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Positive periodic solution for p-Laplacian neutral damped Duffing equation with strong singularities of attractive and repulsive type

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Abstract

This paper is devoted to investigating the following p-Laplacian neutral damped Duffing equation with singularity:

$$(\phi_p(u(t)-cu(t-\tau))')'+Pu'(t)+g(u(t))=e(t),$$

where g has a singularity at $u=0$. Applying the Manásevich–Mawhin theorem on a continuous case of topological degree, we obtain the existence of a positive periodic solution for this equation.

MSC: 34B16; 34B18; 34C25

Keywords: Positive periodic solution; Neutral operator; Attractive and repulsive singularities; Duffing equation

1 Introduction

In recent years, there has been a fairly great amount of work on periodic solutions for Duffing equations with a singularity (see [1–5, 7–11] and the references cited therein), in account of its applications in applied sciences. For example, the Brillouin electron beam focusing problem [2, 8] is to prove the existence of a positive π-periodic solutions of

$$u''+a(1+\cos 2t)u=\frac{1}{u},$$

where a is a positive constant.

In 2012, Cheng and Ren [1] discussed the existence and multiplicity of positive periodic solutions for the following Duffing equation:

$$u''(t)+g(u(t))=p(t), \tag{1.1}$$

where the nonlinear term g has a strong singularity of repulsive type at $u=0$ and satisfies super-linearity condition at $u=+\infty$. It is concluded that there exist infinitely many positive periodic solutions for Eq. (1.1) by applications of the generalized Poincaré–Birkhoff twist theorem. Afterwards, Wang and Ma [11] investigated Eq. (1.1) with g has a strong singularity of repulsive type at $u = 0$ and a semi-linearity condition at $x = +\infty$, using the Poincaré–Birkhoff theorem, the authors proved the existence of infinitely many 2π-periodic solutions.

Proceeding from [1, 2, 8, 11], in this paper, we further consider the following p-Laplacian neutral Duffing equation with singularity:

$$\big(\phi_p\big(u(t) - cu(t-\tau)\big)'\big)' + Pu'(t) + g\big(u(t)\big) = e(t), \tag{1.2}$$

where $p \geq 2$, $\varphi_p(u) = |u|^{p-2}u$ for $u \neq 0$ and $\varphi_p(0) = 0$, c is a constant and $|c| \neq 1$, P, τ are constants and $0 \leq \tau < T$; $e \in L^2(\mathbb{R})$ is a T-periodic function; $g : (0,+\infty) \to \mathbb{R}$ is a L^2-Carathéodory function, the nonlinear term g of (1.2) has a singularity at $u = 0$, i.e., we have a strong singularity of repulsive type (g_1):

$$\lim_{u\to 0^+} g(u) = -\infty, \quad \text{and} \quad \lim_{u\to 0^+} \int_1^u g(s)\,ds = +\infty;$$

or a strong singularity of attractive type (g_2):

$$\lim_{u\to 0^+} g(u) = +\infty, \quad \text{and} \quad \lim_{u\to 0^+} \int_1^u g(s)\,ds = -\infty.$$

Applying topological degree theory [6], we obtain the following conclusions.

Theorem 1.1 *Assume that conditions $|c| < 1$ and (g_1) hold. Suppose the following conditions hold*:

(H_1) *There exist two positive constants d_1, d_2 with $d_1 < d_2$ such that $g(u) - e(t) < 0$ for $(t,u) \in [0,T] \times (0,d_1)$ and $g(u) - e(t) > 0$ for $(t,u) \in [0,T] \times (d_2,+\infty)$.*

Then Eq. (1.2) *has at least one positive T-periodic solution.*

Theorem 1.2 *Assume that the conditions $|c| > 1$, (g_1) and (H_1) hold. Suppose the following condition holds*:

(H_2) *There exist two positive constants α, β such that*

$$g(u) \leq \alpha u^{p-1} + \beta, \quad \textit{for all } u > 0.$$

Then Eq. (1.2) *has at least one positive T-periodic solution if $\alpha^{\frac{1}{p}}(1 + 2|c|)^{\frac{1}{p}} T < |c| - 1$.*

Remark 1.3 It is worth mentioning that the condition of the nonlinear term g is relatively weak in the case that $|c| < 1$, i.e., the nonlinear term g may satisfy the sub-linearity, semi-linearity or super-linearity conditions at $x = \infty$. The nonlinear term g in the case that $|c| > 1$ only satisfies the semi-linearity condition at $u = \infty$. Obviously, our result can be more general.

