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Nontrivial solutions for an integral boundary value problem involving Riemann–Liouville fractional derivatives

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Abstract

In this paper using topological degree we study the existence of nontrivial solutions for a fractional differential equation involving integral boundary conditions. Here, the nonlinear term may be sign-changing and may also depend on the derivatives of the unknown function.

Keywords: Fractional integral boundary value problems; Nontrivial solutions; Topological degree

1 Introduction

We study the existence of nontrivial solutions for the following integral boundary value problem involving Riemann–Liouville fractional derivatives:

$$\begin{cases} D_{0+}^{\beta}D_{0+}^{\alpha}u(t)=f(t,u(t),u'(t),-D_{0+}^{\alpha}u(t)), \quad 0<t<1,\\ u(0)=u'(0)=0, \qquad u'(1)=\int_0^1 g(t)u'(t)\,dt,\\ D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha+1}u(0)=0, \qquad D_{0+}^{\alpha+1}u(1)=\int_0^1 h(t)D_{0+}^{\alpha+1}u(t)\,dt, \end{cases} \tag{1.1}$$

where $\alpha,\beta\in(2,3]$ are two real numbers, D_{0+}^{α}, D_{0+}^{β} are the Riemann–Liouville fractional derivatives, and $f\in C([0,1]\times\mathbb{R}^3,\mathbb{R})$ ($\mathbb{R}=(-\infty,+\infty)$). Moreover, the functions g, h are defined on $[0,1]$ and satisfy the condition:

(H0) $g,h\geq 0$ with $\int_0^1 g(t)t^{\alpha-2}\,dt\in[0,1)$, and $\int_0^1 h(t)t^{\beta-2}\,dt\in[0,1)$.

Fractional-order problems arise naturally in engineering and scientific disciplines such as physics, biophysics, chemistry, control theory, signal and image processing, and aerodynamics; we refer the reader to [1–3]. For example, in [4, 5] the authors introduced a fractional-order model of infection of CD4⁺ T-cells, and the system takes the following form:

$$\begin{cases} D^{\alpha_1}(T)=s-KVT-dT+bI,\\ D^{\alpha_2}(I)=KVT-(b+\delta)I,\\ D^{\alpha_3}(I)=N\delta I-cV, \end{cases}$$

where D^{α_i} are fractional derivatives, $i = 1, 2, 3$. Many results on the existence and multiplicity of solutions (or positive solutions) of nonlinear fractional differential equations can be found for example in [6–37] and the references therein. In [6–11, 19, 20, 26], the authors used the fixed point index theory to study the existence of (positive) solutions for various boundary value problems of fractional differential equations, for example, Bai in [6] obtained positive solutions for the nonlocal fractional-order differential equation boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & \beta u(\eta) = u(1), \end{cases}$$

where $1 < \alpha \le 2$, $0 < \beta\eta^{\alpha-1} < 1$, $0 < \eta < 1$, f is continuous on $[0, 1] \times \mathbb{R}^+$ and satisfies the conditions

$$(\mathrm{H})_{\mathrm{B1}} \quad \liminf_{u\to 0^+} \frac{f(t,u)}{u} > \lambda_1, \qquad \limsup_{u\to+\infty} \frac{f(t,u)}{u} < \lambda_1, \quad \text{uniformly on } t \in [0, 1],$$

and

$$(\mathrm{H})_{\mathrm{B2}} \quad \liminf_{u\to +\infty} \frac{f(t,u)}{u} > \lambda_1, \qquad \limsup_{u\to 0^+} \frac{f(t,u)}{u} < \lambda_1, \quad \text{uniformly on } t \in [0, 1],$$

where λ_1 is the first eigenvalue corresponding of the relevant linear operator. These conditions can also be found in some integer-order differential equations; we refer the reader to [38–52] and the references therein. Integral boundary conditions arise in thermal conduction problems, semiconductor problems and hydrodynamic problems (see [11]) and we refer the reader to [10, 11, 18, 19, 26, 28–30, 33, 34, 39, 43, 50–52] and the references therein. In [26] the authors studied the integral boundary value problem of the nonlinear Hadamard fractional differential equation

$$\begin{cases} D_{\beta}(\varphi_p(D^{\alpha} u(t))) = f(t, u(t)), & 1 < t < e, \\ u(1) = u'(1) = u'(e) = 0, & D^{\alpha} u(1) = 0, \quad \varphi_p(D^{\alpha} u(e)) = \mu \int_1^e \varphi_p(D^{\alpha} u(t)) \frac{dt}{t}, \end{cases}$$

where $\alpha \in (2, 3]$, $\beta \in (1, 2]$ and D^{α}, D^{β} are Hadamard fractional derivatives.

Many papers in the literature also considered sign-changing nonlinearity problems; see [4, 5, 7–9, 13–19, 22–30, 34, 35, 38–52] and the references therein. In [15], the authors studied the fractional differential equation with a singular decreasing nonlinearity and a p-Laplacian operator:

$$\begin{cases} -D_{0+}^{\alpha}(\varphi_p(-D_{0+}^{\gamma} z))(x) = f(x, z(x)), & 0 < x < 1, \\ z(0) = 0, \quad D_{0+}^{\gamma} z(0) = D_{0+}^{\gamma} z(1) = 0, & z(1) = \int_0^1 z(x)\, d\chi(x). \end{cases}$$

Using a double iterative technique, they showed that the above problem has a unique positive solution, and from an iterative technique, they established an appropriate sequence, which converges uniformly to the unique positive solution.

