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More results on integral inequalities for strongly generalized (ϕ, h, s)-preinvex functions

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Abstract
The main goal of this research is to introduce a new form of generalized Hermite–Hadamard and Simpson type inequalities utilizing Riemann–Liouville fractional integral by a new class of preinvex functions which is known as strongly generalized (ϕ, h, s)-preinvex functions in the second sense. It is observed that the derived inequalities are generalizations of the inequalities obtained by W. Liu, W. Wen (Filomat 30(2):333–342, 2016).

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1 Introduction

Convexity plays a focal and major part in mathematical finance, economics, engineering, management sciences, and optimization theory. As of late, a few extensions and generalizations have been considered for classical convexity. A huge speculation of convex functions is that of invex functions presented in [2]. The fundamental properties of the preinvex functions and their use in optimization and mathematical programming issues have been considered in [3–5]. It is realized that the preinvex functions and invex sets may not be convex functions and convex sets, respectively. Another generalization of the convex function, which is known as the φ-convex function presented and examined in [6], is similarly vital. Specifically, these generalizations of the convex functions are very extraordinary and do not contain each other. Another class of nonconvex functions is presented and studied in [7], which incorporates these generalizations as special cases. This class of nonconvex functions is called the φ-preinvex and φ-invex functions. Some well-known integral inequalities like those of Simpson and Hermite–Hadamard type in literature are under discussion. In our opinion, these inequalities have great impact in pure and applied mathematics. Many new extensions and interesting generalizations of these integral inequalities have been studied in recent years. For further details involving Hermite–Hadamard and Simpson type inequalities on different concepts of convex function, the reader is referred to [1, 8–16].

In [1, 17] Wenjun Liu et al. presented the following form of inequalities for MT-convex functions:

Let $f : I \subset R \to R$ be a differentiable mapping on I° (the interior of I) such that $f' \in L_1([u,v])$, where $u, v \in I$ with $u < v$. Then, for all $x \in [u,v]$, $\delta \in [0,1]$, and $\alpha > 0$, we have

$$\begin{aligned} &S_f(x,\delta,\alpha,u,v) \\ &\quad = \frac{(x-u)^{\alpha+1}}{v-u}\int_0^1 \bigl(z^\alpha - \delta\bigr) f'\bigl(zx + (1-z)u\bigr)\,dz \\ &\qquad + \frac{(v-x)^{\alpha+1}}{v-u}\int_0^1 \bigl(\delta - z^\alpha\bigr) f'\bigl(zx + (1-z)v\bigr)\,dz. \end{aligned}$$

Theorem 1 *Let $f : I \subset R \to R$ be a differentiable mapping on I° (the interior of I) such that $f' \in L_1([u,v])$, where $u, v \in I$ with $u < v$. If $|f'|$ is an MT-convex function on $[u,v]$ and $|f'(x)| \le M$ for all $x \in [u,v]$, then we have the following inequality for fractional integrals with $\alpha > 0$:*

$$\begin{aligned} &\bigl|S_f(x,1,\alpha,u,v)\bigr| \\ &\quad = \left|\frac{(x-u)^\alpha f(u) + (v-x)^\alpha f(v)}{v-u} - \frac{\Gamma(\alpha+1)}{v-u}\bigl[J^\alpha_{x^-} f(u) + J^\alpha_{x^+} f(v)\bigr]\right| \\ &\quad \le \frac{M[(x-u)^{\alpha+1} + (v-x)^{\alpha+1}]}{2(v-u)}\left[\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right]. \end{aligned} \tag{1}$$

Proposition 1 *Under the assumption of Theorem 1, putting $x = \frac{u+v}{2}$, we obtain*

$$\begin{aligned} &\left|S_f\left(\frac{u+v}{2},1,\alpha,u,v\right)\right| \\ &\quad = \left|\frac{(v-u)^{\alpha-1}}{2^{\alpha-1}}\frac{f(u)+f(v)}{2} - \frac{\Gamma(\alpha+1)}{v-u}\bigl[J^\alpha_{x^-} f(u) + J^\alpha_{x^+} f(v)\bigr]\right| \\ &\quad \le \frac{M(v-u)^\alpha}{2^{\alpha+1}}\left[\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right]. \end{aligned} \tag{2}$$