Remark 1.4 The condition (g_2) contradicts the condition (g_1). Therefore, the above methods of [1, 11] and condition of Theorems 1.1–1.2 are no long applicable to prove the existence of a positive periodic solutions for Eq. (1.2) with strong singularity of attractive type. We need to give another method and conditions to get over this problem.

Theorem 1.5 *Assume that conditions $|c| < 1$ and (g_2) hold. Furthermore, suppose the following condition holds*:

(H_3) *There exist two positive constants d_3, d_4 with $d_3 < d_4$ such that $g(u) - e(t) > 0$ for $(t,u) \in [0,T] \times (0,d_3)$ and $g(u) - e(t) < 0$ for $(t,u) \in [0,T] \times (d_4,+\infty)$.*

Then Eq. (1.2) has at least one positive T-periodic solution.

Theorem 1.6 *Assume that conditions $|c| > 1$, (g_2) and (H_3) hold. Suppose the following condition holds*:

(H_4) *There exist two positive constants m, n such that*

$$-g(u) \le mu^{p-1} + n, \quad \text{for all } u > 0.$$

Then Eq. (1.2) has at least one positive T-periodic solution if $m^{\frac{1}{p}}(1+2|c|)^{\frac{1}{p}}T < |c| - 1$.

2 Periodic solution for Eq. (1.2) with strong singularity of repulsive type

Firstly, let $A : C_T \to C_T$ be the operator on $C_T := \{u \in C(\mathbb{R},\mathbb{R}) : u(t+T) \equiv u(t)\ \forall t \in \mathbb{R}\}$ given by

$$(Au)(t) := u(t) - cu(t-\tau) \quad \forall u \in C_T, t \in \mathbb{R}.$$

Lemma 2.1 (see [12]) *The operator A has a continuous inverse A^{-1} on C_T, satisfying*

(1) $|[A^{-1}f](t)| \le \frac{\|f\|}{|1-|c||}$, $\forall f \in C_T$, *where* $\|f\| := \max_{t\in\mathbb{R}} |f(t)|$.

(2) $\int_0^T |[A^{-1}f](t)|\,dt \le \frac{1}{|1-|c||}\int_0^T |f(t)|\,dt$, $\forall f \in C_T$.

Secondly, we embed Eq. (1.2) into the following equation family with a parameter $\lambda \in (0,1]$:

$$\big(\phi_p(Au)'(t)\big)' + \lambda Pu'(t) + \lambda g\big(u(t)\big) = \lambda e(t). \tag{2.1}$$

The following lemma is a consequence of Theorem 3.1 of [6].

Lemma 2.2 *Assume that there exist positive constants E_1, E_2, E_3 and $E_1 < E_2$ such that the following conditions hold*:

(1) *Each possible periodic solution u to Eq. (2.1) such that $E_1 < u(t) < E_2$, for all $t \in [0,T]$ and $\|u'\| < E_3$.*

(2) *Each possible solution C to the equation*

$$g(C) - \frac{1}{T}\int_0^T e(t)\,dt = 0$$

satisfies $C \in (E_1,E_2)$.

(3) *We have*

$$\left(g(E_1) - \frac{1}{T}\int_0^T e(t)\,dt\right)\left(g(E_2) - \frac{1}{T}\int_0^T e(t)\,dt\right) < 0.$$

Then Eq. (1.2) has at least one T-periodic solution.

