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Approximation properties of generalized Baskakov–Schurer–Szasz–Stancu operators preserving $e^{-2ax}, a > 0$

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Abstract

The current paper deals with a modified form of the Baskakov–Schurer–Szasz–Stancu operators which preserve e^{-2ax} for $a > 0$. The uniform convergence of the modified operators is shown. The rate of convergence is investigated by using the usual modulus of continuity and the exponential modulus of continuity. Then Voronovskaya-type theorem is given for quantitative asymptotic estimation.

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1 Introduction

Use of linear positive operators has played a crucial role in approximation theory for the last seven decades. In 1950, Szász [22] defined

$$S_s(f;x) = e^{-sx}\sum_{r=0}^{\infty}\frac{(sx)^r}{r!}f\left(\frac{r}{s}\right), \quad s > 0$$

for $x \in [0,\infty)$. In 1957, Baskakov [6] proposed

$$L_s(f;x) = \frac{1}{(1+x)^s}\sum_{r=0}^{\infty}\binom{s+r-1}{r}\frac{x^r}{(1+x)^r}f\left(\frac{r}{s}\right), \quad s \in \mathbb{N}^+, x \in [0,\infty).$$

In 1962, Schurer [20] introduced

$$B_{s,p}(f;x) = \sum_{r=0}^{s+p}\binom{s+p}{r}x^r(1-x)^{s+p-r}f\left(\frac{r}{s}\right),$$

where $x \in [0,1]$ and p is a non-negative integer. In 1983, Stancu [21] studied

$$S_s^{\alpha,\beta}(f;x) = \sum_{r=0}^{s}\binom{s}{r}x^r(1-x)^{s-r}f\left(\frac{r+\alpha}{s+\beta}\right),$$

satisfying the condition $0 \leq \alpha \leq \beta$. Various studies related to these operators, such as Baskakov–Szász type operators [12], Baskakov–Schurer–Szász operators [18], Baskakov–Szász–Stancu operators [17], and q-Baskakov–Schurer–Szász–Stancu operators [19], have been conducted.

In 2010, Aldaz and Render [4] introduced linear and positive operators preserving 1 and e^x. In 2017, Acar et al. [2] examined a modified form of the Szász–Mirakyan operators which reproduces constant and $e^{2ax}, a > 0$. After that, in some studies Szász–Mirakyan operators [5], Baskakov–Szász–Mirakyan-type operators [10], Phillips operators [13], Szász–Mirakyan–Kantorovich operators [11], and Baskakov operators [24] preserving constant and exponential function were examined. In addition, Kajla [16] studied Srivastava–Gupta operators preserving linear functions. On the other hand, Gupta and Tachev [14] found the general estimation in terms of Păltăneaś modulus of continuity.

In 2018, Bodur et al. [7] analyzed Baskakov–Szász–Stancu operators preserving exponential functions. Motivated by this paper, we construct a new generalization of the Baskakov–Schurer–Szász–Stancu operators

$$
\begin{aligned}
M_{s,p}^{\alpha,\beta}(f;x) = (s+p)\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{x^r}{(1+x)^{s+p+r}} \\
\times \int_0^{\infty} e^{-(s+p)t}\frac{(s+p)^r t^r}{r!} f\left(\frac{(s+p)t+\alpha}{s+p+\beta}\right) dt,
\end{aligned}
$$

where s is a positive integer, p is a non-negative integer, and $0 \leq \alpha \leq \beta$. By taking $\alpha = 0$ and $\beta = 0$, we obtain Baskakov–Schurer–Szász operators [18]. In addition, by taking $s + p = u$, we get Baskakov–Szász–Stancu operators [17]. Moreover, by taking $s + p = u$, $\alpha = 0$ and $\beta = 0$, we have Baskakov–Szász operators [12]. We deal with the following modified form:

$$
\begin{aligned}
M_{s}^{\alpha,\beta}(f;x) = (s+p)\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}} \\
\times \int_0^{\infty} e^{-(s+p)t}\frac{(s+p)^r t^r}{r!} f\left(\frac{(s+p)t+\alpha}{s+p+\beta}\right) dt.
\end{aligned}
\tag{1}
$$

Assume that the operators (1) preserve $e^{-2ax}, a > 0$. In that case, we find the function $\nu_{s,p}(x)$ satisfying $M_s^{\alpha,\beta}(e^{-2at};x) = e^{-2ax}$ as follows:

$$
\begin{aligned}
e^{-2ax} &= (s+p)\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}\int_0^{\infty} e^{-(s+p)t}\frac{(s+p)^r t^r}{r!} e^{\frac{-2a((s+p)t+\alpha)}{s+p+\beta}}\, dt \\
&= \frac{s+p+\beta}{s+p+\beta+2a} e^{\frac{-2a\alpha}{s+p+\beta}} \sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\left(\frac{(s+p+\beta)\nu_{s,p}(x)}{s+p+\beta+2a}\right)^r \frac{1}{(1+\nu_{s,p}(x))^{s+p+r}}.
\end{aligned}
$$

By a simple computation, we have

$$
\nu_{s,p}(x) = \frac{s+p+\beta+2a}{2a}\left\{\left(\frac{s+p+\beta+2a}{s+p+\beta} e^{\frac{2a\alpha}{s+p+\beta}-2ax}\right)^{-1/s+p} - 1\right\}. \tag{2}
$$

2 Some auxiliary results

Here, we present some important equalities and auxiliary lemmas, necessary for the proof of the main theorems.

$$\int_0^\infty t^r e^{-At}\,dt = \frac{r!}{A^{r+1}}, \quad A > 0. \tag{3}$$

Negative binomial series is given as follows:

$$(x+a)^{-s} = \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{(-x)^r}{a^{s+r}}. \tag{4}$$

Lemma 1 *Let $v_{s,p}(x)$ be given by (2), then we have*

$$M_s^{\alpha,\beta}\left(e^{-At};x\right) = \frac{s+p+\beta}{s+p+\beta+A} e^{\frac{-A\alpha}{s+p+\beta}} \left(1 + \frac{Av_{s,p}(x)}{s+p+\beta+A}\right)^{-(s+p)}. \tag{5}$$

