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Adaptive wavelet estimations for the derivative of a density in GARCH-type model

Kaikai Cao[1*] and Jingyi Wei[1]

*Correspondence: caokaikai@emails.bjut.edu.cn
[1] Department of Applied Mathematics, Beijing University of Technology, Beijing, P.R. China

Abstract

Recently, Rao investigated the estimations for the derivative of a density in GARCH-type model $S = \sigma^2 Z$ over L^2-risk (Commun. Stat., Theory Methods 46:2396–2410, 2017). This paper extends those estimations to L^p-risk ($1 \le p < \infty$). In addition, we provide a lower bound for this model, which indicates one of our convergence rates to be nearly-optimal.

Keywords: Wavelets; GARCH-type model; Derivative; Besov spaces

1 Introduction

The GARCH-type model

$$S = \sigma^2 Z$$

is considered in this paper, where σ^2 and Z are independent random variables. In practice, we always assume that the density function f_{σ^2} of σ^2 is unknown and $\operatorname{supp} f_{\sigma^2} \subseteq [0,1]$, while the density of Z is known. We want to estimate the first derivative of f_{σ^2} based on n independent and identically distributed (i.i.d.) observed samples $S_1, \ldots, S_n$ of S by wavelet methods, so that we also need suppose the differentiability of f_{σ^2} and $f'_{\sigma^2} \in L^p([0,1])$.

Non-parametric estimations of a density and regression function are widely investigated in the literature [12, 14, 16]. It is well known that the estimations for the derivatives of a density are also important and interesting, which could reflect monotonicity, concavity or convexity properties of density functions. Asymptotic properties of the kernel estimators for a density derivative have been considered earlier in [15], while the wavelet type estimator was discussed in [17].

As usual, we consider the L^p minimax risk (L^p-risk) [13],

$$\inf_{\hat{f}_n} \sup_{f_{\sigma^2} \in \Sigma} E\|\hat{f}_n - f_{\sigma^2}\|_p,$$

where the infimum runs over all possible estimators $\hat{f}_n$ and Σ is a class of functions. Here and after, EX stands for the mathematical expectation of a random variable X and $\|f\|_p$ denotes the ordinary L^p norm.

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In 2012, Chesneau and Doosti [9] investigated the wavelet estimation of density for GARCH model under various dependence structures. Next year, Chesneau [8] studied the wavelet estimation of a density in GARCH-type model leading to upper bounds under L^2-risk. In 2017, Rao [17] considered L^2-risk for the derivative of a density in GARCH-type model over a Besov ball by wavelets.

In this paper, we address to extend Rao's work [17] to L^p-risk ($1 \le p < \infty$). Moreover, we show that one of our convergence rates is nearly-optimal. On the other hand, this work can also be seen as a generalization of multiplicative censoring model. Vardi [18, 19] introduced the multiplicative censoring model which unifies several models including non-parametric inference for renewal processes, non-parametric deconvolution problems and estimation of decreasing density functions. Recently, Abbaszadeh et al. [1] considered the wavelet estimation of a density and its derivatives under L^p-risk ($1 \le p < \infty$) in the multiplicative censoring one. The density estimations for the multiplicative censoring model also can be found in [2, 3] and [6, 7].

This paper is organized as follows. Section 2 briefly describes the Besov ball and wavelet estimators. The theoretical results are given in Sect. 3. Some lemmas are provided in Sect. 4. The proofs are gathered in Sect. 5.

2 Besov ball and estimators

This section describes the Besov ball and wavelet estimators. First, we introduce the Besov ball and its wavelet characterizations.

2.1 Besov ball

Let $W_r^n(\mathbb{R})$ be the Sobolev space with a non-negative integer n,

$$W_r^n(\mathbb{R}) := \{f : f \in L^r(\mathbb{R}), f^{(n)} \in L^r(\mathbb{R})\},$$

and $\|f\|_{W_r^n} := \|f\|_r + \|f^{(n)}\|_r$. Then $L^r(\mathbb{R})$ can be considered as $W_r^0(\mathbb{R})$. For $1 \le r, q \le \infty$ and $s = n + \alpha$ with $\alpha \in (0, 1]$, a Besov space $B_{r,q}^s(\mathbb{R})$ is defined by

$$B_{r,q}^s(\mathbb{R}) := \{f : f \in W_r^n(\mathbb{R}), \|t^{-\alpha}\omega_r^2(f^{(n)}, t)\|_q^* < \infty\}$$

with the norm $\|f\|_{B_{r,q}^s} := \|f\|_{W_r^n} + \|t^{-\alpha}\omega_r^2(f^{(n)}, t)\|_q^*$. Here, $\omega_r^2(f, t) := \sup_{|h| \le t} \|f(\cdot + 2h) - 2f(\cdot + h) + f(\cdot)\|_r$ denotes the smoothness modulus of f and

$$\|h\|_q^* := \begin{cases} (\int_0^{+\infty} |h(t)|^q \frac{dt}{t})^{\frac{1}{q}} & \text{if } 1 \le q < \infty; \\ \operatorname{esssup}_t |h(t)| & \text{if } q = \infty. \end{cases}$$

When $s > 0$ and $1 \le r, q, r' \le \infty$, it is well known that

(i) $B_{r,q}^s \hookrightarrow B_{r,\infty}^s \hookrightarrow B_{\infty,\infty}^{s-\frac{1}{r}}$ for $s > \frac{1}{r}$;
(ii) $B_{r,q}^s \hookrightarrow B_{r',q}^{s'}$ for $r \le r'$ and $s - \frac{1}{r} = s' - \frac{1}{r'}$;
(iii) $B_{\infty,\infty}^s(\mathbb{R})$ is the classical Hölder space $H^s(\mathbb{R})$,

where $A \hookrightarrow B$ stands for a Banach space A continuously embedded in another Banach space B. More precisely, $\|u\|_B \le c_1\|u\|_A$ ($u \in A$) holds for some constant $c_1 > 0$. By (i), $B_{r,q}^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $s > \frac{1}{r}$. All these notations and claims can be found in [13].

In this paper, a Besov ball

$$B^s_{r,q}(M) = \{f \in B^s_{r,q}(\mathbb{R}) : \|f\|_{B^s_{r,q}} \le M\}, \quad M > 0,$$

is considered.

Let ϕ be a scaling function and ψ be the corresponding wavelet function such that

$$\{\phi_{\tau,k}, \psi_{j,k} : j \ge \tau, k \in \mathbb{Z}\}$$

constitutes an orthonormal basis of $L^2(\mathbb{R})$, where τ is a positive integer and $g_{j,k}(x) = 2^{\frac{j}{2}} g(2^j x - k)$ for $g = \phi$ or ψ. Then, for $h \in L^2(\mathbb{R})$,

$$h = \sum_{k\in\Omega_\tau} \alpha_{\tau,k}\phi_{\tau,k} + \sum_{j=\tau}^{\infty}\sum_{k\in\Omega_j} \beta_{j,k}\psi_{j,k} \tag{1}$$

with $\alpha_{j,k} = \langle h, \phi_{j,k}\rangle$, $\beta_{j,k} = \langle h, \psi_{j,k}\rangle$ and

$$\Omega_j = \{k \in \mathbb{Z} : \operatorname{supp} h \cap \operatorname{supp} \phi_{j,k} \ne \emptyset\} \cup \{k \in \mathbb{Z} : \operatorname{supp} h \cap \operatorname{supp} \psi_{j,k} \ne \emptyset\}.$$

In particular, when ϕ, ψ and h have compact supports, the cardinality of Ω_j satisfies $|\Omega_j| \le C2^j$, where $C > 0$ is a constant depending only on the support lengths of ϕ, ψ and h.

As usual, the orthogonal projection operator P_j is given by

$$P_j h = \sum_{k\in\Omega_j} \alpha_{j,k}\phi_{j,k}. \tag{2}$$

When $\phi \in C^m$ (so does ψ) is compactly supported, the identity (1) and (2) hold in L^p sense for $p \ge 1$ [13]. Here and throughout, C^m stands for the set consisting of all m times continuously differentiable functions.