Theorem 2 *Let $f : I \subset R \to R$ be a differentiable mapping on I° (the interior of I) such that $f' \in L_1([u,v])$, where $u, v \in I$ with $u < v$. If $|f'|^q$ is an MT-convex function on $[u,v]$ for $q \ge 1$ and $|f'(x)| \le M$, for all $x \in [u,v]$, then we have the following inequality for fractional integrals with $\alpha > 0$:*

$$\begin{aligned} &\bigl|S_f(x,1,\alpha,u,v)\bigr| \\ &\quad = \left|\frac{(x-u)^\alpha f(u) + (v-x)^\alpha f(v)}{v-u} - \frac{\Gamma(\alpha+1)}{v-u}\bigl[J^\alpha_{x^-} f(u) + J^\alpha_{x^+} f(v)\bigr]\right| \\ &\quad \le \frac{M[(x-u)^{\alpha+1} + (v-x)^{\alpha+1}]}{(v-u)}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{2} - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right]^{1-\frac{1}{q}}. \end{aligned} \tag{3}$$

Theorem 3 *Let $f : I \subset R \to R$ be a differentiable mapping on I° (the interior of I) such that $f' \in L_1([u,v])$, where $u, v \in I$ with $u < v$. If $|f'|^q$ is an MT-convex function on $[u,v]$ for $q \ge 1$ and $|f'(x)| \le M$, for all $x \in [u,v]$, then we have the following inequality for fractional* *integrals with* $\alpha > 0$:

$$\begin{aligned}
&\left|S_f(x,0,\alpha,u,v)\right| \\
&\quad = \left|\frac{(x-u)^{\alpha}+(v-x)^{\alpha}}{v-u}f(x) - \frac{\Gamma(\alpha+1)}{v-u}\left[J^{\alpha}_{x^-}f(u)+J^{\alpha}_{x^+}f(v)\right]\right| \\
&\quad \leq \frac{M[(x-u)^{\alpha+1}+(v-x)^{\alpha+1}]}{(v-u)}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left[\frac{1}{(\alpha p+1)}\right]^{\frac{1}{p}}.
\end{aligned} \tag{4}$$

Theorem 4 *Under the assumption of the above theorem, we obtain*

$$\begin{aligned}
&\left|S_f\left(\frac{u+v}{2},0,\alpha,u,v\right)\right| \\
&\quad = \left|\frac{(v-u)^{\alpha-1}}{2^{\alpha-1}}f\left(\frac{u+v}{2}\right) + \frac{\delta(v-u)^{\alpha-1}}{2^{\alpha-1}}\frac{f(u)+f(v)}{2} - \frac{\Gamma(\alpha+1)}{v-u}\left[J^{\alpha}_{x^-}f(u)+J^{\alpha}_{x^+}f(v)\right]\right| \\
&\quad \leq \frac{M(v-u)^{\alpha}}{2^{\alpha}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{2}{\alpha}\int_0^1(\delta-s)^p s^{\frac{1}{\alpha}-1}\,ds - \frac{1}{\alpha}\int_0^1(\delta-s)^p s^{\frac{1}{\alpha}-1}\,ds\right)^{\frac{1}{p}}.
\end{aligned} \tag{5}$$

