I''_{62}+I'''_{62}.
\end{aligned}
$$

By a similar argument to the proof of I_{21}, we have (2.3), (3.2) and (3.3), then

$$
\begin{aligned}
I'_{62}&=C\sum_{n=1}^{\infty}k_n^{-1}\int_{k_n}^{\infty}\left(t^{-\frac{2}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)^2\mathrm{d}t\\
&\le C\sum_{n=1}^{\infty}k_n^{-1}\left(\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)^2\int_{k_n}^{\infty}t^{-\frac{4}{r}}\,\mathrm{d}t\\
&\le C\sum_{n=1}^{\infty}k_n^{-\frac{4}{r}}\left(\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)^2=C\sum_{n=1}^{\infty}\left(k_n^{-\frac{2}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^2 g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)^2\\
&\le C\sum_{n=1}^{\infty}\left\{\left(\frac{h_n}{k_n}\right)^{\frac{2-r}{r}}k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r+k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{h_n}\right)\right)\right\}^2\\
&\le C\left(\sum_{n=1}^{\infty}\left(\frac{h_n}{k_n}\right)^{\frac{2-r}{r}}\cdot\sup_{n\ge1}k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\right)^2\\
&\quad+C\left\{\sum_{n=1}^{\infty}k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{h_n}\right)\right)\right\}^2\\
&<\infty.
\end{aligned}
$$

Because of (2.7) and (3.3), it is obvious that

$$
\begin{aligned}
I''_{62}&=C\sum_{n=1}^{\infty}k_n^{-1}\left\{\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)\right\}^2\int_{k_n}^{\infty}t^{-2}\,\mathrm{d}t\\
&\le C\left\{\sum_{n=1}^{\infty}k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{k_n}\right)\right)\right\}^2\\
&\le C\left\{\sum_{n=1}^{\infty}k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1-g\left(\frac{|X_{ni}|^r}{h_n}\right)\right)\right\}^2\\
&<\infty.
\end{aligned}
$$

Similar to the proof of I_{22}, then

$$\begin{aligned} I_{62}''' &\le C\sum_{n=1}^{\infty} k_n^{-1}\int_{k_n}^{\infty}\left(\sum_{i=u_n}^{m_n}\frac{\hat{\mathbb{E}}|X_{ni}|^r}{\mu^r\delta^r t}\right)^2 \mathrm{d}t \\ &\le C\sum_{n=1}^{\infty} k_n^{-1}\left(\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\right)^2\int_{k_n}^{\infty} t^{-2}\,\mathrm{d}t \\ &\le C\left(\sum_{n=1}^{\infty} k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\right)^2 \\ &< \infty. \end{aligned}$$

That is to say $I_{62} < \infty$.

By a similar argument to the proof of (3.10) and for I_{63}, t is greater than k_n, it follows that $t^{-\frac{1}{r}} < k_n^{-\frac{1}{r}}$, there exists n such that $h_n \le \delta^r k_n < \delta^r t$, thus $1 - g(\frac{|X_{ni}|}{\delta t^{\frac{1}{r}}}) < 1 - g(\frac{|X_{ni}|}{\delta k_n^{\frac{1}{r}}}) \le 1 - g(\frac{|X_{ni}|^r}{\delta^r k_n}) \le 1 - g(\frac{|X_{ni}|^r}{h_n})$. Then we can get

$$\begin{aligned} &\sup_{t\ge k_n} t^{-\frac{1}{r}}\left|\sum_{i=u_n}^{m_n}(\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_{ni})\right| \\ &\quad\le \sup_{t\ge k_n} t^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}|\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_{ni}| \le \sup_{t\ge k_n} t^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|Y_{ni} - X_{ni}| \\ &\quad= \sup_{t\ge k_n} t^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|Z_{ni}| \le \sup_{t\ge k_n} t^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|\left(1 - g\left(\frac{|X_{ni}|}{\delta t^{\frac{1}{r}}}\right)\right) \\ &\quad\le k_n^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|\left(1 - g\left(\frac{|X_{ni}|}{\delta k_n^{\frac{1}{r}}}\right)\right) \\ &\quad\le k_n^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|\left(\frac{|X_{ni}|}{\mu\delta k_n^{\frac{1}{r}}}\right)^{r-1}\left(1 - g\left(\frac{|X_{ni}|}{\delta k_n^{\frac{1}{r}}}\right)\right) \\ &\quad\le C\cdot k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1 - g\left(\frac{|X_{ni}|^r}{\delta^r k_n}\right)\right) \\ &\quad\le C\cdot k_n^{-1}\sum_{i=u_n}^{m_n}\hat{\mathbb{E}}|X_{ni}|^r\left(1 - g\left(\frac{|X_{ni}|^r}{h_n}\right)\right) \\ &\quad\to 0. \end{aligned}$$

We are conscious of $I_{63} = \sum_{n=1}^{\infty} k_n^{-1}\int_{k_n}^{\infty}\mathbb{V}(t^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}(\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_{ni}) > \frac{1}{3})\,\mathrm{d}t$, on the other hand, we obtain $\sup_{t\ge k_n} t^{-\frac{1}{r}}|\sum_{i=u_n}^{m_n}(\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_{ni})| \to 0$ as $n \to \infty$, then $\sup_{t\ge k_n} t^{-\frac{1}{r}} \times \sum_{i=u_n}^{m_n}(\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_{ni}) \to 0$ as $n \to \infty$, so n is sufficiently large, we know $\mathbb{V}(t^{-\frac{1}{r}}\sum_{i=u_n}^{m_n}(\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_{ni}) > \frac{1}{3}) \le 0$, and we get $I_{63} < \infty$. Combining $I_{61} < \infty$, $I_{62} < \infty$ and $I_{63} < \infty$, then we have $I_6 < \infty$. We use $\{-X_{ni}; u_n \le i \le m_n, n \ge 1\}$ instead of $\{X_{ni}; u_n \le i \le m_n, n \ge 1\}$ in I_6, and by

$\hat{\mathbb{E}}X_{ni} = \hat{\mathcal{E}}X_{ni}$, then we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} \bigl(-X_{ni} - \hat{\mathbb{E}}(-X_{ni})\bigr) > t^{\frac{1}{r}}\right) \mathrm{d}t \\
&\quad = \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} -(X_{ni} - \hat{\mathbb{E}}X_{ni}) > t^{\frac{1}{r}}\right) \mathrm{d}t \\
&\quad < \infty.
\end{aligned}$$

By $I_6 < \infty$, it is obvious that

$$\begin{aligned}
I_5 &= \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\left|\sum_{i=u_n}^{m_n} (X_{ni} - \hat{\mathbb{E}}X_{ni})\right| > t^{\frac{1}{r}}\right) \mathrm{d}t \\
&\le \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} (X_{ni} + \hat{\mathbb{E}}X_{ni}) > t^{\frac{1}{r}}\right) \mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} k_n^{-1} \int_{k_n}^{\infty} \mathbb{V}\left(\sum_{i=u_n}^{m_n} -(X_{ni} - \hat{\mathbb{E}}X_{ni}) > t^{\frac{1}{r}}\right) \mathrm{d}t \\
&< \infty.
\end{aligned}$$

So we obtain $I_5 < \infty$, in other words, (3.9) is proved and the proof is completed. □

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