The following wavelet characterization theorem of Besov space is needed in Sect. 5.

Lemma 2.1 ([13]) *Let a scaling function $\phi \in C^m$ be compactly supported. Then, for $r, q \in [1, +\infty]$, $0 < s < m$ and $h \in L^r(\mathbb{R})$, the following assertions are equivalent*:

(i) $h \in B^s_{r,q}(\mathbb{R})$; (ii) $2^{js}\|P_j h - h\|_r \in l_q$;

(iii) $\|\alpha_{j_0,\cdot}\|_{l_r} + \left\| \{2^{j(s+\frac{1}{2}-\frac{1}{r})}\|\beta_{j_0,\cdot}\|_{l_r}\}_{j\ge j_0} \right\|_{l_q} < \infty.$

In each case,

$$\|h\|_{B^s_{r,q}} \sim \|h\|_{s,r,q} := \|\alpha_{j_0,\cdot}\|_{l_r} + \left\| \{2^{j(s+\frac{1}{2}-\frac{1}{r})}\|\beta_{j_0,\cdot}\|_{l_r}\}_{j\ge j_0} \right\|_{l_q}.$$

Here and afterwards, $A \lesssim B$ means $A \le c_2 B$ for some constant $c_2 > 0$; $A \gtrsim B$ denotes $B \lesssim A$; we also use $A \sim B$ to stand for both $A \lesssim B$ and $A \gtrsim B$.

2.2 Estimators

This part introduces our wavelet estimators for the GARCH-type model $S = \sigma^2 Z$ described earlier. Suppose

$$Z = \prod_{i=1}^{\nu} U_i,$$

where ν is a known positive integer and $U_1, \dots, U_\nu$ are i.i.d. random variables with standard uniform distribution. Clearly, the density function of Z satisfies

$$f_Z(z) = \frac{1}{(\nu-1)!}(-\ln z)^{\nu-1}, \quad 0 \le z \le 1.$$

As in [8, 17], we assume that there exists a known constant C_* such that

$$\sup_{x\in[0,1]} f_s(x) \le C_*, \tag{3}$$

where f_s is the density function of S.

For any $x \in [0,1]$, $h \in C^k([0,1])$, we define

$$T(h)(x) = \bigl(xh(x)\bigr)' = h(x) + xh'(x), \qquad T_k(h)(x) = T\bigl(T_{k-1}(h)\bigr)(x) \tag{4}$$

and

$$G(h)(x) = -xh'(x), \qquad G_k(h)(x) = G\bigl(G_{k-1}(h)\bigr)(x), \tag{5}$$

where k is a positive integer. Then the following lemma holds.

Lemma 2.2 ([8]) *Let G and T be defined as above. Then*

(i) $f_{\sigma^2}(x) = G_\nu(f_s)(x), x \in [0,1]$;

(ii) *For any* $h \in C^\nu([0,1])$,

$$\int_0^1 f_{\sigma^2}(x)h(x)\,dx = \int_0^1 f_s(x)T_\nu(h)(x)\,dx.$$

Next, we will introduce wavelet estimators, which can be found in Ref. [17]. Define

$$\widehat{\alpha}_{j_0,k} = -\frac{1}{n}\sum_{i=1}^{n} T_\nu\bigl((\phi_{j,k})'\bigr)(S_i) \quad \text{and} \quad \widehat{\beta}_{j,k} = -\frac{1}{n}\sum_{i=1}^{n} T_\nu\bigl((\psi_{j,k})'\bigr)(S_i). \tag{6}$$

Here and after, let ϕ be Daubechies' scaling function D_{2N} with large N and ψ be the corresponding wavelet function. It is well known that $\phi, \psi \in C^{\nu+1}$ with N large enough. Furthermore, the linear wavelet estimator is given by

$$\widehat{f'_{\sigma^2}}^{\text{lin}} = \sum_{k\in\Omega_{j_0}} \widehat{\alpha}_{j_0,k}\phi_{j_0,k}, \tag{7}$$

where j_0 is a positive integer which will be chosen later.

In order to get adaptivity, we need the thresholding method [4, 14, 17]. As in [17], let

$$2^{j_1} \sim \left(\frac{n}{\ln n}\right)^{\frac{1}{2(\nu+1)+1}}, \qquad \lambda_j = 2^{(\nu+1)j}\sqrt{\frac{j}{n}}, \qquad \widetilde{\beta}_{j,k} = \widehat{\beta}_{j,k} I\{|\widehat{\beta}_{j,k}| \geq \Upsilon\lambda_j\}$$

with the constants $\Upsilon = c\gamma$, $c > \max\{8C_{\min}, 1\}$ and $\gamma \geq p(2\nu + 3)$. Here,

$$C_{\min} = (\nu+2)! \sum_{u=0}^{\nu} \left[(\nu+1)(\nu+2)! C_* \|\psi^{(u+1)}\|_2^2 + 2\|\psi^{(u+1)}\|_\infty\right] \tag{8}$$

with C_* given in (3). This special choice c is used in Lemma 4.3, while $\gamma \geq p(2\nu + 3)$ is needed in the estimations of Ee_1 and Ee_3 (see Sect. 5). Here, we replace $\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{\ln n}{n}}$ (see [17]) by $\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{j}{n}}$, which is used in the proof of Lemma 4.3. In fact, the universal threshold of classical adaptive density estimation is $\sqrt{\frac{j}{n}}$ (see [11]) and two forms do not influence the convergence rates of our results.

The nonlinear wavelet estimator is given by

$$\widehat{f'_{\sigma^2}}^{\text{non}} = \sum_{k\in\Omega_\tau} \widehat{\alpha}_{\tau,k}\phi_{\tau,k} + \sum_{j=\tau}^{j_1}\sum_{k\in\Omega_j} \widetilde{\beta}_{j,k}\psi_{j,k} \tag{9}$$

with some positive integer τ.

3 Results

This section describes the results in this paper.

Theorem 3.1 *Assume $r \in [1, +\infty)$, $q \in [1, +\infty]$ and $s > \frac{1}{r}$, then, for $p \in [1, +\infty)$, the estimator $\widehat{f'_{\sigma^2}}^{\text{lin}}$ in (7) with $2^{j_0} \sim n^{\frac{1}{2s'+2(\nu+1)+1}}$ satisfies*

$$\sup_{f'_{\sigma^2}\in B^s_{r,q}(M)} E\left\|\widehat{f'_{\sigma^2}}^{\text{lin}} - f'_{\sigma^2}\right\|_p^p \lesssim n^{-\frac{s'p}{2s'+2(\nu+1)+1}},$$

where $s' = s - (\frac{1}{r} - \frac{1}{p})_+$ and $a_+ = \max\{a, 0\}$.

Remark 1 When $p = 2$ and $r \geq 2$, the above estimation shows

$$\sup_{f'_{\sigma^2}\in B^s_{r,q}(M)} E\left\|\widehat{f'_{\sigma^2}}^{\text{lin}} - f'_{\sigma^2}\right\|_2^2 \lesssim n^{-\frac{2s}{2s+2(\nu+1)+1}},$$

which coincides with Theorem 5.1 of Ref. [17].

Remark 2 The condition $s > \frac{1}{r}$ can be replaced by $s' = s - (\frac{1}{r} - \frac{1}{p})_+ > 0$, because the former condition is only used to conclude $B^s_{r,q} \hookrightarrow B^{s'}_{p,q}$ in the proof of Theorem 3.1.

The next theorem gives an adaptive upper bound estimation by the nonlinear wavelet estimator $\widehat{f'_{\sigma^2}}^{\text{non}}$ in (9).

Theorem 3.2 *Let $r \in [1, +\infty)$, $q \in [1, +\infty]$ and $s > \frac{1}{r}$. Then, for $p \in [1, +\infty)$,*

$$\sup_{f'_{\sigma^2} \in B^s_{r,q}(M)} E\left\|\widehat{f'_{\sigma^2}}^{\text{non}} - f'_{\sigma^2}\right\|_p^p \lesssim (\ln n)^p \left(n^{-1} \ln n\right)^{\alpha p}$$

with $\alpha = \min\{\frac{s}{2s+2(\nu+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}\}$.