Theorem 5 *Let* $f: I \subset R \to R$ *be a differentiable mapping on* I° (*the interior of* I) *such that* $f' \in L_1([u,v])$, *where* $a,b \in I$ *with* $u < v$. *If* $|f'|^q$ *is an MT-convex function on* $[u,v]$ *for* $q \geq 1$ *and* $|f'(x)| \leq M$, *for all* $x \in [u,v]$, *then we have the following inequality for fractional integrals with* $\alpha > 0$ *and* $\delta \in [0,1]$:

$$\begin{aligned}
&\left|S_f(x,\delta,\alpha,u,v)\right| \\
&\quad \leq \frac{M[(x-u)^{\alpha+1}+(v-x)^{\alpha+1}]}{v-u}\left(\frac{2\alpha\delta^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\delta\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(2\delta\left(\beta\left(\delta^{\frac{1}{\alpha}};\frac{3}{2},\frac{1}{2}\right)+\beta\left(\delta^{\frac{1}{\alpha}};\frac{1}{2},\frac{3}{2}\right)\right)+\beta\left(\alpha+\frac{3}{2},\frac{1}{2}\right)+\beta\left(\alpha+\frac{1}{2},\frac{3}{2}\right)-\delta\pi\right. \\
&\qquad \left.-2\left(\beta\left(\delta^{\frac{1}{\alpha}};\alpha+\frac{3}{2},\frac{1}{2}\right)+\beta\left(\delta^{\frac{1}{\alpha}};\alpha+\frac{1}{2},\frac{3}{2}\right)\right)\Big/2\right)^{\frac{1}{q}}.
\end{aligned} \tag{6}$$

Fractional calculus was figured in 1695, soon after the advancement of classical calculus. The earliest efficient reviews were credited to Liouville, Riemann, Leibniz, etc. [15, 16, 18–23]. For quite a while, fractional calculus was viewed as a pure mathematical domain without real applications. In any case, in recent decades, such a situation has changed. It has been found that fractional calculus can be useful and even capable, and a diagram of the straightforward history about fractional calculus, particularly with applications, can be found in Machado et al. [24]. Presently, fractional calculus and its applications are experiencing quick advancements with more persuading applications in this real world.

In this paper, we establish a new class of preinvex functions, which are called strongly generalized (ϕ,h,s)-preinvex functions, and some generalizations for these inequalities mentioned above. Before moving towards our main results, first we recall the following definitions.

Definition 1 Let $f \in L_1[u,v]$. The Riemann–Liouville integrals $\int_{u^+}^{\alpha}(f)$ and $\int_{v^-}^{\alpha}(f)$ of order $\alpha > 0$ with $u \geq 0$ are defined by

$$\int_{u^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-z)^{\alpha-1} f(z)\,dz, \quad \text{for } x > u,$$

and

$$\int_{v^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (z-x)^{\alpha-1} f(z)\,dz, \quad \text{for } v > x,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-w} w^{\alpha-1}\,dw$. Here, $\int_{u^+}^0 f(x) = \int_{v^-}^0 f(x) = f(x)$.

In a special case, when $\alpha = 1$ in Definition 1, we get the classical integral.

Here, we present new generalized inequalities using the Riemann–Liouville fractional integral by the class of strongly generalized (ϕ, h, s)-preinvex functions in the second sense.

Definition 2 ([19]) The function f on the invex set $K_{\phi\xi}$ is said to be ϕ-preinvex with respect to ξ and ϕ if

$$f\big(x + ze^{i\phi}\xi(y,x)\big) \leq (1-z)f(x) + zf(y), \quad \forall x, y \in K_{\phi\xi}, z \in [0,1].$$

The function f is said to be ϕ-preconcave if and only if $-f$ is φ-preinvex. Every convex function is a ϕ-preinvex function, but not conversely.

Definition 3 ([5]) The function f on the invex set $K_{\phi\xi}$ is said to be s_ϕ-preinvex with respect to ξ and ϕ if

$$f\big(x + ze^{i\phi}\xi(y,x)\big) \leq (1-z)^s f(x) + z^s f(y), \quad \forall x, y \in K_{\phi\xi}, z \in [0,1], s \in (0,1].$$

2 Main results

First we introduce a new concept named strongly generalized (ϕ, h, s)-preinvex functions in the second sense. It is defined as follows.