Proof Take $f(t) = e^{-At}$, then by using (3) and (4) we obtain

$$\begin{aligned}
&M_s^{\alpha,\beta}\left(e^{-At};x\right) \\
&= (s+p)\sum_{r=0}^{\infty} \binom{s+p+r-1}{r} \frac{(v_{s,p}(x))^r}{(1+v_{s,p}(x))^{s+p+r}} \frac{(s+p)^r}{r!} \int_0^\infty t^r e^{-(s+p)t-(\frac{A(s+p)t+A\alpha}{s+p+\beta})}\,dt \\
&= e^{\frac{-A\alpha}{s+p+\beta}} \sum_{r=0}^{\infty} \binom{s+p+r-1}{r} \frac{(v_{s,p}(x))^r}{(1+v_{s,p}(x))^{s+p+r}} \frac{(s+p)^{r+1}}{r!} \int_0^\infty t^r e^{-\frac{(s+p)(s+p+\beta+A)t}{s+p+\beta}}\,dt \\
&= \frac{s+p+\beta}{s+p+\beta+A} e^{\frac{-A\alpha}{s+p+\beta}} \sum_{r=0}^{\infty} \binom{s+p+r-1}{r} \left(\frac{(s+p+\beta)v_{s,p}(x)}{s+p+\beta+A}\right)^r \frac{1}{(1+v_{s,p}(x))^{s+p+r}} \\
&= \frac{s+p+\beta}{s+p+\beta+A} e^{\frac{-A\alpha}{s+p+\beta}} \left(1 + \frac{Av_{s,p}(x)}{s+p+\beta+A}\right)^{-(s+p)}.
\end{aligned}$$

□

Lemma 2 *Let $e_k(t) = t^k, k = 0,1,2,3,4$. Then we get the following equalities*:

$$\begin{aligned}
M_s^{\alpha,\beta}(e_0;x) &= 1, \\
M_s^{\alpha,\beta}(e_1;x) &= \frac{(s+p)v_{s,p}(x)+\alpha+1}{s+p+\beta}, \\
M_s^{\alpha,\beta}(e_2;x) &= \frac{(s+p)(s+p+1)v_{s,p}^2(x)+(4+2\alpha)(s+p)v_{s,p}(x)+\alpha^2+2\alpha+2}{(s+p+\beta)^2}, \\
M_s^{\alpha,\beta}(e_3;x) &= \frac{(s+p)(s+p+1)(s+p+2)v_{s,p}^3(x)+(9+3\alpha)(s+p)(s+p+1)v_{s,p}^2(x)}{(s+p+\beta)^3} \\
&\quad + \frac{(3\alpha^2+12\alpha+18)(s+p)v_{s,p}(x)+\alpha^3+3\alpha^2+6\alpha+6}{(s+p+\beta)^3},
\end{aligned}$$

$$\begin{aligned}
&M_s^{\alpha,\beta}(e_4;x) \\
&= \frac{(s+p)(s+p+1)(s+p+2)(s+p+3)v_{s,p}^4(x)+(16+4\alpha)(s+p)(s+p+1)(s+p+2)v_{s,p}^3(x)}{(s+p+\beta)^4} \\
&\quad + \frac{(72+36\alpha+6\alpha^2)(s+p)(s+p+1)v_{s,p}^2(x)+(96+72\alpha+24\alpha^2+4\alpha^3)(s+p)v_{s,p}(x)}{(s+p+\beta)^4}
\end{aligned}$$

$$+\frac{24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(s+p+\beta)^4}.$$

Proof Take $f(t)=e_1$, then by using (3) and (4) we have

$$\begin{aligned}
&M_s^{\alpha,\beta}(e_1;x)\\
&=(s+p)\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}\int_0^{\infty}e^{-(s+p)t}\frac{(s+p)^r t^r}{r!}\left(\frac{(s+p)t+\alpha}{s+p+\beta}\right)dt\\
&=\frac{1}{s+p+\beta}\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}\frac{(s+p)^{r+2}}{r!}\int_0^{\infty}t^{r+1}e^{-(s+p)t}\,dt\\
&\quad+\frac{\alpha}{s+p+\beta}\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}\frac{(s+p)^{r+1}}{r!}\int_0^{\infty}t^{r}e^{-(s+p)t}\,dt\\
&=\frac{1}{s+p+\beta}\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}(r+1)\\
&\quad+\frac{\alpha}{s+p+\beta}\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}\\
&=\frac{(s+p)\nu_{s,p}(x)}{s+p+\beta}\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}\\
&\quad+\frac{\alpha+1}{s+p+\beta}\sum_{r=0}^{\infty}\binom{s+p+r-1}{r}\frac{(\nu_{s,p}(x))^r}{(1+\nu_{s,p}(x))^{s+p+r}}\\
&=\frac{(s+p)\nu_{s,p}(x)}{s+p+\beta}\bigl(1+\nu_{s,p}(x)-\nu_{s,p}(x)\bigr)^{-(s+p+1)}+\frac{\alpha+1}{s+p+\beta}\bigl(1+\nu_{s,p}(x)-\nu_{s,p}(x)\bigr)^{-(s+p)}\\
&=\frac{(s+p)\nu_{s,p}(x)}{s+p+\beta}+\frac{\alpha+1}{s+p+\beta}.
\end{aligned}$$

By the same manner, other results can be obtained. □

Lemma 3 *Let us briefly denote $\phi_x^k(t)=(t-x)^k$ for $k=0,1,2,4$. Then we obtain the following equalities for the central moments*:

$$\begin{aligned}
&M_s^{\alpha,\beta}(\phi_x^0;x)=1,\\
&M_s^{\alpha,\beta}(\phi_x^1;x)=\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta}-x,\\
&M_s^{\alpha,\beta}(\phi_x^2;x)\\
&\quad=\frac{(s+p)(s+p+1)\nu_{s,p}^2(x)+(4+2\alpha)(s+p)\nu_{s,p}(x)+\alpha^2+2\alpha+2}{(s+p+\beta)^2}\\
&\qquad-\frac{2x((s+p)\nu_{s,p}(x)+\alpha+1)}{s+p+\beta}+x^2,\\
&M_s^{\alpha,\beta}(\phi_x^4;x)\\
&\quad=\frac{(s+p)(s+p+1)(s+p+2)(s+p+3)\nu_{s,p}^4(x)+(16+4\alpha)(s+p)(s+p+1)(s+p+2)\nu_{s,p}^3(x)}{(s+p+\beta)^4}\\
&\qquad+\frac{(72+36\alpha+6\alpha^2)(s+p)(s+p+1)\nu_{s,p}^2(x)+(96+72\alpha+24\alpha^2+4\alpha^3)(s+p)\nu_{s,p}(x)}{(s+p+\beta)^4}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4}{(s+p+\beta)^4}-4x\biggl(\frac{(s+p)(s+p+1)(s+p+2)v_{s,p}^3(x)}{(s+p+\beta)^3}\\
&+\frac{(9+3\alpha)(s+p)(s+p+1)v_{s,p}^2(x)+(3\alpha^2+12\alpha+18)(s+p)v_{s,p}(x)+\alpha^3+3\alpha^2+6\alpha+6}{(s+p+\beta)^3}\biggr)\\
&+6x^2\biggl(\frac{(s+p)(s+p+1)v_{s,p}^2(x)+(4+2\alpha)(s+p)v_{s,p}(x)+\alpha^2+2\alpha+2}{(s+p+\beta)^2}\biggr)\\
&-4x^3\biggl(\frac{(s+p)v_{s,p}(x)+\alpha+1}{s+p+\beta}\biggr)+x^4.
\end{aligned}
$$

Proof We use the linearity of the $M_s^{\alpha,\beta}$ operators and Lemma 2 $M_s^{\alpha,\beta}(\phi_x^0;x)=M_s^{\alpha,\beta}(e_0;x)$, $M_s^{\alpha,\beta}(\phi_x^1;x)=M_s^{\alpha,\beta}(e_1;x)-xM_s^{\alpha,\beta}(e_0;x)$, $M_s^{\alpha,\beta}(\phi_x^2;x)=M_s^{\alpha,\beta}(e_2;x)-2xM_s^{\alpha,\beta}(e_1;x)+x^2M_s^{\alpha,\beta}(e_0;x)$, $M_s^{\alpha,\beta}(\phi_x^4;x)=M_s^{\alpha,\beta}(e_4;x)-4xM_s^{\alpha,\beta}(e_3;x)+6x^2M_s^{\alpha,\beta}(e_2;x)-4x^3M_s^{\alpha,\beta}(e_1;x)+x^4M_s^{\alpha,\beta}(e_0;x)$. □

Remark 1 Considering the definition of $v_{s,p}(x)$, we obtain the following limits for every $x\in[0,\infty)$ and $0\le\alpha\le\beta$:

$$\lim_{s\to\infty} sM_s^{\alpha,\beta}\bigl(\phi_x^1;x\bigr)=2ax+ax^2 \tag{6}$$

and

$$\lim_{s\to\infty} sM_s^{\alpha,\beta}\bigl(\phi_x^2;x\bigr)=2x+x^2. \tag{7}$$

3 Main results

Let $C^*[0,\infty)$ denote the subspace of all real-valued continuous functions on $[0,\infty)$ with the condition that $\lim_{m\to\infty}f(x)$ exists and is finite, equipped with the uniform norm. The uniform convergence of a sequence of linear positive operators is demonstrated by Boyanov and Veselinov [8]. We present the following theorem according to [8] for the newly constructed operators (1).

Theorem 1 *If the linear positive operators* (1) *satisfy*

$$\lim_{s\to\infty} M_s^{\alpha,\beta}\bigl(e^{-mt};x\bigr)=e^{-mx},\quad m=0,1,2, \tag{8}$$

uniformly in $[0,\infty)$, *then for each* $f\in C^*[0,\infty)$

$$\lim_{s\to\infty} M_s^{\alpha,\beta}(f;x)=f(x) \tag{9}$$

uniformly in $[0,\infty)$.

Proof We have already known that $\lim_{s\to\infty}M_s^{\alpha,\beta}(1;x)=1$. Considering equality (5) with $v_{s,p}(x)$ given in (2), we have

$$
\begin{aligned}
M_s^{\alpha,\beta}\bigl(e^{-t};x\bigr)&=\frac{s+p+\beta}{s+p+\beta+1}e^{\frac{-\alpha}{s+p+\beta}}\biggl(1+\frac{v_{s,p}(x)}{s+p+\beta+1}\biggr)^{-(s+p)}\\
&=e^{-x}+\frac{(1-2a)(2+x)xe^{-x}}{2(s+p)}+\mathcal{O}\bigl((s+p)^{-2}\bigr)
\end{aligned} \tag{10}
$$

and

$$M_s^{\alpha,\beta}\left(e^{-2t};x\right) = \frac{s+p+\beta}{s+p+\beta+2} e^{\frac{-2\alpha}{s+p+\beta}} \left(1 + \frac{2\nu_{s,p}(x)}{s+p+\beta+2}\right)^{-(s+p)}$$
$$= e^{-2x} + \frac{(1-a)(4+2x)xe^{-2x}}{(s+p)} + \mathcal{O}\left((s+p)^{-2}\right). \tag{11}$$

Hence, we prove that

$$\lim_{s\to\infty} M_s^{\alpha,\beta}\left(e^{-mt};x\right) = e^{-mx}, \quad m = 0,1,2,$$

uniformly in $[0,\infty)$. This means that, for any $f \in C^*[0,\infty)$, $\lim_{s\to\infty} M_s^{\alpha,\beta}(f;x) = f(x)$ uniformly in $[0,\infty)$. □

After about four decades later than Boyanov and Veselinov [8], Holhoş [15] studied the uniform convergence of a sequence of linear positive operators. He obtained the following theorem for an effective estimation of the linear positive operators.

Theorem 2 ([15]) *For a sequence of linear positive operators $A_s : C^*[0,\infty) \longrightarrow C^*[0,\infty)$, we have*

$$\|A_s f - f\|_{[0,\infty)} \le \|f\|_{[0,\infty)}\delta_s + (2+\delta_s)\omega^*(f, \sqrt{\delta_s + 2\sigma_s + \rho_s})$$

for every function $f \in C^*[0,\infty)$, where

$$\left\|A_s(e_0) - 1\right\|_{[0,\infty)} = \delta_s,$$
$$\left\|A_s\left(e^{-t}\right) - e^{-x}\right\|_{[0,\infty)} = \sigma_s,$$
$$\left\|A_s\left(e^{-2t}\right) - e^{-2x}\right\|_{[0,\infty)} = \rho_s$$

and $\omega^*(f,\eta) = \sup_{|e^{-x}-e^{-t}|\le\eta, x,t>0} |f(t) - f(x)|$ denotes the modulus of continuity. Here, δ_s, σ_s, and ρ_s tend to zero as $s \to \infty$.