Remark 3 When $sr + (\nu + \frac{3}{2})r - (\nu + \frac{3}{2})p \geq 0$, $\alpha = \frac{s}{2s+2(\nu+1)+1}$. In particular, the above result with $p = 2$ coincides with Theorem 5.2 in [17].

Remark 4 The condition $s > \frac{1}{r}$ in Theorem 3.2 can't be replaced by $s' = s - (\frac{1}{r} - \frac{1}{p})_+ > 0$ for $r \leq p$, since we need $\alpha = \min\{\frac{s}{2s+2(\nu+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}\} \leq \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1} \leq \frac{s-\frac{1}{r}+\frac{1}{p}}{2(\nu+1)+1}$ for the estimation of A_3 in Sect. 5.

Remark 5 Let m be a constant such that $m > s$, and $2^{j_0} \sim n^{\frac{1}{2m+2(\nu+1)+1}}$. Then the number of calculations can be reduced effectively, when the level τ in $\widehat{f'_{\sigma^2}}^{\text{non}}$ is replaced by j_0.

The following theorem shows a lower bound estimation.

Theorem 3.3 *Assume $s > 0$ and $r, q \in [1, +\infty]$, then, for any $p \in [1, +\infty)$,*

$$\inf_{\widehat{f'_{\sigma^2}}} \sup_{f'_{\sigma^2} \in B^s_{r,q}(M)} E\left\|\widehat{f'_{\sigma^2}} - f'_{\sigma^2}\right\|_p^p \gtrsim n^{-\frac{(s-\frac{1}{r}+\frac{1}{p})p}{2(s-\frac{1}{r})+2(\nu+1)+1}},$$

where $\widehat{f'_{\sigma^2}}$ runs over all possible estimators of f'_{σ^2}.

Remark 6 Combining Theorem 3.3 with Theorem 3.2, we find that the convergence rate $\frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}$ is nearly-optimal. As for the other one, we will study it below.

4 Some lemmas

This section is devoted to providing some lemmas, which are needed for the proofs of our theorems.

Lemma 4.1 ([13]) *Let g be a scaling function or a wavelet function with*

$$\sup_{x \in \mathbb{R}} \sum_k \left|g(x-k)\right| < +\infty.$$

Then there exists $C > 0$ such that, for $\lambda = \{\lambda_k\} \in l^p(\mathbb{Z})$ and $1 \leq p \leq \infty$,

$$\left\|\sum_{k \in \mathbb{Z}} \lambda_k g_{jk}\right\|_p \leq C 2^{j(\frac{1}{2}-\frac{1}{p})} \|\lambda\|_{l_p}.$$

We need the well-known Rosenthal's inequality [13], in order to prove Lemma 4.2.

Rosenthal's inequality. Let $X_1, \ldots, X_n$ be independent random variables and $EX_i = 0$. Then

$$E\left|\sum_{i=1}^n X_i\right|^p \leq \begin{cases} C_p[\sum_{i=1}^n E|X_i|^p + (\sum_{i=1}^n E|X_i|^2)^{\frac{p}{2}}], & 2 \leq p < \infty; \\ (\sum_{i=1}^n E|X_i|^2)^{\frac{p}{2}}, & 0 < p < 2, \end{cases}$$

where $C_p > 0$ is a constant.

Lemma 4.2 *Let $\widehat{\alpha}_{j,k}$ and $\widehat{\beta}_{j,k}$ be given by (6). Then, for $p \in (0, +\infty)$,*
(i) $E\widehat{\alpha}_{j,k} = \alpha_{j,k}, E\widehat{\beta}_{j,k} = \beta_{j,k}$;
(ii) $E|\widehat{\alpha}_{j,k} - \alpha_{j,k}|^p \lesssim n^{-\frac{p}{2}} 2^{(\nu+1)jp}, E|\widehat{\beta}_{j,k} - \beta_{j,k}|^p \lesssim n^{-\frac{p}{2}} 2^{(\nu+1)jp}$,
where $\alpha_{j,k} = \langle f'_{\sigma^2}, \phi_{j,k}\rangle$ and $\beta_{j,k} = \langle f'_{\sigma^2}, \psi_{j,k}\rangle$.

Proof (i) One only need prove $E\widehat{\alpha}_{j,k} = \alpha_{j,k}$ and the second one is the same. According to the definition of $\widehat{\alpha}_{j,k}$ in (6), one gets

$$E\widehat{\alpha}_{j,k} = -E\big[T_\nu\big((\phi_{j,k})'\big)(S_1)\big] = -\int_0^1 T_\nu\big((\phi_{j,k})'\big)(x) f_s(x)\,dx$$

thanks to $S_1, \ldots, S_n$ are i.i.d. On the other hand, $f_{\sigma^2}(0) = f_{\sigma^2}(1) = 0$ follows from $\operatorname{supp} f_{\sigma^2} \subseteq [0,1]$ and the continuity of f_{σ^2}. These with Lemma 2.2 imply

$$E\widehat{\alpha}_{j,k} = -\int_0^1 f_{\sigma^2}(x)(\phi_{j,k})'(x)\,dx = -f_{\sigma^2}(x)\phi_{j,k}(x)|_0^1 + \int_0^1 f'_{\sigma^2}(x)\phi_{j,k}(x)\,dx = \alpha_{j,k}.$$

(ii) One also prove the first inequality and the second one is similar. By (6) and the results of (i),

$$\widehat{\alpha}_{j,k} - \alpha_{j,k} = \widehat{\alpha}_{j,k} - E\widehat{\alpha}_{j,k} = \frac{1}{n}\sum_{i=1}^n \big\{E\big[T_\nu\big((\phi_{j,k})'\big)(S_i)\big] - T_\nu\big((\phi_{j,k})'\big)(S_i)\big\}.$$

Let $X_i := E[T_\nu((\phi_{j,k})')(S_i)] - T_\nu((\phi_{j,k})')(S_i)$. Then $X_1, \ldots, X_n$ are i.i.d., $EX_i = 0$ and

$$E|\widehat{\alpha}_{j,k} - \alpha_{j,k}|^p = E\left|\frac{1}{n}\sum_{i=1}^n X_i\right|^p = n^{-p} E\left|\sum_{i=1}^n X_i\right|^p. \tag{10}$$

According to (4),

$$\big|T_\nu\big((\phi_{j,k})'\big)(x)\big| \leq (\nu+2)! \sum_{u=0}^{\nu} \big|x^u (\phi_{j,k})^{(u+1)}(x)\big|. \tag{11}$$

Hence,

$$\begin{aligned} \sup_{x\in[0,1]} \big|T_\nu\big((\phi_{j,k})'\big)(x)\big| &\leq (\nu+2)! \sup_{x\in[0,1]} \sum_{u=0}^{\nu} \big|x^u (\phi_{j,k})^{(u+1)}(x)\big| \\ &\leq (\nu+2)! \sum_{u=0}^{\nu} \sup_{x\in[0,1]} \big|(\phi_{j,k})^{(u+1)}(x)\big| \end{aligned}$$

$$\leq (\nu+2)! \sum_{u=0}^{\nu} \left\| \phi^{(u+1)} \right\|_\infty 2^{(\nu+\frac{3}{2})j}.$$

Clearly,

$$|X_i| \leq E\left|T_\nu\left((\phi_{j,k})'\right)(S_i)\right| + \left|T_\nu\left((\phi_{j,k})'\right)(S_i)\right| \leq C_1 2^{(\nu+\frac{3}{2})j}, \tag{12}$$

where $C_1 = 2(\nu+2)!\sum_{u=0}^{\nu} \|\phi^{(u+1)}\|_\infty$. On the other hand, $\sup_{x\in[0,1]} f_s(x) \leq C_*$ in (3) and $S_i \in [0,1]$ show that

$$E\left|(\phi_{j,k})^{(u+1)}(S_i)\right|^2 = \int_0^1 \left|(\phi_{j,k})^{(u+1)}(x)\right|^2 f_s(x)\,dx \leq C_* \left\|\phi^{(u+1)}\right\|_2^2 2^{(2u+2)j}.$$