Definition 4 The function f on the invex set K is said to be strongly generalized (ϕ, h, s)-preinvex in the second sense with modulus $c > 0$ if it is nonnegative, and for all $u, v \in K$ and $z \times s \in (0,1) \times (0,1]$, the following inequality holds:

$$f\big(v + ze^{i\phi}\xi(u,v)\big) \leq h^s(z)f(u) + h^s(1-z)f(v) - cz(1-z)\big\|e^{i\phi}\xi(u,v)\big\|^2.$$

Notation. Let $f : I \subset R \to R$ be a differentiable mapping on I° (the interior of I), from now on we will consider

$$\begin{aligned}
&\Psi_f\big(x, \delta, \sigma, \alpha, e^{i\phi}\xi(u,v)\big) \\
&= \big(1-\delta^\sigma\big)\left[\frac{\xi(v,x)^\alpha f(v + e^{i\phi}\xi(x,v)) + \xi(x,u)^\alpha f(u + e^{i\phi}\xi(x,u))}{e^{i\phi}\xi(v,u)}\right] f(x) \\
&\quad + \delta^\sigma\left[\frac{\xi(v,x)^\alpha f(v) + \xi(u,x)^\alpha f(u)}{e^{i\phi}\xi(v,u)}\right]
\end{aligned}$$

$$-\frac{\Gamma(\alpha+1)}{e^{i\alpha\phi}\xi(\nu,u)^{\alpha}}\left[J^{\alpha}_{(u+e^{i\phi}\xi(x,u))+}f(u)+J^{\alpha}_{(\nu+e^{i\phi}\xi(x,\nu))+}f(\nu)\right],$$

where $u < u + e^{i\phi}\xi(\nu,u)$, $x \in [u, u + e^{i\phi}\xi(\nu,u)]$, $\delta \in [0,1]$, $\alpha > 0$ and Γ is Euler gamma function.

To get new integral inequalities, first we focus on proving the following lemma.

Lemma 1 *Let $K_{\phi\xi} \subseteq R$ be a ϕ-invex subset with respect to $\phi(\cdot)$ and $\xi\colon K_{\phi\xi} \times K_{\phi\xi} \subseteq R$ with $u < u + e^{i\phi}\xi(\nu,u)$ and $0 \le \phi \le \frac{\pi}{2}$. Suppose that $f : K_{\phi\xi} \to R$ is a differentiable mapping such that $f' \in L([u, u + e^{i\phi}\xi(\nu,u)])$ for all $x \in [u, u + e^{i\phi}\xi(\nu,u)]$, $\delta \times \sigma \in [0,1]$, and $\alpha > 0$, then we have*

$$\begin{aligned}
&\Psi_f\bigl(x,\delta,\sigma,\alpha,e^{i\phi}\xi(\nu,u)\bigr)\\
&\quad=\frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\bigl(z^{\alpha}-\delta^{\sigma}\bigr)f'\bigl(u+ze^{i\phi}\xi(x,u)\bigr)\,dz\\
&\qquad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\bigl(\delta^{\sigma}-z^{\alpha}\bigr)f'\bigl(\nu+ze^{i\phi}\xi(x,\nu)\bigr)\,dz.
\end{aligned}$$

Proof Using integration by parts, we get

$$\begin{aligned}
&\int_0^1\bigl(z^{\alpha}-\delta^{\sigma}\bigr)f'\bigl(u+ze^{i\phi}\xi(x,u)\bigr)\,dz\\
&\quad=\left[\bigl(z^{\alpha}-\delta^{\sigma}\bigr)\frac{f(u+ze^{i\phi}\xi(x,u))}{e^{i\phi}\xi(x,u)}\Big|_0^1-\alpha\int_0^1 z^{\alpha-1}\frac{f(u+ze^{i\phi}\xi(x,u))}{e^{i\phi}\xi(x,u)}\,dz\right]\\
&\quad=\left[\frac{(1-\delta^{\sigma})f(u+e^{i\phi}\xi(x,u))+\delta^{\sigma}f(a)}{e^{i\phi}\xi(x,u)}-\frac{\Gamma(\alpha+1)}{e^{i\alpha\phi}\xi(x,u)^{\alpha}}J^{\alpha}_{(u+e^{i\phi}\xi(x,u))+}f(u)\right].
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
&\int_0^1\bigl(\delta^{\sigma}-z^{\alpha}\bigr)f'\bigl(\nu+ze^{i\phi}\xi(x,\nu)\bigr)\,dz\\
&\quad=\left[\bigl(z^{\alpha}-\delta^{\sigma}\bigr)\frac{f(\nu+ze^{i\phi}\xi(x,u))}{e^{i\phi}\xi(x,u)}\Big|_0^1+\alpha\int_0^1 z^{\alpha-1}\frac{f(\nu+ze^{i\phi}\xi(x,\nu))}{e^{i\phi}\xi(x,\nu)}\,dz\right]\\
&\quad=\left[\frac{(1-\delta^{\sigma})f(\nu+e^{i\phi}\xi(x,\nu))+\delta^{\sigma}f(\nu)}{e^{i\phi}\xi(\nu,x)}-\frac{\Gamma(\alpha+1)}{e^{i\alpha\phi}\xi(\nu,x)^{\alpha}}J^{\alpha}_{(\nu+e^{i\phi}\xi(x,\nu))+}f(\nu)\right].
\end{aligned}$$