In the same manner with the above theorem, we present a quantitative estimation of the Baskakov–Schurer–Szasz–Stancu operators which preserve $e^{-2ax}, a > 0$, as follows.

Theorem 3 *For $f \in C^*[0,\infty)$, we have the following inequality*:

$$\left\|M_s^{\alpha,\beta} f - f\right\|_{[0,\infty)} \le 2\omega^*(f, \sqrt{2\sigma_{s,p} + \rho_{s,p}}), \tag{12}$$

where

$$\left\|M_s^{\alpha,\beta}\left(e^{-t}\right) - e^{-x}\right\|_{[0,\infty)} = \sigma_{s,p},$$
$$\left\|M_s^{\alpha,\beta}\left(e^{-2t}\right) - e^{-2x}\right\|_{[0,\infty)} = \rho_{s,p}.$$

Here, $\sigma_{s,p}$ and $\rho_{s,p}$ tend to zero as $s \to \infty$. So, $M_s^{\alpha,\beta} f$ converges to f uniformly.

Proof The Baskakov–Schurer–Szasz–Stancu operators $M_s^{\alpha,\beta}$ preserve constant. Thus, $\delta_{s,p} = \|M_s^{\alpha,\beta}(e_0) - 1\|_{[0,\infty)} = 0$. In order to calculate $\sigma_{s,p}$, we take into consideration equality (10). So, we obtain

$$M_s^{\alpha,\beta}\big(e^{-t};x\big) - e^{-x} = \frac{(1-2a)(2+x)xe^{-x}}{2(s+p)} + \mathcal{O}\big((s+p)^{-2}\big).$$

Since

$$\sup_{x\in[0,\infty)} xe^{-x} = \frac{1}{e}, \qquad \sup_{x\in[0,\infty)} x^2e^{-x} = \frac{4}{e^2},$$

we achieve

$$\sigma_{s,p} = \big\|M_s^{\alpha,\beta}\big(e^{-t}\big) - e^{-x}\big\|_{[0,\infty)} = \frac{(1-2a)}{e(s+p)} + \frac{2(1-2a)}{e^2(s+p)} + \mathcal{O}\big((s+p)^{-2}\big).$$

In the same way, with the help of equality (11), we have

$$M_s^{\alpha,\beta}\big(e^{-2t};x\big) - e^{-2x} = \frac{(1-a)(4+2x)xe^{-2x}}{(s+p)} + \mathcal{O}\big((s+p)^{-2}\big).$$

By using

$$\sup_{x\in[0,\infty)} xe^{-2x} = \frac{1}{2e}, \qquad \sup_{x\in[0,\infty)} x^2e^{-2x} = \frac{1}{e^2},$$

we get

$$\rho_{s,p} = \big\|M_s^{\alpha,\beta}\big(e^{-2t}\big) - e^{-2x}\big\|_{[0,\infty)} = \frac{2(1-a)}{e(s+p)} + \frac{2(1-a)}{e^2(s+p)} + \mathcal{O}\big((s+p)^{-2}\big).$$

Consequently, $\sigma_{s,p}$ and $\rho_{s,p}$ tend to zero as $s \to \infty$. □

In Sect. 4, we investigate the rate of convergence by using the usual modulus of continuity.

4 The usual modulus of continuity

The class of all bounded and uniform continuous functions f on $[0,\infty)$ is denoted by $C_B[0,\infty)$ endowed with the norm $\|f\|_{C_B} = \sup_{x\geq 0}|f(x)|$. For $f \in C_B[0,\infty)$, the modulus of continuity is given by

$$\omega(f,\delta) := \sup_{0<h<\delta}\ \sup_{x,x+h\in[0,\infty)} \big|f(x+h) - f(x)\big|.$$

The second order modulus of continuity of the function $f \in C_B[0,\infty)$ is defined by

$$\omega_2(f,\delta) := \sup_{0<h<\sqrt{\delta}}\ \sup_{x,x+h\in[0,\infty)} \big|f(x+2h) - 2f(x+h) + f(x)\big|,$$

where $\delta > 0$. Peetre's K-functionals are described as

$$K_2(f,\delta) := \inf_{g\in C_B^2[0,\infty)} \big\{\|f-g\|_{C_B[0,\infty)} + \delta\|g\|_{C_B^2[0,\infty)}\big\}.$$

Here, $C_B^2[0,\infty)$ denotes the space of the functions f, for which f' and f'' belong to $C_B[0,\infty)$. The relation between the second order modulus of continuity and Peetre's K-functional is given by [9]

$$K_2(f,\delta) \le M\omega_2(f,\sqrt{\delta}),$$

where $M > 0$.

Lemma 4 *For $f \in C_B[0,\infty)$, we have $|M_s^{\alpha,\beta}(f;x)| \le \|f\|$.*

Theorem 4 *Let $f \in C_B[0,\infty)$. Then, for all $x \in [0,\infty)$, there exists a positive constant M such that*

$$\left|M_s^{\alpha,\beta}(f;x) - f(x)\right| \le M\omega_2(f,\sqrt{\mu_{s,p}}) + \omega\left(f, \left|\frac{(s+p)v_{s,p}(x)+\alpha+1}{s+p+\beta}\right|\right), \tag{13}$$

where

$$\begin{aligned}\mu_{s,p} = {}& \frac{(s+p)(2s+2p+1)}{(s+p+\beta)^2}v_{s,p}^2(x) + \left(\frac{(4\alpha+6)(s+p)}{(s+p+\beta)^2} - \frac{4x(s+p)}{s+p+\beta}\right)v_{s,p}(x) \\ & + \frac{2\alpha^2+4\alpha+3}{(s+p+\beta)^2} - \frac{4x(\alpha+1)}{s+p+\beta} + 2x^2.\end{aligned} \tag{14}$$

Here, $v_{s,p}(x)$ is the same as in (2).