This with (11) and $S_i \in [0,1]$ leads to

$$\begin{aligned} E\left[T_\nu\left((\phi_{j,k})'\right)(S_i)\right]^2 &\leq \left[(\nu+2)!\right]^2 E\left[\sum_{u=0}^{\nu} \left|S_i^u (\phi_{j,k})^{(u+1)}(S_i)\right|\right]^2 \\ &\leq (\nu+1)\left[(\nu+2)!\right]^2 \sum_{u=0}^{\nu} E\left|(\phi_{j,k})^{(u+1)}(S_i)\right|^2 \\ &\leq (\nu+1)\left[(\nu+2)!\right]^2 C_* \sum_{u=0}^{\nu} \left\|\phi^{(u+1)}\right\|_2^2 2^{(2\nu+2)j}. \end{aligned}$$

Furthermore,

$$E|X_i|^2 \leq E\left[T_\nu\left((\phi_{j,k})'\right)(S_i)\right]^2 \leq C_2 2^{(2\nu+2)j}, \tag{13}$$

where $C_2 = (\nu+1)[(\nu+2)!]^2 C_* \sum_{u=0}^{\nu} \|\phi^{(u+1)}\|_2^2$.

When $0 < p < 2$, by using (10), Jensen's inequality and (13),

$$E|\widehat{\alpha}_{j,k} - \alpha_{j,k}|^p = n^{-p} E\left|\sum_{i=1}^n X_i\right|^p \lesssim n^{-p}\left[\sum_{i=1}^n E|X_i|^2\right]^{\frac{p}{2}} \lesssim n^{-\frac{p}{2}} 2^{(\nu+1)jp}.$$

For the case of $2 \leq p < \infty$, according to Rosenthal's inequality,

$$\begin{aligned} E\left|\sum_{i=1}^n X_i\right|^p &\lesssim \sum_{i=1}^n E|X_i|^p + \left(\sum_{i=1}^n E|X_i|^2\right)^{\frac{p}{2}} \\ &\lesssim n 2^{(\nu+\frac{3}{2})(p-2)j} 2^{(2\nu+2)j} + \left(n 2^{(2\nu+2)j}\right)^{\frac{p}{2}} \\ &\lesssim n^{\frac{p}{2}} 2^{(\nu+1)pj}\left[n^{1-\frac{p}{2}} 2^{(\frac{p}{2}-1)j} + 1\right] \end{aligned}$$

because of (12) and (13). Moreover, $n^{1-\frac{p}{2}} 2^{(\frac{p}{2}-1)j} \leq 1$ follows from $2^j \leq n$ and $p \geq 2$. Then

$$E|\widehat{\alpha}_{j,k} - \alpha_{j,k}|^p = n^{-p} E\left|\sum_{i=1}^n X_i\right|^p \lesssim n^{-\frac{p}{2}} 2^{(\nu+1)pj}$$

due to (10). This completes the proof. □

Bernstein's inequality [13] is necessary in the proof of Lemma 4.3.

Bernstein's inequality. Let $X_1,\dots,X_n$ be i.i.d. random variables, $EX_i = 0$ and $|X_i| \le \|X\|_\infty$ $(i = 1,\dots,n)$. Then, for each $\gamma > 0$,

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\right| > \gamma\right\} \le 2\exp\left(-\frac{n\gamma^2}{2(EX_i^2 + \|X\|_\infty\gamma/3)}\right).$$

Lemma 4.3 *Let $\beta_{j,k}$ be the wavelet coefficient of f'_{σ^2}, $\widehat{\beta}_{j,k}$ be defined in (6) and $\Upsilon = c\gamma$. Then, for any $j > 0$, $j2^j \le n$ and $\gamma \ge 1$, there exists a constant $c \ge \max\{8C_{\min}, 1\}$ such that*

$$P\left\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \Upsilon\lambda_j/2\right\} \lesssim 2^{-\gamma j},$$

where $C_{\min}$ is given by (8).

Proof According to the definition of $\widehat{\beta}_{j,k}$ in (6), one obtains

$$\widehat{\beta}_{j,k} - \beta_{j,k} = \frac{1}{n}\sum_{i=1}^{n}\left\{E\left[T_\nu\left((\psi_{j,k})'\right)(S_i)\right] - T_\nu\left((\psi_{j,k})'\right)(S_i)\right\} = \frac{1}{n}\sum_{i=1}^{n}Y_i,$$

where $Y_i := E[T_\nu((\psi_{j,k})')(S_i)] - T_\nu((\psi_{j,k})')(S_i)$.

Similar to (12) and (13),

$$|Y_i| \le C'_1 2^{(\nu+\frac{3}{2})j} := M \quad \text{and} \quad EY_i^2 \le C'_2 2^{(2\nu+2)j}, \tag{14}$$

where

$$C'_1 = 2(\nu+2)!\sum_{u=0}^{\nu}\left\|\psi^{(u+1)}\right\|_\infty \quad \text{and} \quad C'_2 = (\nu+1)\left[(\nu+2)!\right]^2 C_*\sum_{u=0}^{\nu}\left\|\psi^{(u+1)}\right\|_2^2. \tag{15}$$

Then Bernstein's inequality tells that

$$P\left\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \Upsilon\lambda_j/2\right\} = P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}Y_i\right| > \Upsilon\lambda_j/2\right\} \le 2\exp\left\{-\frac{n(\Upsilon\lambda_j/2)^2}{2(EY_i^2 + M\Upsilon\lambda_j/6)}\right\}. \tag{16}$$

On the other hand, combining with (14), $\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{j}{n}}$ and $j2^j \le n$, one shows

$$\begin{aligned} EY_i^2 + M\Upsilon\lambda_j/6 &\le C'_2 2^{(2\nu+2)j} + \frac{C'_1\Upsilon}{6}2^{(\nu+\frac{3}{2})j}2^{(\nu+1)j}\sqrt{\frac{j}{n}} \\ &= 2^{(2\nu+2)j}\left(C'_2 + \frac{C'_1\Upsilon}{6}\sqrt{\frac{j2^j}{n}}\right) \le \left(C'_2 + C'_1\Upsilon\right)2^{(2\nu+2)j}. \end{aligned}$$

This with (15), $c \ge \max\{8C_{\min}, 1\}$ implies that

$$\frac{n(\Upsilon\lambda_j/2)^2}{2(EY_i^2 + M\Upsilon\lambda_j/6)} \ge \frac{\Upsilon^2 j}{8(C'_2 + C'_1\Upsilon)} = \frac{(c\gamma)^2 j}{8(C'_2 + C'_1 c\gamma)} \ge \gamma j\ln 2 \tag{17}$$

thanks to $j > 0$ and $\gamma > 1$.

Hence, it follows from (16)–(17) that

$$P\{|\widehat{\beta}_{j,k}-\beta_{j,k}|>\Upsilon\lambda_j/2\}\le 2\exp\left\{-\frac{(c\gamma)^2 j}{8(C_2'+C_1'c\gamma)}\right\}\lesssim 2^{-\gamma j},$$

which is the conclusion of Lemma 4.3. □

At the end of this section, we list two more lemmas which will play key roles in the proof of Theorem 3.3.

Lemma 4.4 ([5]) *Let $g\in B^s_{r,q}(\mathbb{R})$ and $f(x)=g(bx)$ $(b\ge 1)$. Then*

$$\|f\|_{B^s_{r,q}}\le b^{s-\frac{1}{r}}\|g\|_{B^s_{r,q}}.$$

To state the last lemma, we need a concept: Let P and Q be two probability measures on $(\varOmega,\aleph)$ and P be absolutely continuous with respect to Q (denoted by $P\ll Q$), the Kullback–Leibler divergence is defined by

$$K(P,Q):=\int_{p\cdot q>0}p(x)\ln\frac{p(x)}{q(x)}\,dx,$$

where p and q are density functions of P,Q, respectively.