Both sides of the above equalities are multiplied by $\frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}$ and $\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}$ analogously, and then adding them, we obtain the required result. This completes the proof. □

Theorem 6 *Let $K_{\phi\xi} \subseteq R$ be a ϕ-invex subset with respect to $\phi(\cdot)$ and $\xi\colon K_{\phi\xi} \times K_{\phi\xi} \subseteq R$ with $u < u + e^{i\phi}\xi(\nu,u)$ and $0 \le \phi \le \frac{\pi}{2}$. Suppose that $f : K_{\phi\xi} \to R$ is a differentiable mapping such that $f' \in L([u, u + e^{i\phi}\xi(\nu,u)])$. If $|f'|$ is strongly generalized (ϕ,h,s)-preinvex in the second sense and $|f'(x)| \le M$, then for all $x \in [u, u + e^{i\phi}\xi(\nu,u)]$, $\delta \times \sigma \in [0,1]$, and $\alpha > 0$, we have*

$$\begin{aligned}
&\bigl|\Psi_f\bigl(x,\delta,\sigma,\alpha,e^{i\phi}\xi(u,\nu)\bigr)\bigr|\\
&\quad\le\frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Bigl[M\bigl(\Psi_1(\delta,\sigma,\alpha,s)+\Psi_2(\delta,\sigma,\alpha,s)\bigr)-c\bigl\|e^{i\phi}\xi(u,x)\bigr\|^2\Psi_3(\delta,\sigma,\alpha)\Bigr]
\end{aligned}$$

$$+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Big[M\big(\Psi_1(\delta,\sigma,\alpha,s)+\Psi_2(\delta,\sigma,\alpha,s)\big)-c\big\|e^{i\phi}\xi(\nu,x)\big\|^2\Psi_3(\delta,\sigma,\alpha)\Big]. \tag{7}$$

Proof Using Lemma 1, the property of modulus, and strongly generalized (ϕ,h,s)-preinvexity in the second sense, we obtain

$$\begin{aligned}
&\big|\Psi_f\big(x,\delta,\sigma,\alpha,e^{i\phi}\xi(u,\nu)\big)\big| \\
&\quad\le \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\big|z^\alpha-\delta^\sigma\big|\big|f'\big(u+e^{i\phi}z\xi(x,u)\big)\big|\,dz \\
&\qquad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\big|z^\alpha-\delta^\sigma\big|\big|f'\big(\nu+e^{i\phi}z\xi(x,\nu)\big)\big|\,dz \\
&\quad\le \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\big|z^\alpha-\delta^\sigma\big|\big(h^s(z)\big|f'(x)\big|+h^s(1-z)\big|f'(u)\big|-cz(1-z)\big\|e^{i\phi}\xi(u,x)\big\|^2\big)\,dz \\
&\qquad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\big(\delta^\sigma-z^\alpha\big)\big(h^s(z)\big|f'(x)\big|+h^s(1-z)\big|f'(u)\big| \\
&\qquad-cz(1-z)\big\|e^{i\phi}\xi(\nu,x)\big\|^2\big)\,dz \\
&\quad=\frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Big[\Psi_1(\delta,\sigma,\alpha,s)\big|f'(x)\big|+\Psi_2(\delta,\sigma,\alpha,s)\big|f'(u)\big|-c\big\|e^{i\phi}\xi(u,x)\big\|^2\Psi_3(\delta,\sigma,\alpha)\Big] \\
&\qquad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Big[\Psi_1(\delta,\sigma,\alpha,s)\big|f'(x)\big|+\Psi_2(\delta,\sigma,\alpha,s)\big|f'(\nu)\big|-c\big\|e^{i\phi}\xi(\nu,x)\big\|^2\Psi_3(\delta,\sigma,\alpha)\Big],
\end{aligned}$$