Proof We define the auxiliary operators $\tilde{M}_s^{\alpha,\beta} : C_B[0,\infty) \to C_B[0,\infty)$

$$\tilde{M}_s^{\alpha,\beta}(g;x) = M_s^{\alpha,\beta}(g;x) + g(x) - g\left(\frac{(s+p)v_{s,p}(x)+\alpha+1}{s+p+\beta}\right), \tag{15}$$

where $v_{s,p}(x)$ is as given by (2). Note that the operators (15) are positive and linear. By using the Taylor expansion for $g \in C_B^2[0,\infty)$, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du, \quad x,t \in [0,\infty). \tag{16}$$

Applying $\tilde{M}_s^{\alpha,\beta}$ operators to the both sides of equation (16) and using Lemma 3, we obtain

$$\begin{aligned}&\left|\tilde{M}_s^{\alpha,\beta}(g;x) - g(x)\right| \\ &\quad = \left|\tilde{M}_s^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| \\ &\quad \le \left|M_s^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| \\ &\qquad + \left|\int_x^{\frac{(s+p)v_{s,p}(x)+\alpha+1}{s+p+\beta}} \left(\frac{(s+p)v_{s,p}(x)+\alpha+1}{s+p+\beta} - u\right)g''(u)\,du\right|.\end{aligned} \tag{17}$$

Further,

$$\left|M_s^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right|$$

$$\leq M_s^{\alpha,\beta}\left(\int_x^t |t-u|\left|g''(u)\right| du; x\right) \leq \left\|g''\right\| M_s^{\alpha,\beta}\left(\phi_x^2; x\right) \tag{18}$$

and

$$\begin{aligned}
&\left|\int_x^{\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta}} \left(\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta} - u\right) g''(u)\, du\right| \\
&\quad \leq \left\|g''\right\| \left(\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta} - x\right)^2.
\end{aligned} \tag{19}$$

Rewrite (18) and (19) in (17), then we have

$$\begin{aligned}
&\left|\tilde{M}_s^{\alpha,\beta}(g;x) - g(x)\right| \\
&\quad \leq \left\|g''\right\| \left(M_s^{\alpha,\beta}\left(\phi_x^2; x\right) + \left(\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta} - x\right)^2\right) \\
&\quad = \left\|g''\right\| \left(\frac{(s+p)(s+p+1)\nu_{s,p}^2(x) + (4+2\alpha)(s+p)\nu_{s,p}(x) + \alpha^2 + 2\alpha + 2}{(s+p+\beta)^2}\right. \\
&\qquad \left. - \frac{2x((s+p)\nu_{s,p}(x)+\alpha+1)}{s+p+\beta} + x^2 + \left(\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta} - x\right)^2\right) \\
&\quad = \left\|g''\right\| \left(\frac{(s+p)(2s+2p+1)}{(s+p+\beta)^2}\nu_{s,p}^2(x) + \left(\frac{(4\alpha+6)(s+p)}{(s+p+\beta)^2} - \frac{4x(s+p)}{s+p+\beta}\right)\nu_{s,p}(x)\right. \\
&\qquad \left. + \frac{2\alpha^2+4\alpha+3}{(s+p+\beta)^2} - \frac{4x(\alpha+1)}{s+p+\beta} + 2x^2\right) \\
&\quad := \left\|g''\right\| \mu_{s,p},
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\mu_{s,p} = {}& \frac{(s+p)(2s+2p+1)}{(s+p+\beta)^2}\nu_{s,p}^2(x) + \left(\frac{(4\alpha+6)(s+p)}{(s+p+\beta)^2} - \frac{4x(s+p)}{s+p+\beta}\right)\nu_{s,p}(x) \\
&+ \frac{2\alpha^2+4\alpha+3}{(s+p+\beta)^2} - \frac{4x(\alpha+1)}{s+p+\beta} + 2x^2.
\end{aligned} \tag{21}$$

By using the auxiliary operators (15) and Lemma 4, we get

$$\left\|\tilde{M}_s^{\alpha,\beta}(f;x)\right\| \leq \left\|M_s^{\alpha,\beta}(f;x)\right\| + 2\|f\| \leq 3\|f\|. \tag{22}$$

From (15), (20), and (22), for every $g \in C_B^2[0,\infty)$, we obtain

$$\begin{aligned}
&\left|M_s^{\alpha,\beta}(f;x) - f(x)\right| \\
&\quad = \left|\tilde{M}_s^{\alpha,\beta}(f;x) - f(x) + f\left(\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta}\right) - f(x)\right. \\
&\qquad \left. + \tilde{M}_s^{\alpha,\beta}(g;x) - \tilde{M}_s^{\alpha,\beta}(g;x) + g(x) - g(x)\right| \\
&\quad \leq \left|\tilde{M}_s^{\alpha,\beta}(f-g;x) - (f-g)(x)\right| + \left|f\left(\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta}\right) - f(x)\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\tilde{M}_s^{\alpha,\beta}(g;x) - g(x)\right| \\
&\leq 4\|f-g\| + \|g''\|\mu_{s,p} + \left|f\left(\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta}\right) - f(x)\right| \\
&\leq K_2(f,\mu_{s,p}) + \omega\left(f, \left|\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta} - x\right|\right) \\
&\leq M\omega_2(f,\sqrt{\mu_{s,p}}) + \omega\left(f, \left|\frac{(s+p)\nu_{s,p}(x)+\alpha+1}{s+p+\beta} - x\right|\right).
\end{aligned}
\tag{23}
$$

□

Remark 2 We see that $\mu_{s,p} = \frac{x(2+x)}{s+p} + O((s+p)^{-2}) \to 0$, when $s \to \infty$. This result guarantees the convergence of Theorem 4.

In Sect. 5, we obtain the rate of convergence by using the exponential modulus of continuity.

5 The exponential modulus of continuity

For $f \in C[0,\infty)$, the exponential growth of order $B > 0$ is given by

$$
\|f\|_B := \sup_{x\in[0,\infty)} \left|f(x)e^{-Bx}\right| < \infty. \tag{24}
$$

The first order modulus of continuity of functions with exponential growth is defined as

$$
\omega_1(f,\delta,B) = \sup_{\substack{h\leq\delta \\ x\in[0,\infty)}} \left|f(x)-f(x+h)\right|e^{-Bx}. \tag{25}
$$

Let $f \in \operatorname{Lip}(c,B)$ for some $0 < c \leq 1$. Then, for each $\delta < 1$,

$$
\omega_1(f,\delta,B) \leq M\delta^c. \tag{26}
$$

Let K be a subspace of $C[0,\infty)$ which contains functions f with exponential growth, $\|f\|_B < \infty$.