Lemma 4.5 (Fano's lemma, [10]) *Let $(\varOmega,\aleph,P_k)$ be probability measurable spaces and $A_k\in\aleph$, $k=0,1,\dots,m$. If $A_k\cap A_v=\emptyset$ for $k\neq v$, then*

$$\sup_{0\le k\le m}P_k(A_k^c)\ge\min\left\{\frac{1}{2},\sqrt{m}\exp(-3e^{-1}-\kappa_m)\right\},$$

where A^c stands for the complement of A and $\kappa_m=\inf_{0\le v\le m}\frac{1}{m}\sum_{k\neq v}K(P_k,P_v)$.

5 Proofs of results

In this section, we will prove our main results.

5.1 Proofs of upper bounds

We rewrite Theorem 3.1 as follows before giving its proof.

Theorem 5.1 *Assume $r\in[1,+\infty)$, $q\in[1,+\infty]$ and $s>\frac{1}{r}$, then, for $p\in[1,+\infty)$, the estimator $\widehat{f'_{\sigma^2}}^{\mathrm{lin}}$ in (7) with $2^{j_0}\sim n^{\frac{1}{2s'+2(v+1)+1}}$ satisfies*

$$\sup_{f'_{\sigma^2}\in B^s_{r,q}(M)}E\left\|\widehat{f'_{\sigma^2}}^{\mathrm{lin}}-f'_{\sigma^2}\right\|_p^p\lesssim n^{-\frac{s'p}{2s'+2(v+1)+1}},$$

where $s'=s-(\frac{1}{r}-\frac{1}{p})_+$ and $a_+=\max\{a,0\}$.

Proof When $r>p$, $s'=s-(\frac{1}{r}-\frac{1}{p})_+=s$. Denote $\varOmega=\operatorname{supp}(\widehat{f'_{\sigma^2}}^{\mathrm{lin}}-f'_{\sigma^2})$. Then

$$E\left\|\widehat{f'_{\sigma^2}}^{\mathrm{lin}}-f'_{\sigma^2}\right\|_p^p=E\int\left|\widehat{f'_{\sigma^2}}^{\mathrm{lin}}-f'_{\sigma^2}\right|^p dx$$

$$\leq E\Big[\int_{\Omega}\big(|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}|^p\big)^{\frac{r}{p}}\,dx\Big]^{\frac{p}{r}}\Big(\int_{\Omega}1\,dx\Big)^{1-\frac{p}{r}}\lesssim E\big(\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_r^r\big)^{\frac{p}{r}}$$

due to the Hölder inequality. Furthermore, according to Jensen's inequality and $\frac{p}{r}<1$,

$$\sup_{f'_{\sigma^2}\in B^s_{r,q}(M)}E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_p^p\lesssim\sup_{f'_{\sigma^2}\in B^s_{r,q}(M)}\big(E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_r^r\big)^{\frac{p}{r}}.\tag{18}$$

By $s'=s-\frac{1}{r}+\frac{1}{p}\leq s$ and $r\leq p$, one finds $B^s_{r,q}\hookrightarrow B^{s'}_{p,q}$. Hence,

$$\sup_{f'_{\sigma^2}\in B^s_{r,q}(M)}E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_p^p\lesssim\sup_{f'_{\sigma^2}\in B^{s'}_{p,q}(M)}E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_p^p.\tag{19}$$

Next, one only need estimate $\sup_{f'_{\sigma^2}\in B^{s'}_{p,q}(M)}E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_p^p$ by (18) and (19). Note that

$$\begin{aligned}E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_p^p&\leq E\big[\|\widehat{f'_{\sigma^2}}^{\text{lin}}-P_{j_0}f'_{\sigma^2}\|_p+\|P_{j_0}f'_{\sigma^2}-f'_{\sigma^2}\|_p\big]^p\\&\lesssim E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-P_{j_0}f'_{\sigma^2}\|_p^p+\|P_{j_0}f'_{\sigma^2}-f'_{\sigma^2}\|_p^p.\end{aligned}\tag{20}$$

Combining $2^{j_0}\sim n^{\frac{1}{2s'+2(\nu+1)+1}}$, $f'_{\sigma^2}\in B^{s'}_{p,q}(M)$ with Lemma 2.1, one concludes

$$\|P_{j_0}f'_{\sigma^2}-f'_{\sigma^2}\|_p^p\lesssim 2^{-j_0s'p}\lesssim n^{-\frac{s'p}{2s'+2(\nu+1)+1}}.\tag{21}$$

On the other hand,

$$E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-P_{j_0}f'_{\sigma^2}\|_p^p\lesssim 2^{j_0(\frac{p}{2}-1)}\sum_{k\in\Omega_{j_0}}E|\widehat{\alpha}_{j_0,k}-\alpha_{j_0,k}|^p\lesssim n^{-\frac{p}{2}}2^{(\nu+1+\frac{1}{2})j_0p}\tag{22}$$

thanks to Lemma 4.1 and Lemma 4.2. Then it follows

$$E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-P_{j_0}f'_{\sigma^2}\|_p^p\lesssim n^{-\frac{s'p}{2s'+2(\nu+1)+1}}$$

from $2^{j_0}\sim n^{\frac{1}{2s'+2(\nu+1)+1}}$. This with (20) and (21) leads to

$$\sup_{f'_{\sigma^2}\in B^{s'}_{p,q}(M)}E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_p^p\lesssim n^{-\frac{s'p}{2s'+2(\nu+1)+1}}.\tag{23}$$

Combining (23) with (18) and (19), one finds that

$$\sup_{f'_{\sigma^2}\in B^s_{r,q}(M)}E\|\widehat{f'_{\sigma^2}}^{\text{lin}}-f'_{\sigma^2}\|_p^p\lesssim n^{-\frac{s'p}{2s'+2(\nu+1)+1}}.$$

The proof is done. □

Now, the upper bound of nonlinear wavelet estimator (Theorem 3.2) is restated below.

Theorem 5.2 *Let $r \in [1,+\infty), q \in [1,+\infty]$ and $s > \frac{1}{r}$. Then, for any $p \in [1,+\infty)$, the estimator $\widehat{f'_{\sigma^2}}^{\mathrm{non}}$ in (9) satisfies*

$$\sup_{f'_{\sigma^2} \in B^s_{r,q}(M)} E\big\|\widehat{f'_{\sigma^2}}^{\mathrm{non}} - f'_{\sigma^2}\big\|_p^p \lesssim (\ln n)^p \big(n^{-1}\ln n\big)^{\alpha p},$$

where $\alpha = \min\{\frac{s}{2s+2(v+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(v+1)+1}\}$.

Proof When $r > p$, similar to (18),

$$\sup_{f'_{\sigma^2} \in B^s_{r,q}(M)} E\big\|\widehat{f'_{\sigma^2}}^{\mathrm{non}} - f'_{\sigma^2}\big\|_p^p \lesssim \sup_{f'_{\sigma^2} \in B^s_{r,q}(M)} \big(E\big\|\widehat{f'_{\sigma^2}}^{\mathrm{non}} - f'_{\sigma^2}\big\|_r^r\big)^{\frac{p}{r}}.$$

Hence, it suffices to establish the result for $r \le p$. According to (1), (2) and (9), $E\|\widehat{f'_{\sigma^2}}^{\mathrm{non}} - f'_{\sigma^2}\|_p^p \lesssim A_1 + A_2 + A_3$, where

$$A_1 = E\bigg\|\sum_{k\in\Omega_\tau}(\widehat{\alpha}_{j,k} - \alpha_{j,k})\phi_{j,k}\bigg\|_p^p; \qquad A_2 = E\bigg\|\sum_{j=\tau}^{j_1}\sum_{k\in\Omega_j}(\widetilde{\beta}_{j,k} - \beta_{j,k})\psi_{j,k}\bigg\|_p^p \quad \text{and}$$

$$A_3 = \big\|P_{j_1+1}f'_{\sigma^2} - f'_{\sigma^2}\big\|_p^p.$$

Next, one proves $A_1 + A_2 + A_3 \lesssim (\ln n)^p (n^{-1}\ln n)^{\alpha p}$ for $f'_{\sigma^2} \in B^s_{r,q}(M)$ and $r \le p$.