2.1 Proof of Theorem 1.1

Proof of Theorem 1.1 Firstly, integrating both sides of Eq. (2.1) over $[0, T]$, we get

$$\int_0^T \bigl[g\bigl(u(t)\bigr) - e(t)\bigr]\,dt = 0. \tag{2.2}$$

In view of the mean value theorem of integrals, there exists a point $\xi \in (0, T)$ such that

$$g\bigl(u(\xi)\bigr) - e(\xi) = 0.$$

From condition (H_1) and $u(t)$ being continuous, we have

$$d_1 \le u(\xi) \le d_2. \tag{2.3}$$

Multiplying both sides of Eq. (2.1) by $(Au)'(t)$ and integrating from 0 to T, we deduce

$$\begin{aligned} &\int_0^T \bigl(\phi_p\bigl((Au)'(t)\bigr)\bigr)'(Au)'(t)\,dt + \lambda P \int_0^T u'(t)(Au)'(t)\,dt + \lambda \int_0^T g\bigl(u(t)\bigr)(Au)'(t)\,dt \\ &\quad = \lambda \int_0^T e(t)(Au)'(t)\,dt. \end{aligned} \tag{2.4}$$

Moreover,

$$\int_0^T \bigl(\phi_p(Au)'(t)\bigr)'(Au)'(t)\,dt = \int_0^T (Au)'(t)\,d\phi_p(Au)'(t) = 0 \tag{2.5}$$

and

$$\begin{aligned} \int_0^T g\bigl(u(t)\bigr)(Au)'(t)\,dt &= \int_0^T g\bigl(u(t)\bigr)\bigl(Au'\bigr)(t)\,dt \\ &= \int_0^T g\bigl(u(t)\bigr)\bigl(u'(t) - cu'(t-\tau)\bigr)\,dt \\ &= \int_0^T g\bigl(u(t)\bigr)\,du(t) - c\int_0^T g\bigl(u(t)\bigr)\,du(t-\tau) \\ &= -c\int_0^T g\bigl(u(t)\bigr)\,du(t) \\ &= 0, \end{aligned} \tag{2.6}$$

since $(Au)'(t) = (Au')(t)$ and $du(t) = \frac{du(t-\tau)}{d(t-\tau)}\,dt = du(t-\tau)$.

Substituting Eqs. (2.5) and (2.6) into (2.4), we obtain

$$P\int_0^T u'(t)\bigl(Au'\bigr)(t)\,dt = \int_0^T e(t)\bigl(Au'\bigr)(t)\,dt. \tag{2.7}$$

From Eq. (2.7), we arrive at

$$\left|P\int_0^T \bigl|u'(t)\bigr|^2\,dt\right| = \left|Pc\int_0^T u'(t)u'(t-\tau)\,dt + \int_0^T e(t)\bigl(Au'\bigr)(t)\,dt\right|.$$

From the Hölder inequality and $\int_0^T |u'(t-\tau)|^2\,dt = \int_0^T |u'(t)|^2\,dt$, we see that

$$\begin{aligned} |P|\int_0^T \left|u'(t)\right|^2 dt &\le |P||c|\int_0^T \left|u'(t)\right|\left|u'(t-\tau)\right| dt + \left(1+|c|\right)\int_0^T \left|e(t)\right|\left|u'(t)\right| dt \\ &\le |P||c|\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \left|u'(t-\tau)\right|^2 dt\right)^{\frac{1}{2}} \\ &\quad + \left(1+|c|\right)\left(\int_0^T \left|e(t)\right|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \left|u'(t)\right| dt\right)^{\frac{1}{2}} \\ &= |P||c|\int_0^T \left|u'(t)\right|^2 dt + \left(1+|c|\right)\|e\|_2\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}, \end{aligned}$$

where $\|e\|_2 := (\int_0^T |e(t)|^2\,dt)^{\frac{1}{2}}$. Obviously, $|P| - |P||c| > 0$, since $|c| < 1$. Hence, we deduce

$$\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} \le \frac{(1+|c|)\|e\|_2}{|P| - |P||c|} := M_1'. \tag{2.8}$$

From Eqs. (2.3), (2.8) and the Hölder inequality, we see that

$$\begin{aligned} u(t) = u(\xi) + \int_\xi^t u'(s)\,ds &\le d_2 + \sqrt{T}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &\le d_2 + \sqrt{T}\frac{(1+|c|)\|e\|_2}{|P| - |P||c|} := M_1. \end{aligned} \tag{2.9}$$

On the other hand, from Eq. (2.2), it is clear that

$$\begin{aligned} \int_0^T \left|g\left(u(t)\right)\right| dt &= \int_{g(u(t))\ge 0} g\left(u(t)\right) dt - \int_{g(u(t))\le 0} g\left(u(t)\right) dt \\ &= 2\int_{g(u(t))\ge 0} g\left(u(t)\right) dt - \int_0^T e(t)\,dt. \end{aligned} \tag{2.10}$$