In this paper we use topological degree theory to consider the existence of nontrivial solutions for (1.1). The novelty is twofold: (1) the nonlinearity depends on the unknown function u and its integer-order, fractional-order derivatives u', $-D_{0+}^{\alpha}u$, (2) the nonlinearity can be unbounded on $[0,1] \times \mathbb{R}^3$, which improves results on semipositone problems (i.e., boundedness from below); see [5, 21, 42, 43, 45].

2 Preliminaries

We present some definitions and notations from fractional calculus theory involving Riemann–Liouville fractional derivatives; for details see the books [1–3].

Definition 2.1 (see [1–3]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to (-\infty,+\infty)$ is given by

$$D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1}f(s)\,ds,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2.2 (see [1–3]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,+\infty) \to (-\infty,+\infty)$ is given by

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds,$$

provided that the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.3 (see [5, Lemma 2.3]) *Let $\alpha > 0$, then, for $u, D_{0+}^{\alpha}u \in C(0,1) \cap L(0,1)$, we have*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_Nt^{\alpha-N}, \quad \text{for some } c_i \in \mathbb{R}, i = 1,2,\ldots,N,$$

where N is the smallest integer greater than or equal to α.

Lemma 2.4 *Suppose that* (H0) *holds. If let $-D_{0+}^{\alpha}u := v$, then the fractional boundary value problem*

$$\begin{cases} -D_{0+}^{\alpha}u(t) = v(t), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u'(1) = \int_0^1 g(t)u'(t)\,dt, \end{cases} \tag{2.1}$$

can be transformed into its equivalent Hammerstein integral equation, which takes the form

$$u(t) = \int_0^1 G_1(t,s)v(s)\,ds, \tag{2.2}$$

where

$$G_1(t,s) = \widetilde{G}_1(t,s) + \frac{t^{\alpha-1}}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\int_0^1 g(t)\widetilde{G}_2(t,s)\,dt,$$

$$\widetilde{G}_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0\le t\le s\le 1,\end{cases}\tag{2.3}$$

$$\widetilde{G}_2(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-2}(1-s)^{\alpha-2}-(t-s)^{\alpha-2}, & 0\le s\le t\le 1,\\ t^{\alpha-2}(1-s)^{\alpha-2}, & 0\le t\le s\le 1.\end{cases}$$

Proof From Lemma 2.3 we have

$$u(t)=-I^{\alpha}_{0+}v(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3},\quad \text{for } c_i\in\mathbb{R}, i=1,2,3.$$

Note that $u(0)=u'(0)=0$, and we obtain $c_2=c_3=0$. Then we have

$$u(t)=-I^{\alpha}_{0+}v(t)+c_1t^{\alpha-1}=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds+c_1t^{\alpha-1}.$$

Therefore, from the condition $u'(1)=\int_0^1 g(t)u'(t)\,dt$, we have the equation

$$-\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds+c_1=-\int_0^1 g(t)\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds\,dt+c_1\int_0^1 g(t)t^{\alpha-2}\,dt.$$

Solving this, we have

$$\begin{aligned}c_1&=\frac{1}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\left[\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds-\int_0^1 g(t)\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds\,dt\right]\\&=\frac{1}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\left[\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds-\int_0^1 g(t)\int_s^1\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,dt\,ds\right].\end{aligned}$$

As a result, we obtain

$$\begin{aligned}u(t)&=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds+\frac{1}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\int_0^1\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds\\&\quad-\frac{t^{\alpha-1}}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\int_0^1 g(t)\int_s^1\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,dt\,ds\\&=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds+\int_0^1\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds-\int_0^1\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds\\&\quad+\frac{1}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\int_0^1\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,ds\\&\quad-\frac{t^{\alpha-1}}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\int_0^1 g(t)\int_s^1\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}v(s)\,dt\,ds\\&=\int_0^1\widetilde{G}_1(t,s)v(s)\,ds+\frac{t^{\alpha-1}}{1-\int_0^1 g(t)t^{\alpha-2}\,dt}\int_0^1\int_0^1 g(t)\widetilde{G}_2(t,s)\,dt\,v(s)\,ds\\&=\int_0^1 G_1(t,s)v(s)\,ds.\end{aligned}$$

This completes the proof. □

For convenience, let

$$G_2(t,s) = \frac{\partial}{\partial t} G_1(t,s) = (\alpha-1)\left[\widetilde{G}_2(t,s) + \frac{t^{\alpha-2}}{1-\int_0^1 g(t)t^{\alpha-2}\,dt} + \int_0^1 g(t)\widetilde{G}_2(t,s)\,dt\right], \quad \text{for } t,s\in[0,1].$$

Lemma 2.5 *Suppose that* (H0) *holds. If* α, β, f *are as in* (1.1), *then the fractional boundary value problem* (1.1) *is equivalent to the Hammerstein integral equation*

$$v(t) = \int_0^1 H_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)v(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v(\tau)\,d\tau, v(s)\right) ds, \tag{2.4}$$

where

$$\begin{aligned} H_1(t,s) &= \widetilde{H}_1(t,s) + \frac{t^{\beta-1}}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 h(t)\widetilde{H}_2(t,s)\,dt, \\ \widetilde{H}_1(t,s) &= \frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1}(1-s)^{\beta-2} - (t-s)^{\beta-1}, & 0\le s\le t\le 1, \\ t^{\beta-1}(1-s)^{\beta-2}, & 0\le t\le s\le 1, \end{cases} \\ \widetilde{H}_2(t,s) &= \frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-2}(1-s)^{\beta-2} - (t-s)^{\beta-2}, & 0\le s\le t\le 1, \\ t^{\beta-2}(1-s)^{\beta-2}, & 0\le t\le s\le 1. \end{cases} \end{aligned} \tag{2.5}$$