Theorem 5 *Let $M_s^{\alpha,\beta} : K \to C[0,\infty)$ be the sequence of linear positive operators preserving e^{-2ax}, $a > 0$. We assume that $M_s^{\alpha,\beta}$ satisfy*

$$
M_s^{\alpha,\beta}\left((t-x)^2e^{Bt};x\right) \leq C_a(B,x)M_s^{\alpha,\beta}\left(\phi_x^2;x\right) \tag{27}
$$

for fixed $x \in [0,\infty)$ and for $B > 0$. Additionally, if $f \in C^2[0,\infty) \cap K$, $0 < c \leq 1$, and $f'' \in \operatorname{Lip}(c,B)$, then for fixed $x \in [0,\infty)$, we have

$$
\begin{aligned}
&\left|M_s^{\alpha,\beta}(f;x) - f(x) - f'(x)M_s^{\alpha,\beta}\left(\phi_x^1;x\right) - \frac{f''(x)}{2}M_s^{\alpha,\beta}\left(\phi_x^2;x\right)\right| \\
&\quad\leq M_s^{\alpha,\beta}\left(\phi_x^2;x\right)\left(\frac{\sqrt{C_a(2B,x)}}{2} + \frac{C_a(B,x)}{2} + e^{2Bx}\right)\omega_1\left(f'', \sqrt{\frac{M_s^{\alpha,\beta}(\phi_x^4;x)}{M_s^{\alpha,\beta}(\phi_x^2;x)}}, B\right).
\end{aligned}
$$

Proof We begin with the Taylor expansion of the function $f \in C^2[0,\infty)$ at $x \in [0,\infty)$.

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + H_2(f;t,x), \tag{28}$$

where $H_2(f;t,x) = \frac{(f''(\eta)-f''(x))(t-x)^2}{2}$ is the remainder term. Here, η is between t and x. Applying the operators $M_s^{\alpha,\beta}$ to equality (28), we obtain

$$\begin{aligned}
&\left| M_s^{\alpha,\beta}(f;x) - f(x) - f'(x)M_s^{\alpha,\beta}\big(\phi_x^1;x\big) - \frac{f''(x)}{2}M_s^{\alpha,\beta}\big(\phi_x^2;x\big)\right| \\
&\quad = \left|M_s^{\alpha,\beta}\big(H_2(f;t,x);x\big)\right| \\
&\quad \le M_s^{\alpha,\beta}\big(\left|H_2(f;t,x)\right|;x\big).
\end{aligned} \tag{29}$$

Here,

$$H_2(f;t,x) = \frac{(f''(\eta)-f''(x))(t-x)^2}{2} \le \frac{(t-x)^2}{2}\begin{cases} e^{Bx}\omega_1(f'',h,B), & |t-x| \le h, \\ e^{Bx}\omega_1(f'',kh,B), & h \le |t-x| \le kh. \end{cases}$$

Tachev et al. [23] proved that, for each $h > 0$ and $k \in \mathbb{N}$,

$$\omega_1(f,kh,B) \le ke^{B(k-1)h}\omega_1(f,h,B). \tag{30}$$

By using inequality (30), we get

$$\begin{aligned}
&\frac{e^{Bx}(t-x)^2}{2}\omega_1\big(f'',kh,B\big) \\
&\quad \le \frac{e^{Bx}(t-x)^2}{2}ke^{B(k-1)h}\omega_1\big(f'',h,B\big) \\
&\quad \le \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)e^{Bx}e^{B|t-x|}\omega_1\big(f'',h,B\big) \\
&\quad \le \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)\big(e^{Bt}+e^{2Bx}\big)\omega_1\big(f'',h,B\big).
\end{aligned}$$

Therefore,

$$\left|H_2(f;t,x)\right| \le \frac{(t-x)^2}{2}\left(\frac{|t-x|}{h}+1\right)\big(e^{Bt}+e^{2Bx}\big)\omega_1\big(f'',h,B\big). \tag{31}$$

Applying the operators $M_s^{\alpha,\beta}$ to inequality (31), we write

$$\begin{aligned}
&M_s^{\alpha,\beta}\big(\left|H_2(f;t,x)\right|;x\big) \\
&\quad \le \frac{1}{2}M_s^{\alpha,\beta}\left(\left(\frac{|t-x|^3}{h}+|t-x|^2\right)\big(e^{Bt}+e^{2Bx}\big);x\right)\omega_1\big(f'',h,B\big) \\
&\quad = \left(\frac{1}{2h}M_s^{\alpha,\beta}\big(|t-x|^3e^{Bt};x\big) + \frac{1}{2}M_s^{\alpha,\beta}\big(|t-x|^2e^{Bt};x\big)\right. \\
&\qquad \left. + \frac{e^{2Bx}}{2h}M_s^{\alpha,\beta}\big(|t-x|^3;x\big) + \frac{e^{2Bx}}{2}M_s^{\alpha,\beta}\big(|t-x|^2;x\big)\right)\omega_1\big(f'',h,B\big).
\end{aligned}$$

By some computations we obtain

$$
\begin{aligned}
&M_s^{\alpha,\beta}\left(|t-x|^2e^{Bt};x\right)\\
&= M_s^{\alpha,\beta}\left(t^2e^{Bt};x\right) - 2xM_s^{\alpha,\beta}\left(te^{Bt};x\right) + x^2M_s^{\alpha,\beta}\left(e^{Bt};x\right)\\
&= e^{\frac{B\alpha}{s+p+\beta}}\frac{(s+p)(s+p+1)(s+p+\beta)^3\nu_{s,p}(x)^2}{(s+p+\beta-B)^5}\left(1-\frac{B\nu_{s,p}(x)}{s+p+\beta-B}\right)^{-(s+p+2)}\\
&\quad + e^{\frac{B\alpha}{s+p+\beta}}\left(\frac{4(s+p)(s+p+\beta)^2\nu_{s,p}(x)}{(s+p+\beta-B)^4} + \frac{2\alpha(s+p)(s+p+\beta)\nu_{s,p}(x)}{(s+p+\beta-B)^3}\right.\\
&\quad \left. - \frac{2x(s+p)(s+p+\beta)^2\nu_{s,p}(x)}{(s+p+\beta-B)^3}\right)\left(1-\frac{B\nu_{s,p}(x)}{s+p+\beta-B}\right)^{-(s+p+1)}\\
&\quad + e^{\frac{B\alpha}{s+p+\beta}}\left(\frac{2(s+p+\beta)}{(s+p+\beta-B)^3} + \frac{2\alpha}{(s+p+\beta-B)^2} + \frac{\alpha^2}{(s+p+\beta)(s+p+\beta-B)}\right.\\
&\quad \left. - \frac{2x(s+p+\beta)}{(s+p+\beta-B)^2} - \frac{2x\alpha}{(s+p+\beta-B)} + \frac{x^2(s+p+\beta)}{(s+p+\beta-B)}\right)\left(1-\frac{B\nu_{s,p}(x)}{s+p+\beta-B}\right)^{-(s+p)}\\
&= e^{Bx}\left(1+\frac{B(12+12(1+2a+B)x+4(1+6a+3B)x^2+3(2a+B)x^3)}{2(2+x)(s+p)}\right.\\
&\quad \left. + O\left((s+p)^{-2}\right)\right)M_s^{\alpha,\beta}\left(\phi_x^2;x\right).
\end{aligned}
$$