Case (I). If $\bar{e} := \frac{1}{T}\int_0^T e(t)\,dt \le 0$, from Eq. (2.10), we have

$$\int_0^T \left|g\left(u(t)\right)\right| dt \le 2\int_0^T \left(g^+\left(u(t)\right) - e(t)\right) dt,$$

where $g^+(u) := \max\{g(u), 0\}$. Since $g^+(u(t)) - e(t) \ge 0$, from condition (H_1), we know $u(t) \ge d_2$. Then we deduce

$$\begin{aligned} \int_0^T \left|g\left(u(t)\right)\right| dt &\le 2\int_0^T g^+\left(u(t)\right) dt + \int_0^T \left|e(t)\right| dt \\ &\le 2T\left\|g_{M_1}^+\right\| + T^{\frac{1}{2}}\|e\|_2, \end{aligned} \tag{2.11}$$

where $\|g^+_{M_1}\| := \max_{d_2\le u\le M_1} g^+(u)$. As $(Au)(0) = (Au)(T)$, there exists a point $t_1 \in [0,T]$ such that $(Au)'(t_1) = 0$, from Eqs. (2.2), (2.8) and (2.11), we have

$$\begin{aligned}\left\|\phi_p\big((Au)'\big)\right\| &= \max_{t\in[t_1,t_1+T]}\left\{\left|\int_{t_1}^{t}\big(\phi_p\big((Au)'(s)\big)\big)'\,ds\right|\right\}\\ &\le |P|\int_0^T\left|u'(t)\right|dt+\int_0^T\left|g\big(u(t)\big)\right|dt+\int_0^T\left|e(t)\right|dt\\ &\le |P|T^{\frac{1}{2}}\left(\int_0^T\left|u'(t)\right|^2dt\right)^{\frac{1}{2}}+\int_0^T\left|g\big(u(t)\big)\right|dt+\int_0^T\left|e(t)\right|dt\\ &\le |P|T^{\frac{1}{2}}M_1'^{\frac{1}{2}}+2T\left\|g^+_{M_1}\right\|+2T^{\frac{1}{2}}\|e\|_2 := M_2'. \end{aligned}\tag{2.12}$$

We claim that there exists a positive constant $M_2^{**} > M_2' + 1$ such that, for all $t \in \mathbb{R}$,

$$\left\|(Au)'\right\| \le M_2^{**}.\tag{2.13}$$

In fact, if x' is not bounded, there exists a positive constant M_2'' such that $\|u'\| > M_2''$ for some $u' \in \mathbb{R}$. Therefor, it is clear that

$$\left\|\phi_p(Au)'\right\| = \left\|\phi_p\big(Au'\big)\right\| = \left\|Au'\right\|^{p-1} = \big(1+|c|\big)^{p-1}\left\|u'\right\|^{p-1} \ge \big(1+|c|\big)^{p-1}M_2''^{p-1} := M_2^*.$$

Then we have a contradiction. So, Eq. (2.13) holds. By Lemma 2.1 and Eq. (2.13), we get

$$\begin{aligned}\left\|u'\right\| &= \left\|A^{-1}Au'\right\| = \left\|A^{-1}(Au)'\right\|\\ &\le \frac{\|(Au)'\|}{1-|c|}\\ &\le \frac{M_2^{**}}{1-|c|} := M_2,\end{aligned}\tag{2.14}$$

since $|c| < 1$.

Case (II). If $\bar{e} > 0$, from Eq. (2.10), we obtain

$$\int_0^T\left|g\big(u(t)\big)\right|dt \le 2\int_0^T g^+\big(u(t)\big)\,dt.$$

Since $g^+(u(t)) \ge 0$, from condition (H_1), we know that there exists a positive constant d_2^* such that $u(t) \ge d_2^*$. Therefore, we see that

$$\begin{aligned}\int_0^T\left|g\big(u(t)\big)\right|dt &\le 2\int_0^T g^+\big(u(t)\big)\,dt\\ &\le 2T\left\|g^+_M\right\|,\end{aligned}$$

where $\|g^+_M\| := \max_{d_2^*\le u\le M_1} g^+(x)$. Similarly, we deduce $\|u'\| \le M_2$.