Proof Substituting $-D_{0+}^{\alpha}u = v$ into (1.1), we have

$$\begin{cases} -D_{0+}^{\beta}v(t) = f(t, \int_0^1 G_1(t,s)v(s)\,ds, \int_0^1 G_2(t,s)v(s)\,ds, v(t)), & 0<t<1, \\ v(0) = v'(0) = 0, \qquad v'(1) = \int_0^1 h(t)v'(t)\,dt. \end{cases} \tag{2.6}$$

Using $\widetilde{f}(t)$ to replace $f(t,\cdot,\cdot,\cdot)$, and from Lemma 2.3 we obtain

$$v(t) = -I_{0+}^{\beta}\widetilde{f}(t) + c_1t^{\beta-1} + c_2t^{\beta-2} + c_3t^{\beta-3}, \quad \text{for } c_i\in\mathbb{R}, i=1,2,3.$$

Note that $v(0) = v'(0) = 0$ implies $c_2 = c_3 = 0$. Hence,

$$v(t) = -\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\widetilde{f}(s)\,ds + c_1t^{\beta-1}.$$

From the condition $v'(1) = \int_0^1 h(t)v'(t)\,dt$, we get

$$-\int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds + c_1 = -\int_0^1 h(t)\int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds\,dt + c_1\int_0^1 h(t)t^{\beta-2}\,dt.$$

Solving this equation, we obtain

$$\begin{aligned} c_1 &= \frac{1}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds \\ &\quad - \frac{1}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 h(t)\int_0^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds\,dt \end{aligned}$$

$$= \frac{1}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds$$
$$- \frac{1}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 h(t)\int_s^1 \frac{(t-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,dt\,ds.$$

Consequently, we have

$$\begin{aligned}
v(t) &= -\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\widetilde{f}(s)\,ds + \frac{1}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 \frac{t^{\beta-1}(1-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds \\
&\quad - \frac{t^{\beta-1}}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 h(t)\int_s^1 \frac{(t-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,dt\,ds \\
&= -\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\widetilde{f}(s)\,ds + \int_0^1 \frac{t^{\beta-1}(1-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds - \int_0^1 \frac{t^{\beta-1}(1-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds \\
&\quad + \frac{1}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 \frac{t^{\beta-1}(1-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,ds \\
&\quad - \frac{t^{\beta-1}}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1 h(t)\int_s^1 \frac{(t-s)^{\beta-2}}{\Gamma(\beta)}\widetilde{f}(s)\,dt\,ds \\
&= \int_0^1 \widetilde{H}_1(t,s)\widetilde{f}(s)\,ds + \frac{t^{\beta-1}}{1-\int_0^1 h(t)t^{\beta-2}\,dt}\int_0^1\int_0^1 h(t)\widetilde{H}_2(t,s)\,dt\widetilde{f}(s)\,ds \\
&= \int_0^1 H_1(t,s)\widetilde{f}(s)\,ds.
\end{aligned}$$

This completes the proof. □

Lemma 2.6 *The functions* G_i, H_1 $(i=1,2)$ *satisfy the properties*:

(i) $G_i(t,s), H_1(t,s) \geq 0$ *for* $t,s \in [0,1]\times[0,1]$,

(ii) $t^{\beta-1}\phi_\beta(s) \leq H_1(t,s) \leq \phi_\beta(s)$ *for* $t,s\in[0,1]\times[0,1]$, *where* $\phi_\beta(s) = \frac{s(1-s)^{\beta-2}}{\Gamma(\beta)} + \frac{\int_0^1 h(t)\widetilde{H}_2(t,s)\,dt}{1-\int_0^1 h(t)t^{\beta-2}\,dt}$, $s\in[0,1]$.

Proof We only prove (ii). From [20] we have

$$t^{\beta-1}\frac{1}{\Gamma(\beta)}s(1-s)^{\beta-2} \leq \widetilde{H}_1(t,s) \leq \frac{1}{\Gamma(\beta)}s(1-s)^{\beta-2}, \quad \text{for } t,s\in[0,1]\times[0,1].$$

Combining this with (2.5), we easily obtain the inequalities in (ii). This completes the proof. □

Let $E := C[0,1]$, $\|v\| := \max_{t\in[0,1]}|v(t)|$ (here $v\in E$), $P := \{v\in E : v(t)\geq 0, \forall t\in[0,1]\}$. Then $(E,\|\cdot\|)$ is a real Banach space, and P is a cone on E. Now, we define an operator $A : E\to E$ as follows:

$$(Av)(t) := \int_0^1 H_1(t,s)f\left(s, \int_0^1 G_1(s,\tau)v(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v(\tau)\,d\tau, v(s)\right)ds, \qquad (2.7)$$

for all $v\in E$. Moreover, we note that the continuity of G_1, G_2, H_1 and f implies that $A : E\to E$ is a completely continuous operator. Note that the existence of solutions of (2.6) is equivalent to that of fixed points of A, and then from (1.1) and (2.6) ($-D_{0+}^{\alpha}u = v$), we see that if there exists a $\overline{y} \in E$ such that $A\overline{y} = \overline{y}$, then $\overline{y}$ is a solution for (1.1). Therefore, in what follows we study the existence of fixed points of A. For this purpose we need to define a linear operator $L_{a,b,c} : E \to E$ as follows:

$$(L_{a,b,c}v)(t) := \int_0^1 H_{a,b,c}(t,s)v(s)\,ds, \quad \forall v \in E, \tag{2.8}$$

where $H_{a,b,c}(t,s) := aH_3(t,s) + bH_2(t,s) + cH_1(t,s)$, $\forall t,s \in [0,1]$, with $a,b,c \geq 0$ and $a^2 + b^2 + c^2 \neq 0$; here

$$H_2(t,s) = \int_0^1 H_1(t,\tau)G_2(\tau,s)\,d\tau, \qquad H_3(t,s) = \int_0^1 H_1(t,\tau)G_1(\tau,s)\,d\tau, \quad \forall t,s \in [0,1].$$