where we used the fact

$$\begin{aligned}
&\Psi_1(\delta,\sigma,\alpha,s)=\int_0^1\big|z^\alpha-\delta^\sigma\big|h^s(z)\,dz, \\
&\Psi_2(\delta,\sigma,\alpha,s)=\int_0^1\big|z^\alpha-\delta^\sigma\big|h^s(1-z)\,dz, \\
&\Psi_3(\delta,\sigma,\alpha)=\int_0^1\big|z^\alpha-\delta^\sigma\big|z(1-z)\,dz \\
&\qquad=2\delta^\sigma\beta\big(\delta^{\frac{\sigma}{\alpha}},2,2\big)-2\beta\big(\delta^{\frac{\sigma}{\alpha}},\alpha+2,2\big)+\beta(\alpha+2,2)-\delta^\sigma\beta(2,2).
\end{aligned}$$

Hence the proof. □

Remark 1 On letting $s=1$, $\xi(u,\nu)=u-\nu$, $\phi=c=\sigma=0$, $x=\frac{u+\nu}{2}$, and $h(z)=\frac{\sqrt{z}}{2\sqrt{1-z}}$ in Theorem 6, then inequality (7) reduces to inequality (2).

Theorem 7 *Let $K_{\phi\xi}\subseteq R$ be a ϕ-invex subset with respect to $\phi(\cdot)$ and $\xi\colon K_{\phi\xi}\times K_{\phi\xi}\subseteq R$ with $u<u+e^{i\phi}\xi(\nu,u)$ for $0\le\phi\le\frac{\pi}{2}$. Suppose that $f:K_{\phi\xi}\to R$ is a differentiable mapping such that $f'\in L([u,u+e^{i\phi}\xi(\nu,u)])$. If $|f'|$ is strongly generalized (ϕ,h,s)-preinvex in the second sense with $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $|f'(x)|\le M$, then for all $x\in[u,u+e^{i\phi}\xi(\nu,u)]$, $\delta\times\sigma\in(0,1]\times(0,1]$, and $\alpha>0$, we have*

$$\begin{aligned}
&\big|\Psi_f\big(x,\delta,\sigma,\alpha,e^{i\phi}\xi(u,\nu)\big)\big| \\
&\quad\le\frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Psi_6(\delta,\sigma,\alpha)^{\frac{1}{p}}\Big[M^q(\Psi_4+\Psi_5)-c\big\|e^{i\phi}\xi(u,x)\big\|^2\beta(2,2)\Big]^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Psi_6(\delta,\sigma,\alpha)^{\frac{1}{p}}\left[M^q(\Psi_4+\Psi_5)-c\left\|e^{i\phi}\xi(\nu,x)\right\|^2\beta(2,2)\right]^{\frac{1}{q}}. \tag{8}$$