Multiplying both sides of Eq. (2.1) by $u'(t)$ and integrating on the interval $[\xi, t]$, where $\xi \in [0, T]$ is defined in Eq. (2.3), we get

$$\begin{aligned}\lambda \int_{u(\xi)}^{u(t)} g(v)\,dv &= \lambda \int_{\xi}^{t} g\big(u(s)\big)u'(s)\,ds \\ &= -\int_{\xi}^{t} \big(\phi_p\big((Au)'(s)\big)\big)'u'(s)\,ds - \lambda P \int_{\xi}^{t} |u'(s)|^2\,ds \\ &\quad + \lambda \int_{\xi}^{t} e(s)u'(s)\,ds. \end{aligned} \tag{2.15}$$

Furthermore, from Eqs. (2.12), (2.8) and (2.14), we get

$$\begin{aligned}\lambda \left| \int_{u(\xi)}^{u(t)} g(v)\,dv \right| &\le \int_{\xi}^{t} \big|\big(\phi_p\big((Au)'(s)\big)\big)'\big|\big|u'(s)\big|\,ds + \lambda P \int_{\xi}^{t} \big|u'(s)\big|^2\,ds \\ &\quad + \lambda \int_{\xi}^{t} \big|e(s)\big|\big|u'(s)\big|\,ds \\ &\le \lambda M_2 M_2' + \lambda |P| \big(M_1'\big)^2 + \lambda M_2 T^{\frac{1}{2}} \|e\|_2 := \lambda M_3'.\end{aligned}$$

From the repulsive condition (g_1), we know that there exists a constant $M_3 > 0$ such that

$$u(t) \ge M_3, \quad \forall t \in [\xi, T]. \tag{2.16}$$

Similarly, we can discuss $t \in [0, \xi]$.

From Eqs. (2.3), (2.9), (2.14) and (2.16), it is obvious that a periodic solution u to Eq. (2.1) satisfies

$$E_1 < u(t) < E_2, \qquad \|u'\| < E_3.$$

Then the condition (1) of Lemma 2.2 is satisfied. For a possible solution C to the equation

$$g(C) - \frac{1}{T}\int_0^T e(t)\,dt = 0,$$

we have $C \in (E_1, E_2)$. Hence, the condition (2) of Lemma 2.2 holds. Finally, it is clear that the condition (3) of Lemma 2.2 is also satisfied. In fact, from condition (H_1), we can get

$$g(E_1) - \frac{1}{T}\int_0^T e(t)\,dt < 0$$

and

$$g(E_2) - \frac{1}{T}\int_0^T e(t)\,dt > 0.$$

Using Lemma 2.2, it is concluded that Eq. (1.2) has at least one positive periodic solution. □

2.2 Proof of Theorem 1.2

Proof of Theorem 1.2 The same strategy and notation are followed as in the proof of Theorem 1.1. Then we see that

$$u(t) \le d_2 + \int_0^T \left|u'(t)\right| dt. \tag{2.17}$$

Multiplying both sides of Eq. (2.1) by $(Au)(t)$ and integrating on the interval $[0, T]$, it is clear that

$$\begin{aligned}&\int_0^T \bigl(\phi_p\bigl((Au)'(t)\bigr)\bigr)'(Au)(t)\,dt + \lambda P\int_0^T u'(t)(Au)(t)\,dt + \lambda\int_0^T g\bigl(u(t)\bigr)(Au)(t)\,dt\\ &\quad = \lambda\int_0^T e(t)(Au)(t)\,dt.\end{aligned} \tag{2.18}$$

Substituting $\int_0^T (\phi_p(Au)'(t))'(Au)(t)\,dt = -\int_0^T |(Au)'(t)|^p\,dt$ and $P\int_0^T u'(t)u(t)\,dt = 0$ into Eq. (2.18), we have

$$\begin{aligned}\int_0^T \left|(Au)'(t)\right|^p dt = {}&-\lambda Pc\int_0^T u'(t)u(t-\tau)\,dt + \lambda\int_0^T g\bigl(u(t)\bigr)\bigl(u(t) - cu(t-\tau)\bigr)\,dt\\ &- \lambda\int_0^T e(t)\bigl(u(t) - cu(t-\tau)\bigr)\,dt.\end{aligned} \tag{2.19}$$