Moreover, we know that the continuity of H_i ($i = 1,2,3$) implies that $L_{a,b,c}$ is a completely continuous operator and $L_{a,b,c}(P) \subset P$. Let $r(L_{a,b,c})$ denote the spectral radius of $L_{a,b,c}$, and from Gelfand's theorem we see that $r(L_{a,b,c}) > 0$ (the proof is standard; see [20, Lemma 5]).

Let $P_0 = \{v \in P : v(t) \geq t^{\beta-1}\|v\|, \forall t \in [0,1]\}$. Then if we define an operator $(A_1v)(t) = \int_0^1 H_1(t,s)v(s)\,ds$, where H_1 is defined by (2.5), and from Lemma 2.6(ii) we have

$$A_1(P) \subset P_0. \tag{2.9}$$

Indeed, if $v \in P$, Lemma 2.6(ii) implies that

$$\int_0^1 t^{\beta-1}\phi_\beta(s)v(s)\,ds \leq \int_0^1 H_1(t,s)v(s)\,ds \leq \int_0^1 \phi_\beta(s)v(s)\,ds,$$

and

$$(A_1v)(t) \geq t^{\beta-1}\|A_1v\|.$$

Lemma 2.7 (see [53, Theorem 19.3]) *Let P be a reproducing cone in a real Banach space E and let $L : E \to E$ be a compact linear operator with $L(P) \subset P$. Let $r(L)$ be the spectral radius of L. If $r(L) > 0$, then there exists $\varphi \in P \setminus \{0\}$ such that $L\varphi = r(L)\varphi$.*

Therefore, from Lemma 2.7 we see that there exists $\varphi_{a,b,c} \in P \setminus \{0\}$ such that

$$L_{a,b,c}\varphi_{a,b,c} = r(L_{a,b,c})\varphi_{a,b,c}. \tag{2.10}$$

In what follows, we prove that

$$\varphi_{a,b,c} \in P_0. \tag{2.11}$$

Indeed, from (2.10) we have

$$\begin{aligned}\varphi_{a,b,c}(t) &= \frac{1}{r(L_{a,b,c})}(L_{a,b,c}\varphi_{a,b,c})(t) = \frac{1}{r(L_{a,b,c})}\int_0^1 H_{a,b,c}(t,s)\varphi_{a,b,c}(s)\,ds \\ &= \frac{1}{r(L_{a,b,c})}\int_0^1 \big[aH_3(t,s) + bH_2(t,s) + cH_1(t,s)\big]\varphi_{a,b,c}(s)\,ds, \quad t \in [0,1].\end{aligned}$$

Using Lemma 2.6(ii) and the definitions of H_i ($i = 1,2,3$), we have

$$\|\varphi_{a,b,c}\| \le \frac{1}{r(L_{a,b,c})}\int_0^1 \left[a\int_0^1 \phi_\beta(\tau)G_1(\tau,s)\,d\tau + b\int_0^1 \phi_\beta(\tau)G_2(\tau,s)\,d\tau + c\phi_\beta(s)\right] \times \varphi_{a,b,c}(s)\,ds$$

and

$$\begin{aligned}\varphi_{a,b,c}(t) &\ge \frac{1}{r(L_{a,b,c})}\int_0^1 \left[a\int_0^1 t^{\beta-1}\phi_\beta(\tau)G_1(\tau,s)\,d\tau + b\int_0^1 t^{\beta-1}\phi_\beta(\tau)G_2(\tau,s)\,d\tau \right.\\ &\quad \left. + ct^{\beta-1}\phi_\beta(s)\right]\varphi_{a,b,c}(s)\,ds\\ &\ge t^{\beta-1}\|\varphi_{a,b,c}\|.\end{aligned}$$

Therefore, (2.11) is true.

Lemma 2.8 (see [54]) *Let E be a Banach space and Ω a bounded open set in E. Suppose that $A:\Omega\to E$ is a continuous compact operator. If there exists $u_0 \in E\setminus\{0\}$ such that*

$$u - Au \ne \mu u_0, \quad \forall u \in \partial\Omega, \mu \ge 0,$$

then the topological degree $\deg(I-A,\Omega,0) = 0$.

Lemma 2.9 (see [54]) *Let E be a Banach space and Ω a bounded open set in E with $0\in\Omega$. Suppose that $A:\Omega\to E$ is a continuous compact operator. If*

$$Au \ne \mu u, \quad \forall u \in \partial\Omega, \mu \ge 1,$$

then the topological degree $\deg(I-A,\Omega,0) = 1$.

3 Main results

Let $\alpha_i,\beta_i,\gamma_i \ge 0$ ($i=1,2$) with $\alpha_1^2+\beta_1^2+\gamma_1^2 \ne 0$, $\alpha_2^2+\beta_2^2+\gamma_2^2\ne 0$, and $r^{-1}(L_{\alpha_i,\beta_i,\gamma_i}) = \lambda_{\alpha_i,\beta_i,\gamma_i}$ for $i=1,2$. Now, we list our assumptions for f as follows:

(H1) $f \in C([0,1]\times\mathbb{R}^3,\mathbb{R})$.