By the same arguments as (22),

$$A_1 \lesssim 2^{\tau(\frac{p}{2}-1)} \sum_{k\in\Omega_\tau} E|\widehat{\alpha}_{\tau,k} - \alpha_{\tau,k}|^p \lesssim n^{-\frac{p}{2}} 2^{(v+1+\frac{1}{2})\tau p} \sim n^{-\frac{p}{2}} \lesssim \big(n^{-1}\ln n\big)^{\alpha p}$$

thanks to $\alpha = \min\{\frac{s}{2s+2(v+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(v+1)+1}\} < \frac{1}{2}$.

Note that $f'_{\sigma^2} \in B^s_{r,q} \hookrightarrow B^{s-\frac{1}{r}+\frac{1}{p}}_{p,q}$ for $r \le p$. This with Lemma 2.1 and $2^{j_1} \sim (\frac{n}{\ln n})^{\frac{1}{2(v+1)+1}}$ shows

$$A_3 = \big\|P_{j_1+1}f'_{\sigma^2} - f'_{\sigma^2}\big\|_p^p \lesssim 2^{-j_1(s-\frac{1}{r}+\frac{1}{p})p} \lesssim \left(\frac{\ln n}{n}\right)^{\frac{(s-\frac{1}{r}+\frac{1}{p})p}{2(v+1)+1}} \lesssim \big(n^{-1}\ln n\big)^{\alpha p},$$

because $s > \frac{1}{r}$ and $\alpha = \min\{\frac{s}{2s+2(v+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(v+1)+1}\} \le \frac{s-\frac{1}{r}+\frac{1}{p}}{2(v+1)+1}$.

To estimate A_2, define

$$\widehat{B}_j = \{k : |\widehat{\beta}_{j,k}| \ge \Upsilon\lambda_j\}; \qquad B_j = \left\{k : |\beta_{j,k}| \ge \frac{1}{2}\Upsilon\lambda_j\right\} \quad \text{and} \quad C_j = \{k : |\beta_{j,k}| \ge 2\Upsilon\lambda_j\}.$$

Then $E\|\sum_{j=\tau}^{j_1}\sum_{k\in\Omega_j}(\widetilde{\beta}_{j,k} - \beta_{j,k})\psi_{j,k}\|_p^p \lesssim (\ln n)^{p-1}\sum_{i=1}^4 Ee_i$ by Lemma 4.1, where

$$e_1 = \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\widehat{\beta}_{j,k} - \beta_{j,k}|^p I\big\{k \in \widehat{B}_j \cap B_j^c\big\};$$

$$e_2 = \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\widehat{\beta}_{j,k} - \beta_{j,k}|^p I\{k \in \widehat{B}_j \cap B_j\};$$

$$e_3 = \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\{k \in \widehat{B}_j^c \cap C_j\};$$

$$e_4 = \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\{k \in \widehat{B}_j^c \cap C_j^c\}.$$

By the Hölder inequality and $\{k \in \widehat{B}_j \cap B_j^c\} \subseteq \{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \Upsilon\lambda_j/2\}$,

$$E|\widehat{\beta}_{j,k} - \beta_{j,k}|^p I\{k \in \widehat{B}_j \cap B_j^c\} \le \left(E|\widehat{\beta}_{j,k} - \beta_{j,k}|^{2p}\right)^{\frac{1}{2}} P^{\frac{1}{2}}\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \Upsilon\lambda_j/2\}.$$

This with Lemma 4.2 and Lemma 4.3 shows that

$$Ee_1 \lesssim \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} n^{-\frac{p}{2}} 2^{j[(\nu+1)p-\frac{\gamma}{2}]} \lesssim n^{-\frac{p}{2}} 2^{\tau(\nu p+\frac{3}{2}p-\frac{\gamma}{2})} \lesssim \left(\frac{\ln n}{n}\right)^{\alpha p},$$

where one uses $\gamma > p(2\nu + 3)$ and $\alpha = \min\{\frac{s}{2s+2(\nu+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}\} < \frac{1}{2}$.

From $k \in \widehat{B}_j^c \cap C_j$, one finds $|\widehat{\beta}_{j,k} - \beta_{j,k}| > \Upsilon\lambda_j$ and $|\beta_{j,k}| \le |\widehat{\beta}_{j,k} - \beta_{j,k}| + |\widehat{\beta}_{j,k}| \le 2|\widehat{\beta}_{j,k} - \beta_{j,k}|$. On the other hand, $\{k \in \widehat{B}_j^c \cap C_j\} \subseteq \{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \Upsilon\lambda_j\} \subseteq \{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \Upsilon\lambda_j/2\}$. Therefore, it follows from the same arguments as Ee_1 that

$$Ee_3 \lesssim \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\widehat{\beta}_{j,k} - \beta_{j,k}|^p I\{k \in \widehat{B}_j^c \cap C_j\} \lesssim \left(\frac{\ln n}{n}\right)^{\alpha p}.$$

Next, one estimates Ee_2 and Ee_4. Define

$$\omega = sr + \left(\nu + \frac{3}{2}\right)r - \left(\nu + \frac{3}{2}\right)p, \qquad 2^{j_0^*} \sim \left(\frac{n}{\ln n}\right)^{\frac{1}{2s+2(\nu+1)+1}}, \\ 2^{j_1^*} \sim \left(\frac{n}{\ln n}\right)^{\frac{1}{2(s-\frac{1}{r})+2(\nu+1)+1}}. \tag{24}$$

Then, by $s > \frac{1}{r}$ and $2^{j_1} \sim (\frac{n}{\ln n})^{\frac{1}{2(\nu+1)+1}}$,

$$0 < \frac{1}{2s + 2(\nu + 1) + 1}, \frac{1}{2(s - \frac{1}{r}) + 2(\nu + 1) + 1} < \frac{1}{2(\nu + 1) + 1} \quad \text{and} \quad \tau < j_0^*, j_1^* < j_1.$$

When $\omega \ge 0$, one writes down

$$e_2 = \left(\sum_{j=\tau}^{j_0^*} + \sum_{j=j_0^*}^{j_1}\right) 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\widehat{\beta}_{j,k} - \beta_{j,k}|^p I\{k \in \widehat{B}_j \cap B_j\} := e_{21} + e_{22}. \tag{25}$$

According to (22),

$$Ee_{21} \lesssim \sum_{j=\tau}^{j_0^*} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} E|\widehat{\beta}_{j,k} - \beta_{j,k}|^p \lesssim n^{-\frac{p}{2}} 2^{(\nu+1+\frac{1}{2})j_0^* p}. \tag{26}$$

Note that $\frac{2|\beta_{jk}|}{\Upsilon\lambda_j} \geq 1$, $\sum_k |\beta_{jk}|^r \lesssim 2^{-j(s+\frac{1}{2}-\frac{1}{r})r}$ from $k \in B_j$, $f'_{\sigma^2} \in B^s_{r,q}(M)$ and Lemma 2.1; On the other hand, Lemma 4.2 tells $E|\widehat{\beta}_{j,k} - \beta_{j,k}|^p \lesssim 2^{(\nu+1)jp} n^{-\frac{p}{2}}$. These with $\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{j}{n}}$ and $\omega = sr + (\nu+\frac{3}{2})r - (\nu+\frac{3}{2})p \geq 0$ lead to

$$\begin{aligned} Ee_{22} &\lesssim \sum_{j=j_0^*}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} E|\widehat{\beta}_{j,k} - \beta_{j,k}|^p \left(\frac{|\beta_{jk}|}{\Upsilon\lambda_j}\right)^r \\ &\lesssim \sum_{j=j_0^*}^{j_1} 2^{j(\frac{p}{2}-1)} 2^{(\nu+1)jp} n^{-\frac{p}{2}} \lambda_j^{-r} \sum_k |\beta_{jk}|^r \lesssim 2^{-j_0^*\omega} n^{-\frac{p-r}{2}}. \end{aligned} \tag{27}$$

Combining (25)–(27) with $2^{j_0^*} \sim (\frac{n}{\ln n})^{\frac{1}{2s+2(\nu+1)+1}}$, one obtains

$$Ee_2 = Ee_{21} + Ee_{22} \lesssim n^{-\frac{p}{2}} 2^{(\nu+1+\frac{1}{2})j_0^* p} + 2^{-j_0^*\omega} n^{-\frac{p-r}{2}} \lesssim \left(\frac{\ln n}{n}\right)^{\frac{sp}{2s+2(\nu+1)+1}} = \left(\frac{\ln n}{n}\right)^{\alpha p}$$

because of $\omega \geq 0$ and $\alpha = \min\{\frac{s}{2s+2(\nu+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}\} = \frac{s}{2s+2(\nu+1)+1}$.