Furthermore, we deduce

$$\int_0^T u'(t)u(t-\tau)\,dt = \int_0^T u(t-\tau)\,du(t) = \int_0^T u(t-\tau)\,du(t-\tau) = 0. \tag{2.20}$$

From condition (H_2) and $u(t) > 0$, we see that

$$\int_0^T g\bigl(u(t)\bigr)u(t)\,dt \le \alpha\int_0^T \bigl(u(t)\bigr)^p\,dt + \beta\int_0^T u(t)\,dt. \tag{2.21}$$

Substituting Eqs. (2.20) and (2.21) into (2.19), applying the Hölder inequality, we obtain

$$\begin{aligned}\int_0^T \left|(Au)'(t)\right|^p dt &\le \alpha\int_0^T \bigl(u(t)\bigr)^p\,dt + \beta\int_0^T u(t)\,dt + |c|\int_0^T \left|g\bigl(u(t)\bigr)\right|\left|u(t-\tau)\right| dt\\ &\quad + \bigl(1+|c|\bigr)\|u\|\int_0^T \left|e(t)\right| dt\\ &\le \alpha T\|u\|^p + \beta T\|u\| + |c|\|u\|\int_0^T \left|g\bigl(u(t)\bigr)\right| dt\\ &\quad + \bigl(1+|c|\bigr)\|u\|T^{\frac{1}{2}}\|e\|_2.\end{aligned} \tag{2.22}$$

From Eq. (2.10) and condition (H_2), we get

$$\begin{aligned}\int_0^T \left|g\bigl(u(t)\bigr)\right| dt &= 2\int_{g(u(t))\ge 0} g\bigl(u(t)\bigr)\,dt - \int_0^T e(t)\,dt\\ &\le 2\alpha\|u\|^{p-1}T + 2\beta T + \|e\|_2 T^{\frac{1}{2}}.\end{aligned} \tag{2.23}$$

Substituting Eqs. (2.17) and (2.23) into (2.22), we see that

$$\begin{aligned}
&\int_0^T \left|(Au)'(t)\right|^p dt \\
&\quad \le \alpha\left(1+2|c|\right)T\|u\|^p + \left(1+2|c|\right)\left(\beta T + T^{\frac{1}{2}}\|e\|_2\right)\|u\| \\
&\quad \le \alpha\left(1+2|c|\right)T\left(d_2 + \int_0^T \left|u'(t)\right| dt\right)^p + N_1\left(d_2 + \int_0^T \left|u'(t)\right| dt\right) \\
&\quad = \alpha\left(1+2|c|\right)T\left(\left(\int_0^T \left|u'(t)\right| dt\right)^p + pd_2\left(\int_0^T \left|u'(t)\right| dt\right)^{p-1} + \cdots + d_2^p\right) \\
&\qquad + N_1\left(d_2 + \int_0^T \left|u'(t)\right| dt\right),
\end{aligned} \tag{2.24}$$

where $N_1 := (1+2|c|)(\beta T + T^{\frac{1}{2}}\|e\|_2)$. Applying Lemma 2.1, we have

$$\begin{aligned}
\int_0^T \left|u'(t)\right| dt &= \int_0^T \left|\left(A^{-1}Au'\right)(t)\right| dt \\
&\le \frac{\int_0^T |(Au)'(t)|\, dt}{|c|-1} \\
&\le \frac{T^{\frac{1}{q}}(\int_0^T |(Au)'(t)|^p\, dt)^{\frac{1}{p}}}{|c|-1},
\end{aligned} \tag{2.25}$$

since $\frac{1}{p} + \frac{1}{q} = 1$ and $|c| > 1$. We apply the inequality

$$(x+y)^k \le x^k + y^k, \quad \text{for } x, y > 0, 0 < k < 1.$$

Substituting Eqs. (2.24) into (2.25), we have

$$\begin{aligned}
\int_0^T \left|u'(t)\right| dt \le{}& \frac{\alpha^{\frac{1}{p}}(1+2|c|)^{\frac{1}{p}} T \int_0^T |u'(t)|\, dt}{|c|-1} \\
&+ \frac{\alpha^{\frac{1}{p}}(1+2|c|)^{\frac{1}{p}}(pd_2)^{\frac{1}{p}} T(\int_0^T |u'(t)|\, dt)^{\frac{p-1}{p}}}{|c|-1} \\
&+ \cdots + \frac{T^{\frac{1}{q}}(N_1^{\frac{1}{p}} + (pd_2^{p-1})^{\frac{1}{p}})(\int_0^T |u'(t)|\, dt)^{\frac{1}{p}} + T^{\frac{1}{p}}(d_2 + (N_1 d_2)^{\frac{1}{p}})}{|c|-1}.
\end{aligned}$$