(H2) There exist two nonnegative functions $b(t), c(t) \in C[0,1]$ with $c(t)\not\equiv 0$ and a function $K(x_1,x_2,x_3)\in C[\mathbb{R}^3,\mathbb{R}^+]$ such that

$$f(t,x_1,x_2,x_3) \ge -b(t) - c(t)K(x_1,x_2,x_3), \quad \forall x_i \in \mathbb{R}, t\in[0,1], i=1,2,3.$$

(H3) $\lim_{\alpha_1|x_1|+\beta_1|x_2|+\gamma_1|x_3|\to+\infty} \frac{K(x_1,x_2,x_3)}{\alpha_1|x_1|+\beta_1|x_2|+\gamma_1|x_3|} = 0$.

(H4) $\liminf_{\alpha_1|x_1|+\beta_1|x_2|+\gamma_1|x_3|\to+\infty} \frac{f(t,x_1,x_2,x_3)}{\alpha_1|x_1|+\beta_1|x_2|+\gamma_1|x_3|} > \lambda_{\alpha_1,\beta_1,\gamma_1}$, uniformly for $t\in[0,1]$.

(H5) $\limsup_{\alpha_2|x_1|+\beta_2|x_2|+\gamma_2|x_3|\to 0} \frac{|f(t,x_1,x_2,x_3)|}{\alpha_2|x_1|+\beta_2|x_2|+\gamma_2|x_3|} < \lambda_{\alpha_2,\beta_2,\gamma_2}$, uniformly for $t\in[0,1]$.

We now present our main result.

Theorem 3.1 *Suppose that* (H0)–(H5) *hold. Then* (1.1) *has at least one nontrivial solution.*

Proof From (H4) there exist $\varepsilon_0 > 0$ and $X_0 > 0$ such that

$$f(t,x_1,x_2,x_3) \geq (\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0)\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big),$$
$$\forall t \in [0,1], \alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3| > X_0. \tag{3.1}$$

For any given ε with $\varepsilon_0 - \|c\|\varepsilon > 0$, and from (H3) there exists $X_1 > X_0$ such that

$$K(x_1,x_2,x_3) \leq \varepsilon\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big), \quad \forall \alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3| > X_1. \tag{3.2}$$

It follows from (H2), (3.1), (3.2) that

$$\begin{aligned} f(t,x_1,x_2,x_3) &\geq (\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0)\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big) - b(t) - c(t)K(x_1,x_2,x_3) \\ &\geq (\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0)\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big) \\ &\quad - b(t) - \varepsilon c(t)\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big) \\ &\geq \big(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon c(t)\big)\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big) - b(t) \\ &\geq \big(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\big)\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big) \\ &\quad - \|b\|, \quad \forall \alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3| > X_1. \end{aligned} \tag{3.3}$$

Let $C_{X_1} = (\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|)X_1 + \max_{0\leq t\leq 1,\alpha_1|x_1|+\beta_1|x_2|+\gamma_1|x_3|\leq X_1} |f(t,x_1,x_2,x_3)|$, $K^* = \max_{\alpha_1|x_1|+\beta_1|x_2|+\gamma_1|x_3|\leq X_1} K(x_1,x_2,x_3)$. Then it easy to see that

$$\begin{aligned} f(t,x_1,x_2,x_3) &\geq \big(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\big)\big(\alpha_1|x_1| + \beta_1|x_2| + \gamma_1|x_3|\big) \\ &\quad - b(t) - C_{X_1}, \quad \forall (t,x_1,x_2,x_3) \in [0,1] \times \mathbb{R}^3. \end{aligned} \tag{3.4}$$

Note that ε can be chosen arbitrarily small, and we let

$$\begin{aligned} R > \max\Bigg\{ &\frac{(\|b\| + \|c\|K^* + C_{X_1})\int_0^1 \phi_h(s)\,ds}{1 - \varepsilon M_{\alpha_1,\beta_1,\gamma_1}\|c\|}, \\ &\frac{(\lambda_{\alpha_1,\beta_1,\gamma_1} + 2\varepsilon_0 - 2\|c\|\varepsilon)(\|b\| + \|c\|K^* + C_{X_1})\int_0^1 \phi_h(s)\,ds}{\varepsilon_0 - \|c\|\varepsilon - \varepsilon\|c\|M_{\alpha_1,\beta_1,\gamma_1}(\lambda_{\alpha_1,\beta_1,\gamma_1} + 2\varepsilon_0 - 2\|c\|\varepsilon)}\Bigg\}, \end{aligned}$$

where $M_{\alpha_1,\beta_1,\gamma_1} = \int_0^1 \phi_h(s)(\alpha_1\int_0^1 G_1(s,\tau)\,d\tau + \beta_1\int_0^1 G_2(s,\tau)\,d\tau + \gamma_1)\,ds$, and $\phi_h(s) = \frac{(1-s)^{\beta-2}}{\Gamma(\beta)} + \frac{\int_0^1 h(t)\widetilde{H}_2(t,s)\,dt}{1-\int_0^1 h(t)t^{\beta-2}\,dt}$, $s \in [0,1]$.