Proof Using Lemma 1 and the Holder integral inequality, we obtain

$$\begin{aligned}
&\left|\Psi_f\left(x,\delta,\sigma,\alpha,e^{i\phi}\xi(u,\nu)\right)\right| \\
&\leq \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\left|z^\alpha-\delta^\sigma\right|\left|f'\left(u+ze^{i\phi}\xi(x,u)\right)\right|dz \\
&\quad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\int_0^1\left|\delta^\sigma-z^\alpha\right|\left|f'\left(\nu+ze^{i\phi}\xi(x,\nu)\right)\right|dz. \\
&\leq \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|^p dz\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(u+ze^{i\phi}\xi(x,u)\right)\right|^q dz\right)^{\frac{1}{q}} \\
&\quad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\left(\int_0^1\left|\delta^\sigma-z^\alpha\right|^p dz\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\nu+ze^{i\phi}\xi(x,\nu)\right)\right|^q dz\right)^{\frac{1}{q}} \\
&\leq \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\left(\int_0^1\left|\delta^\sigma-z^\alpha\right|^p dz\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\left(\begin{array}{c}h^s(z)|f'(x)|^q\\+h^s(1-z)|f'(u)|^q-cz(1-z)\|e^{i\phi}\xi(u,x)\|^2\end{array}\right)dz\right)^{\frac{1}{q}} \\
&\quad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\left(\int_0^1\left|\delta^\sigma-z^\alpha\right|^p dz\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\left(\begin{array}{c}h^s(z)|f'(x)|^q\\+h^s(1-z)|f'(u)|^q-cz(1-z)\|e^{i\phi}\xi(\nu,x)\|^2\end{array}\right)dz\right)^{\frac{1}{q}} \\
&=\frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Psi_6(\delta,\sigma,\alpha)^{\frac{1}{p}}\left[M^q(\Psi_4+\Psi_5)-c\left\|e^{i\phi}\xi(u,x)\right\|^2\beta(2,2)\right]^{\frac{1}{q}} \\
&\quad+\frac{\xi(\nu,x)^{\alpha+1}}{e^{i\phi}\xi(\nu,u)}\Psi_6(\delta,\sigma,\alpha)^{\frac{1}{p}}\left[M^q(\Psi_4+\Psi_5)-c\left\|e^{i\phi}\xi(\nu,x)\right\|^2\beta(2,2)\right]^{\frac{1}{q}},
\end{aligned}$$

where we used the fact

$$\begin{aligned}
&\Psi_4=\int_0^1 h^s(z)\,dz, \\
&\Psi_5=\int_0^1 h^s(1-z)\,dz, \\
&\Psi_6(\delta,\sigma,\alpha)=\int_0^1\left|z^\alpha-\delta^\sigma\right|dz=\frac{1+2\alpha\delta^{\sigma(\frac{1+\alpha}{\alpha})}}{1+\alpha}-\delta^\sigma.
\end{aligned}$$

This completes the proof. □

Remark 2 On letting $\delta=s=1$, $\xi(u,\nu)=u-\nu$, $\phi=c=0$, and $h(z)=\frac{\sqrt{z}}{2\sqrt{1-z}}$ in Theorem 7, inequality (8) reduces to inequality (3).

Remark 3 On letting $s=1$, $\xi(u,\nu)=u-\nu$, $\phi=c=\delta=0$, and $h(z)=\frac{\sqrt{z}}{2\sqrt{1-z}}$ in Theorem 7, inequality (8) reduces to inequality (4).

Remark 4 On letting $\sigma = s = 1$, $\xi(u,v) = u - v$, $\phi = c = 0$, $x = \frac{u+v}{2}$, and $h(z) = \frac{\sqrt{z}}{2\sqrt{1-z}}$ in Theorem 7, inequality (8) reduces to inequality (5).

Theorem 8 *Let $f : I = [u, u + e^{i\phi}\xi(v,u)] \subset [0,\infty) \to R$ be a differentiable mapping on I° such that $f' \in L_1([u, u + e^{i\phi}\xi(v,u)])$. If $|f'|^q$ is strongly generalized (ϕ,h,s)-preinvex and $|f'(x)| \le M$ for all $x \in [u, u + e^{i\phi}\xi(v,u)]$, $\delta \times \sigma \in [0,1] \times (0,1]$, and $\alpha > 0$, we have*