Since $\alpha^{\frac{1}{p}}(1+2|c|)^{\frac{1}{p}} T < |c| - 1$, we know that there exists a positive constant M_1' such that

$$\int_0^T \left|u'(t)\right| dt \le M_1'. \tag{2.26}$$

From Eqs. (2.17) and (2.26), we have

$$\|u\| \le d_2 + \frac{1}{2}\int_0^T \left|u'(t)\right| dt \le d_2 + \frac{M_1'}{2} := M_1.$$

The proof is left for the reader, being the same as that of Theorem 1.1. □

3 Periodic solution for Eq. (1.2) with strong singularity of attractive type

3.1 Proof of Theorem 1.5

Proof of Theorem 1.5 We follow the same strategy and notation as in the proof of Theorem 1.1. We can get

$$u(t) \leq M_1.$$

From Eq. (2.10), we have

$$\begin{aligned} \int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt &= \int_{g(u(t))\geq 0} g\bigl(u(t)\bigr)\,dt - \int_{g(u(t))\leq 0} g\bigl(u(t)\bigr)\,dt \\ &= -2\int_{g(u(t))\leq 0} g^-\bigl(u(t)\bigr)\,dt + \int_0^T e(t)\,dt, \end{aligned} \tag{3.1}$$

where $g^- := \min\{g(u), 0\}$.

Case (I). If $\bar{e} \geq 0$, from Eq. (3.1), we obtain

$$\int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt \leq -2\int_0^T \bigl(g^-\bigl(u(t)\bigr) - e(t)\bigr)\,dt.$$

Since $g^-(u(t)) - e(t) \leq 0$, from condition (H_3), we know that $u(t) \geq d_4$. Then we deduce

$$\begin{aligned} \int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt &\leq -2\int_0^T g^-\bigl(u(t)\bigr)\,dt + \int_0^T \bigl|e(t)\bigr|\,dt \\ &\leq 2T\bigl\|g^-_{M_1}\bigr\| + T^{\frac{1}{2}}\|e\|_2, \end{aligned} \tag{3.2}$$

where $\|g^-_{M_1}\| := \max_{d_4\leq u\leq M_1}(-g^-(u))$. From Eqs. (2.12) and (3.2), we see that

$$\begin{aligned} \bigl\|\phi_p\bigl((Au)'\bigr)\bigr\| &\leq |p|\int_0^T \bigl|u'(t)\bigr|\,dt + \int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt \\ &\leq |p|T^{\frac{1}{2}}M_1'^{\frac{1}{2}} + 2T\bigl\|g^-_{M_1}\bigr\| + 2T^{\frac{1}{2}}\|e\|_2 := M_2'. \end{aligned} \tag{3.3}$$

Case (II). If $\bar{e} < 0$, from Eq. (3.1), we arrive at

$$\int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt \leq -2\int_0^T g^-\bigl(u(t)\bigr)\,dt.$$

Since $g^-(u(t)) \leq 0$, from condition (H_3), we know that there exists a positive constant d_4^* such that $u(t) \geq d_4^*$. Therefore, we have

$$\begin{aligned} \int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt &\leq -2\int_0^T g^-\bigl(u(t)\bigr)\,dt \\ &\leq 2T\bigl\|g^-_M\bigr\|, \end{aligned}$$

where $\|g^-_M\| := \max_{d_4^*\leq u\leq M_1}(-g^-(u))$. Similarly, we can get $|\phi_p((Au)'(t))| \leq M_2'$.