Now we prove that

$$v - Av \neq \mu\varphi_{\alpha_1,\beta_1,\gamma_1}, \quad \forall v \in \partial B_R, \mu \geq 0, \tag{3.5}$$

where $\varphi_{\alpha_1,\beta_1,\gamma_1}$ is the positive eigenfunction of $L_{\alpha_1,\beta_1,\gamma_1}$ corresponding to the eigenvalue $\lambda_{\alpha_1,\beta_1,\gamma_1}$, and then $\varphi_{\alpha_1,\beta_1,\gamma_1} = \lambda_{\alpha_1,\beta_1,\gamma_1}L_{\alpha_1,\beta_1,\gamma_1}\varphi_{\alpha_1,\beta_1,\gamma_1}$ and $\varphi_{\alpha_1,\beta_1,\gamma_1} \in P_0$ by (2.11).

Suppose (3.5) is not true. Then there exists $v_0 \in \partial B_R$ and $\mu_0 > 0$ such that

$$v_0 - Av_0 = \mu_0\varphi_{\alpha_1,\beta_1,\gamma_1}. \tag{3.6}$$

Let

$$\begin{aligned}\widetilde{\nu}(t) &= \int_0^1 H_1(t,s)\biggl[b(s) + c(s)K\biggl(\int_0^1 G_1(s,\tau)\nu_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)\nu_0(\tau)\,d\tau, \nu_0(s)\biggr) \\ &\quad + C_{X_1}\biggr]\,ds, \quad \forall t \in [0,1]. \end{aligned} \tag{3.7}$$

Then we have

$$\begin{aligned}\widetilde{\nu}(t) &\le \int_0^1 H_1(t,s)\biggl[b(s) + c(s)\biggl[\varepsilon\biggl(\alpha_1\biggl|\int_0^1 G_1(s,\tau)\nu_0(\tau)\,d\tau\biggr| \\ &\quad + \beta_1\biggl|\int_0^1 G_2(s,\tau)\nu_0(\tau)\,d\tau\biggr| + \gamma_1\bigl|\nu_0(s)\bigr|\biggr) + K^*\biggr] + C_{X_1}\biggr]\,ds \\ &\le \int_0^1 t^{\beta-1}\phi_h(s)\biggl[b(s) + c(s)\biggl[\varepsilon\biggl(\alpha_1\int_0^1 G_1(s,\tau)\bigl|\nu_0(\tau)\bigr|\,d\tau \\ &\quad + \beta_1\int_0^1 G_2(s,\tau)\bigl|\nu_0(\tau)\bigr|\,d\tau + \gamma_1\bigl|\nu_0(s)\bigr|\biggr) + K^*\biggr] + C_{X_1}\biggr]\,ds \\ &\le t^{\beta-1}\int_0^1 \phi_h(s)\biggl[\|b\| + \|c\|\biggl[\varepsilon\biggl(\alpha_1\int_0^1 G_1(s,\tau)\,d\tau \\ &\quad + \beta_1\int_0^1 G_2(s,\tau)\,d\tau + \gamma_1\biggr)\|\nu_0\| + K^*\biggr] + C_{X_1}\biggr]\,ds \\ &\le t^{\beta-1}\bigl(\|b\| + \|c\|K^* + C_{X_1}\bigr)\int_0^1 \phi_h(s)\,ds \\ &\quad + t^{\beta-1}\varepsilon\|c\|\int_0^1 \phi_h(s)\biggl(\alpha_1\int_0^1 G_1(s,\tau)\,d\tau + \beta_1\int_0^1 G_2(s,\tau)\,d\tau + \gamma_1\biggr)\|\nu_0\|\,ds \\ &\le \bigl(\|b\| + \|c\|K^* + C_{X_1}\bigr)\int_0^1 \phi_h(s)\,ds \\ &\quad + \varepsilon\|c\|R\int_0^1 \phi_h(s)\biggl(\alpha_1\int_0^1 G_1(s,\tau)\,d\tau + \beta_1\int_0^1 G_2(s,\tau)\,d\tau + \gamma_1\biggr)\,ds. \end{aligned} \tag{3.8}$$

Consequently, we have

$$\begin{aligned}\|\widetilde{\nu}\| &\le \bigl(\|b\| + \|c\|K^* + C_{X_1}\bigr)\int_0^1 \phi_h(s)\,ds \\ &\quad + \varepsilon\|c\|R\int_0^1 \phi_h(s)\biggl(\alpha_1\int_0^1 G_1(s,\tau)\,d\tau + \beta_1\int_0^1 G_2(s,\tau)\,d\tau + \gamma_1\biggr)\,ds \\ &< R. \end{aligned} \tag{3.9}$$

Note that $\widetilde{\nu} \in P_0$, and then from (2.9), $\varphi_{\alpha_1,\beta_1,\gamma_1} \in P_0$, and

$$\begin{aligned}\nu_0(t) + \widetilde{\nu}(t) &= A\nu_0(t) + \mu_0\varphi_{\alpha_1,\beta_1,\gamma_1}(t) + \widetilde{\nu}(t) \\ &= \int_0^1 H_1(t,s)\biggl[f\biggl(s, \int_0^1 G_1(s,\tau)\nu_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)\nu_0(\tau)\,d\tau, \nu_0(s)\biggr)\end{aligned}$$

$$+ b(s) + c(s)K\left(\int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau, v_0(s)\right) + C_{X_1}\Bigg]\,ds$$
$$+ \mu_0\varphi_{\alpha_1,\beta_1,\gamma_1}(t),$$

we have

$$v_0 + \widetilde{v} \in P_0, \tag{3.10}$$

using the fact that

$$f\left(s, \int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau, v_0(s)\right) + b(s)$$
$$+ c(s)K\left(\int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau, v_0(s)\right) + C_{X_1} \in P.$$