Since $s+p\geq 1$,

$$
M_s^{\alpha,\beta}\left(|t-x|^2e^{Bt};x\right) \leq C_a(B,x)M_s^{\alpha,\beta}\left(\phi_x^2;x\right). \tag{32}
$$

We have the following inequalities with the help of Cauchy–Schwarz inequality:

$$
\begin{aligned}
&M_s^{\alpha,\beta}\left(|t-x|^3e^{Bt};x\right)\\
&\leq \sqrt{M_s^{\alpha,\beta}\left(|t-x|^2e^{2Bt};x\right)}\sqrt{M_s^{\alpha,\beta}\left(|t-x|^4;x\right)}\\
&\leq \sqrt{C_a(2B,x)M_s^{\alpha,\beta}\left(\phi_x^2;x\right)}\sqrt{M_s^{\alpha,\beta}\left(\phi_x^4;x\right)},
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
&M_s^{\alpha,\beta}\left(|t-x|^3;x\right)\\
&\leq \sqrt{M_s^{\alpha,\beta}\left(|t-x|^4;x\right)}\sqrt{M_s^{\alpha,\beta}\left(|t-x|^2;x\right)}\\
&\leq \sqrt{M_s^{\alpha,\beta}\left(\phi_x^4;x\right)}\sqrt{M_s^{\alpha,\beta}\left(\phi_x^2;x\right)}.
\end{aligned} \tag{34}
$$

Thus, by using inequalities (32), (33), and (34) in (29), we write

$$
\begin{aligned}
&\left|M_s^{\alpha,\beta}(f;x) - f(x) - f'(x)M_s^{\alpha,\beta}\left(\phi_x^1;x\right) - \frac{f''(x)}{2}M_s^{\alpha,\beta}\left(\phi_x^2;x\right)\right|\\
&\leq \left(\frac{1}{2h}\sqrt{C_a(2B,x)M_s^{\alpha,\beta}\left(\phi_x^2;x\right)}\sqrt{M_s^{\alpha,\beta}\left(\phi_x^4;x\right)}\right.\\
&\quad + \frac{1}{2}C_a(B,x)M_s^{\alpha,\beta}\left(\phi_x^2;x\right) + \frac{e^{2Bx}}{2h}\sqrt{M_s^{\alpha,\beta}\left(\phi_x^4;x\right)}\sqrt{M_s^{\alpha,\beta}\left(\phi_x^2;x\right)}
\end{aligned}
$$

$$+\frac{e^{2Bx}}{2}M_s^{\alpha,\beta}\left(\phi_x^2;x\right)\Big)\omega_1\left(f'',h,B\right). \tag{35}$$

Finally, when we choose $h=\sqrt{\frac{M_s^{\alpha,\beta}(\phi_x^4;x)}{M_s^{\alpha,\beta}(\phi_x^2;x)}}$ and substitute it in (35), we obtain

$$\begin{aligned}&\left|M_s^{\alpha,\beta}(f;x)-f(x)-f'(x)M_s^{\alpha,\beta}\left(\phi_x^1;x\right)-\frac{f''(x)}{2}M_s^{\alpha,\beta}\left(\phi_x^2;x\right)\right|\\&\quad\le M_s^{\alpha,\beta}\left(\phi_x^2;x\right)\left(\frac{\sqrt{C_a(2B,x)}}{2}+\frac{C_a(B,x)}{2}+e^{2Bx}\right)\omega_1\left(f'',\sqrt{\frac{M_s^{\alpha,\beta}(\phi_x^4;x)}{M_s^{\alpha,\beta}(\phi_x^2;x)}},B\right).\end{aligned}$$

Note that, for fixed $x\in[0,\infty)$, $\frac{M_s^{\alpha,\beta}(\phi_x^4;x)}{M_s^{\alpha,\beta}(\phi_x^2;x)}=\frac{5x(2+x)}{s+p}+O((s+p)^{-2})\to 0$ as $s\to\infty$. This result guarantees the convergence of Theorem 5. □

In Sect. 6, we give the Voronovskaya-type theorem to examine the asymptotic behavior of the constructed operators (1). For the quantitative Voronovskaya-type theorems, we refer to the pioneering works [1] and [3].

6 Voronovskaya-type theorem

Theorem 6 *For $f,f''\in C^*[0,\infty)$ and $x\in[0,\infty)$, we have the inequality*

$$\begin{aligned}&\left|s\left(M_s^{\alpha,\beta}(f;x)-f(x)\right)-\left(2ax+ax^2\right)f'(x)-\left(x+\frac{x^2}{2}\right)f''(x)\right|\\&\quad\le\left|r_{s,p}(x)\right|\left|f'(x)\right|\\&\qquad+\left|t_{s,p}(x)\right|\left|f''(x)\right|+2\left(2t_{s,p}(x)+2x+x^2+z_{s,p}(x)\right)\omega^*\left(f'',s^{-1/2}\right),\end{aligned}$$

where

$$\begin{aligned}r_{s,p}(x)&=sM_s^{\alpha,\beta}\left(\phi_x^1;x\right)-\left(2ax+ax^2\right),\\t_{s,p}(x)&=\frac{s}{2}M_s^{\alpha,\beta}\left(\phi_x^2;x\right)-\left(x+\frac{x^2}{2}\right),\\z_{s,p}(x)&=s^2\sqrt{M_s^{\alpha,\beta}\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{M_s^{\alpha,\beta}\left(\phi_x^4;x\right)}.\end{aligned}$$