$$\begin{aligned}
&\left|\Psi_f\left(x,\delta,\sigma,\alpha,e^{i\phi}\xi(u,v)\right)\right| \\
&\quad\le \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\Psi_6(\delta,\sigma,\alpha)\right)^{\frac{1}{p}}\Big[M^q\left(\Psi_1(\delta,\sigma,\alpha,s)+\Psi_2(\delta,\sigma,\alpha,s)\right) \\
&\qquad - c\left\|e^{i\phi}\xi(u,x)\right\|^2\beta(2,2)\Big]^{\frac{1}{q}} \\
&\qquad + \frac{\xi(v,x)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\Psi_6(\delta,\sigma,\alpha)\right)^{\frac{1}{p}}\Big[M^q\left(\Psi_1(\delta,\sigma,\alpha,s)+\Psi_2(\delta,\sigma,\alpha,s)\right) \\
&\qquad - c\left\|e^{i\phi}\xi(v,x)\right\|^2\beta(2,2)\Big]^{\frac{1}{q}}.
\end{aligned} \tag{9}$$

Proof Using Lemma 1, the property of modulus and power mean inequality, we have

$$\begin{aligned}
&\left|\Psi_f\left(x,\delta,\sigma,\alpha,e^{i\phi}\xi(u,v)\right)\right| \\
&\quad\le \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\int_0^1\left(z^\alpha-\delta^\sigma\right)\left|f'\left(u+ze^{i\phi}\xi(x,u)\right)\right|dz \\
&\qquad + \frac{\xi(v,x)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\int_0^1\left(\delta^\sigma-z^\alpha\right)\left|f'\left(v+ze^{i\phi}\xi(x,v)\right)\right|dz \\
&\quad\le \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|dz\right)^{1-\frac{1}{q}}\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|\left|f'\left(u+ze^{i\phi}\xi(x,u)\right)\right|dz\right)^{\frac{1}{q}} \\
&\qquad + \frac{\xi(v,x)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|dz\right)^{1-\frac{1}{q}}\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|\left|f'\left(v+ze^{i\phi}\xi(x,v)\right)\right|dz\right)^{\frac{1}{q}} \\
&\quad\le \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|dz\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|\begin{pmatrix} h^s(z)M^q \\ +h^s(1-z)M^q-cz(1-z)\|e^{i\phi}\xi(u,x)\|^2\end{pmatrix}dz\right)^{\frac{1}{q}} \\
&\qquad + \frac{\xi(v,x)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|dz\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\int_0^1\left|z^\alpha-\delta^\sigma\right|\begin{pmatrix} h^s(z)M^q \\ +h^s(1-z)M^q-cz(1-z)\|e^{i\phi}\xi(v,x)\|^2\end{pmatrix}dz\right)^{\frac{1}{q}} \\
&\quad= \frac{\xi(x,u)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\Psi_6(\delta,\sigma,\alpha)\right)^{\frac{1}{p}}\Big[M^q\left(\Psi_1(\delta,\sigma,\alpha,s)+\Psi_2(\delta,\sigma,\alpha,s)\right) \\
&\qquad - c\left\|e^{i\phi}\xi(u,x)\right\|^2\beta(2,2)\Big]^{\frac{1}{q}} \\
&\qquad + \frac{\xi(v,x)^{\alpha+1}}{e^{i\phi}\xi(v,u)}\left(\Psi_6(\delta,\sigma,\alpha)\right)^{\frac{1}{p}}\Big[M^q\left(\Psi_1(\delta,\sigma,\alpha,s)+\Psi_2(\delta,\sigma,\alpha,s)\right)
\end{aligned}$$

$$-c\left\|e^{i\phi}\xi(\nu,x)\right\|^2\beta(2,2)\Big]^{\frac{1}{q}}.$$

Hence the proof. □

Remark 5 On letting $\sigma = s = 1$, $\xi(u,\nu) = u - \nu$, $\phi = c = 0$, $x = \frac{u+\nu}{2}$, and $h(z) = \frac{\sqrt{z}}{2\sqrt{1-z}}$ in Theorem 8, inequality (9) reduces to inequality (6).

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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