When $\omega = sr + (\nu+\frac{3}{2})r - (\nu+\frac{3}{2})p < 0$, $\alpha = \min\{\frac{s}{2s+2(\nu+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}\} = \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}$. Define $p_1 = (1-2\alpha)p$. Then $r \leq p_1 \leq p$ follows from

$$\omega < 0 \quad \text{and} \quad r \leq p_1 = (1-2\alpha)p = \frac{2(\nu+1)p + p - 2}{2(s-\frac{1}{r}) + 2(\nu+1) + 1} \leq p.$$

Moreover, $\sum_k |\beta_{jk}|^{p_1} \leq (\sum_k |\beta_{jk}|^r)^{\frac{p_1}{r}} \lesssim 2^{-j(s+\frac{1}{2}-\frac{1}{r})p_1}$ thanks to $r \leq p_1$, $f'_{\sigma^2} \in B^s_{r,q}(M)$ and Lemma 2.1. This with (27) and $\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{j}{n}}$ shows

$$\begin{aligned} Ee_2 &\lesssim \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} E|\widehat{\beta}_{j,k} - \beta_{j,k}|^p \left(\frac{|\beta_{jk}|}{\Upsilon\lambda_j}\right)^{p_1} \\ &\lesssim \sum_{j=\tau}^{j_1} 2^{j(\frac{p}{2}-1)} 2^{(\nu+1)jp} n^{-\frac{p}{2}} \lambda_j^{-p_1} \sum_k |\beta_{jk}|^{p_1} \\ &\lesssim n^{-\frac{p-p_1}{2}} \sum_{j=\tau}^{j_1} 2^{j[\frac{p}{2}-1+(\nu+1)(p-p_1)-(s+\frac{1}{2}-\frac{1}{r})p_1]} \lesssim \ln n \left(\frac{\ln n}{n}\right)^{\alpha p} \end{aligned}$$

due to $\frac{p-p_1}{2} = \alpha p$ and $\frac{p}{2} - 1 + (\nu+1)(p-p_1) - (s+\frac{1}{2}-\frac{1}{r})p_1 = 0$.

Finally, one estimates Ee_4. When $\omega = sr + (\nu + \frac{3}{2})r - (\nu + \frac{3}{2})p \geq 0$, $\alpha = \min\{\frac{s}{2s+2(\nu+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}\} = \frac{s}{2s+2(\nu+1)+1}$. Furthermore,

$$e_4 = \left(\sum_{j=\tau}^{j_0^*} + \sum_{j=j_0^*+1}^{j_1}\right) 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\left\{k \in \widehat{B}_j^c \cap C_j^c\right\} := e_{41} + e_{42}, \tag{28}$$

where j_0^* is given by (24).

Since $|\beta_{j,k}| \leq 2\Upsilon\lambda_j \lesssim 2^{(\nu+1)j}(jn^{-1})^{\frac{1}{2}}$ holds by $k \in C_j^c$ and $\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{j}{n}}$, one concludes that

$$\begin{aligned} Ee_{41} &= E\sum_{j=\tau}^{j_0^*} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\left\{k \in \widehat{B}_j^c \cap C_j^c\right\} \\ &\lesssim \sum_{j=\tau}^{j_0^*} 2^{j(\frac{p}{2}-1)} 2^j 2^{(\nu+1)jp}\left(jn^{-1}\right)^{\frac{p}{2}} \lesssim 2^{(\nu+1+\frac{1}{2})j_0^* p}\left(\frac{\ln n}{n}\right)^{\frac{p}{2}}. \end{aligned} \tag{29}$$

On the other hand,

$$\begin{aligned} Ee_{42} &= E\sum_{j=j_0^*+1}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\left\{k \in \widehat{B}_j^c \cap C_j^c\right\} \\ &\lesssim \sum_{j=j_0^*+1}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p \left(\lambda_j|\beta_{j,k}|^{-1}\right)^{p-r} \\ &\lesssim \sum_{j=j_0^*+1}^{j_1} 2^{j(\frac{p}{2}-1)} \lambda_j^{p-r} \sum_k |\beta_{j,k}|^r \end{aligned}$$

due to $|\beta_{j,k}| \leq 2\Upsilon\lambda_j$ and $r \leq p$.

Clearly, $\|\beta_{j,\cdot}\|_{l_r} \lesssim 2^{-j(s-\frac{1}{r}+\frac{1}{2})}$ by $f'_{\sigma^2} \in B^s_{r,q}(M)$ and Lemma 2.1. This with $\lambda_j = 2^{(\nu+1)j}\sqrt{\frac{j}{n}}$ implies that

$$Ee_{42} \lesssim \sum_{j=j_0^*+1}^{j_1} 2^{j(\frac{p}{2}-1)} 2^{(\nu+1)(p-r)j} 2^{-j(sr+\frac{r}{2}-1)} \left(\frac{\ln n}{n}\right)^{\frac{p-r}{2}} \lesssim \left(\frac{\ln n}{n}\right)^{\frac{p-r}{2}} 2^{-j_0^*\omega}, \tag{30}$$

because $\omega = sr + (\nu + \frac{3}{2})r - (\nu + \frac{3}{2})p \geq 0$.

According to (28)–(30) and $2^{j_0^*} \sim (\frac{n}{\ln n})^{\frac{1}{2s+2(\nu+1)+1}}$, one obtains

$$Ee_4 = Ee_{41} + Ee_{42} \lesssim 2^{(\nu+1+\frac{1}{2})j_0^* p}\left(\frac{\ln n}{n}\right)^{\frac{p}{2}} + \left(\frac{\ln n}{n}\right)^{\frac{p-r}{2}} 2^{-j_0^*\omega} \lesssim \left(\frac{\ln n}{n}\right)^{\alpha p}$$

by $\alpha = \frac{s}{2s+2(\nu+1)+1}$.

For the case of $\omega = sr + (\nu + \frac{3}{2})r - (\nu + \frac{3}{2})p < 0$. Let

$$e_4 = \left(\sum_{j=\tau}^{j_1^*} + \sum_{j=j_1^*+1}^{j_1}\right) 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\left\{k \in \widehat{B}_j^c \cap C_j^c\right\} := e'_{41} + e'_{42}, \tag{31}$$

where j_1^* is given by (24). Similar to (30),

$$Ee'_{41} = \sum_{j=\tau}^{j_1^*} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\{k \in \widehat{B}_j^c \cap C_j^c\} \lesssim \left(\frac{\ln n}{n}\right)^{\frac{p-r}{2}} 2^{-j_1^*\omega} \tag{32}$$

thanks to $\omega < 0$.

To estimate Ee'_{42}, one observes that $\|\beta_{j,\cdot}\|_{l_p} \lesssim \|\beta_{j,\cdot}\|_{l_r} \lesssim 2^{-j(s-\frac{1}{r}+\frac{1}{2})}$ by $r \le p$, $f'_{\sigma^2} \in B^s_{r,q}(M)$ and Lemma 2.1. Hence,

$$\begin{aligned} Ee'_{42} &= \sum_{j=j_1^*+1}^{j_1} 2^{j(\frac{p}{2}-1)} \sum_{k\in\Omega_j} |\beta_{j,k}|^p I\{k \in \widehat{B}_j^c \cap C_j^c\} \lesssim \sum_{j=j_1^*+1}^{j_1} 2^{j(\frac{p}{2}-1)} \|\beta_{j,\cdot}\|_{l_r}^p \\ &\lesssim \sum_{j=j_1^*+1}^{j_1} 2^{j(\frac{p}{2}-1)} 2^{-j(s-\frac{1}{r}+\frac{1}{2})p} \lesssim 2^{-j_1^*(s-\frac{1}{r}+\frac{1}{p})p} \end{aligned} \tag{33}$$

because of $s > \frac{1}{r}$.