Therefore, (2.10), (3.4) and (3.7) enable us to obtain

$$\begin{aligned}
&Av_0(t) + \widetilde{v}(t) \\
&= \int_0^1 H_1(t,s)f\left(s, \int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau, v_0(s)\right)ds \\
&\quad + \int_0^1 H_1(t,s)\left[b(s) + c(s)K\left(\int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau, v_0(s)\right)\right. \\
&\quad \left. + C_{X_1}\right]ds \\
&\geq \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\right)\int_0^1 H_1(t,s)\left[\alpha_1\left|\int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau\right|\right. \\
&\quad \left. + \beta_1\left|\int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau\right| + \gamma_1\left|v_0(s)\right|\right]ds \\
&\quad - \int_0^1 H_1(t,s)\left[b(s) + C_{X_1}\right]ds \\
&\quad + \int_0^1 H_1(t,s)\left[b(s) + c(s)K\left(\int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau, \int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau, v_0(s)\right)\right. \\
&\quad \left. + C_{X_1}\right]ds \\
&\geq \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\right)\int_0^1 H_1(t,s)\left[\alpha_1\left|\int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau\right|\right. \\
&\quad \left. + \beta_1\left|\int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau\right| + \gamma_1\left|v_0(s)\right|\right]ds \\
&\geq \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\right)\left|\alpha_1\int_0^1 H_1(t,s)\int_0^1 G_1(s,\tau)v_0(\tau)\,d\tau\,ds\right. \\
&\quad \left. + \beta_1\int_0^1 H_1(t,s)\int_0^1 G_2(s,\tau)v_0(\tau)\,d\tau\,ds + \gamma_1\int_0^1 H_1(t,s)v_0(s)\,ds\right| \\
&\geq \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\right)\left|\alpha_1\int_0^1 H_3(t,\tau)v_0(\tau)\,d\tau + \beta_1\int_0^1 H_2(t,\tau)v_0(\tau)\,d\tau\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \gamma_1 \int_0^1 H_1(t,s)v_0(s)\,ds \Big| \\
&= \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\right) \left| \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)v_0(s)\,ds \right| \\
&\geq \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \varepsilon\|c\|\right) \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)v_0(s)\,ds.
\end{aligned}
\tag{3.11}
$$

From the definition of operator $L_{\alpha_1,\beta_1,\gamma_1}$, we get

$$
\begin{aligned}
&\left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \|c\|\varepsilon\right) \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)v_0(s)\,ds \\
&\quad = \lambda_{\alpha_1,\beta_1,\gamma_1} \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)\left(v_0(s) + \widetilde{v}(s)\right)ds + \left(\varepsilon_0 - \|c\|\varepsilon\right) \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)v_0(s)\,ds \\
&\qquad - \lambda_{\alpha_1,\beta_1,\gamma_1} \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)\widetilde{v}(s)\,ds \\
&\quad = \lambda_{\alpha_1,\beta_1,\gamma_1} L_{\alpha_1,\beta_1,\gamma_1}(v_0 + \widetilde{v})(t) + \left(\varepsilon_0 - \|c\|\varepsilon\right) \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)\left(v_0(s) + \widetilde{v}(s)\right)ds \\
&\qquad - \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \|c\|\varepsilon\right) \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)\widetilde{v}(s)\,ds.
\end{aligned}
\tag{3.12}
$$

From (3.10), we get $v_0(t) + \widetilde{v}(t) \geq t^{\beta-1}\|v_0 + \widetilde{v}\| \geq t^{\beta-1}(\|v_0\| - \|\widetilde{v}\|)$, $t \in [0,1]$, and hence from (3.8) we have

$$
\begin{aligned}
&\left(\varepsilon_0 - \|c\|\varepsilon\right) \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)\left(v_0(s) + \widetilde{v}(s)\right)ds \\
&\qquad - \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \|c\|\varepsilon\right) \int_0^1 H_{\alpha_1,\beta_1,\gamma_1}(t,s)\widetilde{v}(s)\,ds \\
&\quad \geq \left(\varepsilon_0 - \|c\|\varepsilon\right)\left(R - \|\widetilde{v}\|\right) \int_0^1 s^{\beta-1} H_{\alpha_1,\beta_1,\gamma_1}(t,s)\,ds - \left(\lambda_{\alpha_1,\beta_1,\gamma_1} + \varepsilon_0 - \|c\|\varepsilon\right) \\
&\qquad \times \left[\left(\|b\| + \|c\|K^* + C_{X_1}\right) \int_0^1 \phi_h(s)\,ds + \varepsilon\|c\|RM_{\alpha_1,\beta_1,\gamma_1}\right] \int_0^1 s^{\beta-1} H_{\alpha_1,\beta_1,\gamma_1}(t,s)\,ds \\
&\quad \geq 0.
\end{aligned}
\tag{3.13}
$$

Combining (3.11), (3.12) and (3.13), and we obtain

$$
Av_0(t) + \widetilde{v}(t) \geq \lambda_{\alpha_1,\beta_1,\gamma_1} L_{\alpha_1,\beta_1,\gamma_1}(v_0 + \widetilde{v})(t), \quad t \in [0,1]. \tag{3.14}
$$

Therefore, using (3.6) and (3.14) we have

$$
v_0 + \widetilde{v} = Av_0 + \mu_0 \varphi_{\alpha_1,\beta_1,\gamma_1} + \widetilde{v} \geq \lambda_{\alpha_1,\beta_1,\gamma_1} L_{\alpha_1,\beta_1,\gamma_1}(v_0 + \widetilde{v}) + \mu_0 \varphi_{\alpha_1,\beta_1,\gamma_1} \geq \mu_0 \varphi_{\alpha_1,\beta_1,\gamma_1}.
$$