The proof left for the reader, being the same as that of Theorem 1.1. □

3.2 Proof of Theorem 1.6

Proof of Theorem 1.6 The same strategy and notation are followed as in the proof of Theorem 1.2. We only consider $\int_0^T |g(u(t))|\,dt$. From Eqs. (2.23), (3.1) and condition (H_4), we deduce

$$\begin{aligned}\int_0^T \left|g\big(u(t)\big)\right| dt &= -2\int_{g(u)\le 0} g^{-1}\big(u(t)\big)\,dt + \int_0^T e(t)\,dt \\ &\le 2m\|u\|^{p-1}T + 2nT + T^{\frac{1}{2}}\|e\|_2. \end{aligned}\tag{3.4}$$

From Eqs. (2.24) and (3.1), we obtain

$$\begin{aligned}\int_0^T \left|(Au)'(t)\right|^p dt \le{}& m\big(1+2|c|\big)T\left(\left(\int_0^T \left|u'(t)\right| dt\right)^p + \cdots + d_4^p\right) \\ &+ N_1\left(d_4 + \int_0^T \left|u'(t)\right| dt\right).\end{aligned}$$

The proof left for the reader, being the same as that of Theorem 1.1. □

4 Examples

Example 4.1 Consider the following neutral Duffing equation with strong singularity of repulsive type:

$$\left(\phi_p\left(u(t) - \frac{1}{2}u(t-\tau)\right)'\right)' + Pu'(t) + \sum_{i=1}^n u^{2i}(t) - \frac{1}{u^\mu} = e^{\cos t},\tag{4.1}$$

where τ is a positive constant and $0 < \tau < T$, P and μ are constants and $\mu \ge 1$, n is an integer.

Comparing Eqs. (4.1) to (1.2), we know that $g(u) = \sum_{i=1}^n u^{2i}(t) - \frac{1}{u^\mu}$, $T = 2\pi$, $c = \frac{1}{2} < 1$. Obviously, there exist constants $d_1 = 0.1$ and $d_2 = 1$ such that condition (H_1) holds. In fact, $\lim_{u\to 0^+}\int_1^u g(s)\,ds = \lim_{u\to 0^+}\int_1^u (\sum_{i=1}^n u^{2i}(t) - \frac{1}{u^\mu})\,ds = +\infty$, thus, the condition ($H_2$) holds. Therefore, applying Theorem 1.1, we know that Eq. (4.1) has at least one positive 2π-periodic solution.

Example 4.2 Consider the following neutral Duffing equation with strong singularity of attractive type:

$$\big(\phi_p\big(u(t) - 40u(t-\tau)\big)'\big)' + 8u'(t) - 16u^3 + \frac{1}{u} = e^{\sin 2t},\tag{4.2}$$

where $p = 4$, τ is a constant and $0 \le \tau < T$.

It is clear that $T = \pi$, $P = 8$, $c = 40 > 1$, $g(u) = 16u^3 - \frac{1}{u}$, $e(t) = e^{\sin 2t}$. Obviously, it is easy to see that there exist constants $d_3 = \frac{1}{4}$ and $d_4 = \frac{3}{5}$ such that condition (H_3) holds. $\lim_{u\to 0^+}\int_1^u g(s)\,ds = \lim_{u\to 0^+}\int_1^u (-16s^3 + \frac{1}{s})\,ds = -\infty$, thus, the condition ($g_2$) holds. Consider $-g(u) \le 16u^3 + 1$, where $m = 16$, $n = 1$. So, condition (H_4) is satisfied. Next, it is verified that

$$\frac{m^{\frac{1}{p}}(1+2|c|)^{\frac{1}{p}}T}{|c|-1} = \frac{2\times 3\times \pi}{39} < 1.$$

Therefore, using Theorem 1.6, it is concluded that Eq. (4.2) has at least one positive π-periodic solution.

5 Conclusion

In this paper, using the Manásevich–Mawhin theorem on the continuous case of topological degree, we discuss the existence of a positive periodic solution for the p-Laplacian neutral Duffing equation (1.2). The nonlinear term g satisfies the strong singularity of attractive and repulsive type at $u = 0$ and may obey the sub-linearity, semi-linearity or super-linearity conditions at $u = \infty$. First, we obtain the existence of a positive periodic solution for Eq. (1.2) with a strong singularity of repulsive type. Afterwards, we prove the existence of a positive periodic solution for Eq. (1.2) with a strong singularity of attractive type. Our results improve and extend the results in [1, 2, 8, 11].

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Abbreviations
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