Proof By the Taylor expansion for a function f, we write

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+k(t,x)(t-x)^2, \tag{36}$$

where

$$k(t,x):=\frac{f''(\xi)-f''(x)}{2}.$$

Here, $k(t,x)$ is the remainder term and ξ is a number between x and t. Applying the $M_s^{\alpha,\beta}$ operators to (36), we obtain

$$M_s^{\alpha,\beta}(f;x)-f(x)=f'(x)M_s^{\alpha,\beta}\left(\phi_x^1;x\right)+\frac{f''(x)}{2}M_s^{\alpha,\beta}\left(\phi_x^2;x\right)+M_s^{\alpha,\beta}\left(k(t,x)\phi_x^2;x\right).$$

Then

$$\begin{aligned}&\left|s\left[M_s^{\alpha,\beta}(f;x)-f(x)\right]-\left(2ax+ax^2\right)f'(x)-\left(x+\frac{x^2}{2}\right)f''(x)\right|\\&\quad\le\left|sM_s^{\alpha,\beta}\left(\phi_x^1;x\right)-\left(2ax+ax^2\right)\right|\left|f'(x)\right|+\frac{1}{2}\left|sM_s^{\alpha,\beta}\left(\phi_x^2;x\right)-\left(2x+x^2\right)\right|\left|f''(x)\right|\\&\qquad+\left|sM_s^{\alpha,\beta}\left(k(t,x)\phi_x^2;x\right)\right|.\end{aligned}$$

We briefly denote that $r_{s,p}(x) := sM_s^{\alpha,\beta}(\phi_x^1;x) - (2ax+ax^2)$ and $t_{s,p}(x) := \frac{s}{2}M_s^{\alpha,\beta}(\phi_x^2;x) - (x+\frac{x^2}{2})$. Thus,

$$\begin{aligned}&\left|s\left[M_s^{\alpha,\beta}(f;x)-f(x)\right]-\left(2ax+ax^2\right)f'(x)-\left(x+\frac{x^2}{2}\right)f''(x)\right|\\&\quad\le\left|r_{s,p}(x)\right|\left|f'(x)\right|+|t_{s,p}|\left|f''(x)\right|\\&\qquad+\left|sM_s^{\alpha,\beta}\left(k(t,x)\phi_x^2;x\right)\right|.\end{aligned}$$

Note that by using equalities (6) and (7), $r_{s,p}(x)$ and $t_{s,p}(x)$ go to zero as $s\to\infty$. Now, we deal with the term $|sM_s^{\alpha,\beta}(k(t,x)\phi_x^2;x)|$.

$$\left|f(t)-f(x)\right|\le\left(1+\frac{(e^{-x}-e^{-t})^2}{\eta^2}\right)\omega^*(f,\eta).$$

Using this inequality, we have

$$\left|k(t,x)\right|\le\left(1+\frac{(e^{-x}-e^{-t})^2}{\eta^2}\right)\omega^*\left(f'',\eta\right).$$

For $\eta>0$, if $|e^{-x}-e^{-t}|>\eta$, then $|k(t,x)|\le\frac{2(e^{-x}-e^{-t})^2}{\eta^2}\omega^*(f'',\eta)$ and if $|e^{-x}-e^{-t}|\le\eta$, then $|k(t,x)|\le 2\omega^*(f'',\eta)$. Thus, we write $|k(t,x)|\le 2(1+\frac{(e^{-x}-e^{-t})^2}{\eta^2})\omega^*(f'',\eta)$. Therefore,

$$\begin{aligned}&\left|sM_s^{\alpha,\beta}\left(k(t,x)\phi_x^2;x\right)\right|\\&\quad\le sM_s^{\alpha,\beta}\left(\left|k(t,x)\right|\phi_x^2;x\right)\\&\quad\le 2s\omega^*\left(f'',\eta\right)M_s^{\alpha,\beta}\left(\phi_x^2;x\right)+\frac{2s}{\eta^2}\omega^*\left(f'',\eta\right)M_s^{\alpha,\beta}\left(\left(e^{-x}-e^{-t}\right)^2\phi_x^2;x\right)\\&\quad\le 2s\omega^*\left(f'',\eta\right)M_s^{\alpha,\beta}\left(\phi_x^2;x\right)\\&\qquad+\frac{2s}{\eta^2}\omega^*\left(f'',\eta\right)\sqrt{M_s^{\alpha,\beta}\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{M_s^{\alpha,\beta}\left(\phi_x^4;x\right)}.\end{aligned}$$

If we choose $\eta=1/\sqrt{s}$ and $z_{s,p} := \sqrt{s^2M_s^{\alpha,\beta}((e^{-x}-e^{-t})^4;x)}\sqrt{s^2M_s^{\alpha,\beta}(\phi_x^4;x)}$, we get

$$\begin{aligned}&\left|s\left(M_s^{\alpha,\beta}(f;x)-f(x)\right)-\left(2ax+ax^2\right)f'(x)-\left(x+\frac{x^2}{2}\right)f''(x)\right|\\&\quad\le\left|r_{s,p}(x)\right|\left|f'(x)\right|\\&\qquad+\left|t_{s,p}(x)\right|\left|f''(x)\right|+\left(4t_{s,p}(x)+4x+2x^2+2z_{s,p}(x)\right)\omega^*\left(f'',s^{-1/2}\right).\end{aligned}$$

□

Remark 3 We obtain the following result by some calculations:

$$\lim_{s\to\infty} s^2 M_s^{\alpha,\beta}\left(\phi_x^4;x\right) = 3x^2(2+x)^2. \tag{37}$$

Additionally, we get the following result:

$$\lim_{s\to\infty} s^2 M_s^{\alpha,\beta}\left(\left(e^{-t}-e^{-x}\right)^4;x\right) = 3x^2(2+x)^2 e^{-4x}. \tag{38}$$

We give the following corollary as a result of Theorem 6 and Remark 3.

Corollary 1 *Suppose that* $f, f'' \in C^*[0,\infty)$ *and* $x \in [0,\infty)$. *Then the equality*

$$\lim_{s\to\infty} s\left(M_s^{\alpha,\beta}(f;x) - f(x)\right) = \left(2ax + ax^2\right)f'(x) + \left(x + \frac{x^2}{2}\right)f''(x) \tag{39}$$

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