Define

$$
\mu^* = \sup\{\mu > 0 : v_0 + \widetilde{v} \geq \mu \varphi_{\alpha_1,\beta_1,\gamma_1}\}.
$$

It is easy to see that $\mu^* \geq \mu_0$ and $v_0 + \widetilde{v} \geq \mu^* \varphi_{\alpha_1,\beta_1,\gamma_1}$. From $\varphi_{\alpha_1,\beta_1,\gamma_1} = \lambda_{\alpha_1,\beta_1,\gamma_1} L_{\alpha_1,\beta_1,\gamma_1} \times \varphi_{\alpha_1,\beta_1,\gamma_1}$, we obtain

$$\lambda_{\alpha_1,\beta_1,\gamma_1} L_{\alpha_1,\beta_1,\gamma_1}(v_0 + \widetilde{v}) \geq \lambda_{\alpha_1,\beta_1,\gamma_1} L_{\alpha_1,\beta_1,\gamma_1} \mu^* \varphi_{\alpha_1,\beta_1,\gamma_1} = \mu^* \varphi_{\alpha_1,\beta_1,\gamma_1}.$$

Hence

$$v_0 + \widetilde{v} \geq \lambda_{\alpha_1,\beta_1,\gamma_1} L_{\alpha_1,\beta_1,\gamma_1}(v_0 + \widetilde{v}) + \mu_0 \varphi_{\alpha_1,\beta_1,\gamma_1} \geq (\mu_0 + \mu^*) \varphi_{\alpha_1,\beta_1,\gamma_1},$$

which contradicts the definition of μ^*. Therefore, (3.5) holds, and from Lemma 2.8 we obtain

$$\deg(I - A, B_R, 0) = 0. \tag{3.15}$$

From (H5) there exist $0 < \varepsilon_1 < \lambda_{\alpha_2,\beta_2,\gamma_2}$ and $0 < r < R$ such that

$$\left|f(t, x_1, x_2, x_3)\right| \leq (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1)\left(\alpha_2|x_1| + \beta_2|x_2| + \gamma_2|x_3|\right),$$

for all $x_i \in \mathbb{R}$, $i = 1, 2, 3$, $t \in [0, 1]$ with $0 \leq \alpha_2|x_1| + \beta_2|x_2| + \gamma_2|x_3| < r$. Consequently, we obtain

$$\begin{aligned}
\left|(Av)(t)\right| &\leq (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) \int_0^1 H_1(t,s)\left(\alpha_2\left|\int_0^1 G_1(s,\tau)v(\tau)\,d\tau\right|\right.\\
&\quad \left. + \beta_2\left|\int_0^1 G_2(s,\tau)v(\tau)\,d\tau\right| + \gamma_2\left|v(s)\right|\right) ds \\
&\leq (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) \int_0^1 H_1(t,s)\left(\alpha_2 \int_0^1 G_1(s,\tau)\left|v(\tau)\right| d\tau\right.\\
&\quad \left. + \beta_2 \int_0^1 G_2(s,\tau)\left|v(\tau)\right| d\tau + \gamma_2\left|v(s)\right|\right) ds \\
&= (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) \int_0^1 H_{\alpha_2,\beta_2,\gamma_2}(t,s)\left|v(s)\right| ds \\
&= (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1)\left(L_{\alpha_2,\beta_2,\gamma_2}|v|\right)(t), \quad \forall t \in [0,1], v \in E, \|v\| \leq r.
\end{aligned}$$

Now for this r, we prove that

$$Av \neq \lambda v, \quad \forall v \in \partial B_r, \lambda \geq 1. \tag{3.16}$$

Assume the contrary. Then there exist $v_0 \in \partial B_r$ and $\lambda_0 \geq 1$ such that $Av_0 = \lambda_0 v_0$. Let $\omega(t) = |v_0(t)|$. Then $\omega \in \partial B_r \cap P$ and

$$\omega \leq \frac{1}{\lambda_0}(\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) L_{\alpha_2,\beta_2,\gamma_2}\omega \leq (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) L_{\alpha_2,\beta_2,\gamma_2}\omega.$$

By induction, we have $\omega \leq (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1)^n L^n_{\alpha_2,\beta_2,\gamma_2}\omega$, for $n = 1, 2, \ldots$. As a result, we have

$$\|\omega\| \leq (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1)^n \left\|L^n_{\alpha_2,\beta_2,\gamma_2}\right\| \|\omega\|,$$

and thus

$$1 \le (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1)^n \left\| L^n_{\alpha_2,\beta_2,\gamma_2} \right\|.$$

Therefore, by Gelfand's theorem, we have

$$(\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) r(L_{\alpha_2,\beta_2,\gamma_2}) = (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) \lim_{n\to\infty} \sqrt[n]{\left\| L^n_{\alpha_2,\beta_2,\gamma_2} \right\|} \ge 1.$$

This contradicts

$$(\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon_1) r(L_{\alpha_2,\beta_2,\gamma_2}) = 1 - \varepsilon_1 r(L_{\alpha_2,\beta_2,\gamma_2}) < 1.$$

Thus (3.16) holds and from Lemma 2.9 we have

$$\deg(I - A, B_r, 0) = 1. \tag{3.17}$$

Now (3.15) and (3.17) imply that

$$\deg(I - A, B_R \setminus \overline{B}_r, 0) = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = -1.$$

Therefore the operator A has at least one fixed point in $B_R \setminus \overline{B}_r$. Equivalently, (1.1) has at least one nontrivial solution. This completes the proof. □

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Authors' contributions
The authors conceived of the study, drafted the manuscript, and approved the final manuscript.

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