Combining (31)–(33) with $2^{j_1^*} \sim (\frac{n}{\ln n})^{\frac{1}{2(s-\frac{1}{r})+2(\nu+1)+1}}$, one knows

$$Ee_4 = Ee'_{41} + Ee'_{42} \lesssim \left(\frac{\ln n}{n}\right)^{\frac{p-r}{2}} 2^{-j_1^*\omega} + 2^{-j_1^*(s-\frac{1}{r}+\frac{1}{p})p} \lesssim \left(\frac{\ln n}{n}\right)^{\alpha p}$$

thanks to $\omega < 0$ and $\alpha = \min\{\frac{s}{2s+2(\nu+1)+1}, \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}\} = \frac{s-\frac{1}{r}+\frac{1}{p}}{2(s-\frac{1}{r})+2(\nu+1)+1}$. This completes the proof of Theorem 5.2. □

5.2 Proof of lower bound

Finally, we are in a position to state and prove the lower bound estimation.

Theorem 5.3 *Assume $s > 0$ and $r, q \in [1, +\infty]$, then, for any $p \in [1, +\infty)$,*

$$\inf_{\widehat{f}'_{\sigma^2}} \sup_{f'_{\sigma^2}\in B^s_{r,q}(M)} E\|\widehat{f}'_{\sigma^2} - f'_{\sigma^2}\|_p^p \gtrsim n^{-\frac{(s-\frac{1}{r}+\frac{1}{p})p}{2(s-\frac{1}{r})+2(\nu+1)+1}},$$

where $\widehat{f}'_{\sigma^2}$ runs over all possible estimators of f'_{σ^2}.

Proof It is sufficient to construct density functions h_k such that $h'_k \in B^s_{r,q}(M)$ and

$$\sup_k E\|\widehat{f}'_{\sigma^2} - h'_k\|_p^p \gtrsim n^{-\frac{(s-\frac{1}{r}+\frac{1}{p})p}{2(s-\frac{1}{r})+2(\nu+1)+1}}.$$

Define $g(x) = Cm(x)$, where $m \in C_0^\infty$ with $\operatorname{supp} m \subseteq [0,1]$, $\int_{\mathbb{R}} m(x)\,dx = 0$ and $C > 0$ is a constant. Let C_0^∞ stand for the set of all infinitely many times differentiable and compactly supported functions. Furthermore, one chooses a density function h_0 satisfying $h_0 \in B^{s+1}_{r,q}(\frac{M}{2})$, $\operatorname{supp} h_0 \subseteq [0,1]$ and $h_0(x) \ge M_1 > 0$ for $x \in [\frac{1}{2}, \frac{3}{4}]$.

Take $a_j = 2^{-j(s-\frac{1}{r}+\frac{1}{2}+\nu+1)}$ and

$$h_1(x) = h_0(x) + a_j G_\nu(g_{j,l})(x), \tag{34}$$

where G is given by (5) and $g_{j,l}(x) = 2^{\frac{j}{2}} g(2^j x - l)$ with $l = 2^{j-1}$.

First, one checks that h_1 is a density function. Since $\operatorname{supp} g_{j,l} \subseteq [\frac{1}{2}, \frac{3}{4}]$ by $\operatorname{supp} m \subseteq [0,1]$ and j large enough, one finds $h_1(x) \geq 0$ for $x \notin [\frac{1}{2}, \frac{3}{4}]$. It is easy to calculate that

$$G_\nu(g_{j,l})(x) = (-1)^\nu \sum_{u=1}^{\nu} C_u x^u (g_{j,l})^{(u)}(x), \tag{35}$$

where $C_u > 0$ is a constant. Then, for $x \in [\frac{1}{2}, \frac{3}{4}]$ and large j,

$$\begin{aligned} h_1(x) &\geq M_1 - \left| a_j \sum_{u=1}^{\nu} C_u x^u (g_{j,l})^{(u)}(x) \right| \\ &\geq M_1 - a_j 2^{\frac{j}{2}} \sum_{u=1}^{\nu} C_u 2^{uj} \left\| g^{(u)}\left(2^j \cdot - l\right) \right\|_\infty \\ &\geq M_1 - 2^{-j(s-\frac{1}{r}+1)} \sum_{u=1}^{\nu} C_u \left\| g^{(u)} \right\|_\infty \geq 0 \end{aligned} \tag{36}$$

thanks to $h_0(x)|_{[\frac{1}{2},\frac{3}{4}]} \geq M_1$ and $a_j = 2^{-j(s-\frac{1}{r}+\frac{1}{2}+\nu+1)}$. On the other hand, $\int_{\mathbb{R}} g(x)\,dx = \int_{\mathbb{R}} Cm(x)\,dx = 0$ and $\operatorname{supp} g_{j,l} \subseteq [\frac{1}{2}, \frac{1}{2}+2^{-j}]$ by $\operatorname{supp} m \subseteq [0,1]$ and $l = 2^{j-1}$. This with $m \in C_0^\infty$ and $g(x) = Cm(x)$ shows

$$\begin{aligned} \int x^u (g_{j,l})^{(u)}(x)\,dx &= x^u (g_{j,l})^{(u-1)}(x)\Big|_{\frac{1}{2}}^{\frac{1}{2}+2^{-j}} - u \int x^{u-1} (g_{j,l})^{(u-1)}(x)\,dx \\ &= \cdots = (-1)^m \frac{u!}{(u-m)!} \int x^{u-m} (g_{j,l})^{(u-m)}(x)\,dx \\ &= (-1)^u u! \int g_{j,l}(x)\,dx = 0 \end{aligned}$$

for any $u \in \{1, \dots, \nu\}$. Therefore,

$$\int h_1(x)\,dx = \int h_0(x)\,dx + (-1)^\nu a_j \sum_{u=1}^{\nu} C_u \int x^u (g_{j,l})^{(u)}(x)\,dx = 1.$$

From this with (36) one concludes that h_1 is a density function.

Next, one shows $h_0', h_1' \in B^s_{r,q}(M)$. Clearly, $h_0' \in B^s_{r,q}(M)$ by $h_0 \in B^{s+1}_{r,q}(\frac{M}{2})$. Hence, one only needs prove $h_1' \in B^s_{r,q}(M)$.

By (34) and (35),

$$\|h_1\|_{B^{s+1}_{r,q}} \leq \|h_0\|_{B^{s+1}_{r,q}} + a_j \sum_{u=1}^{\nu} C_u \left\| x^u (g_{j,l})^{(u)}(x) \right\|_{B^{s+1}_{r,q}}. \tag{37}$$

On the other hand, for each $\tau \in \{0,\dots,u\}$,

$$\left\| \left[2^j\left(x-\frac{1}{2}\right)\right]^{u-\tau} g^{(u)}\left[2^j\left(x-\frac{1}{2}\right)\right]\right\|_{B^{s+1}_{r,q}} \le 2^{j(s+1-\frac{1}{r})}\left\|x^{u-\tau}g^{(u)}(x)\right\|_{B^{s+1}_{r,q}}$$

because of Lemma 4.4. Combining this with $l=2^{j-1}$ and $[2^j(x-\frac{1}{2}+\frac{1}{2})]^u=\sum_{\tau=0}^{u}C_u^\tau[2^j(x-\frac{1}{2})]^{u-\tau}2^{-\tau}2^{\tau j}$, one obtains

$$\begin{aligned}\left\|x^u(g_{j,l})^{(u)}(x)\right\|_{B^{s+1}_{r,q}} &= 2^{\frac{j}{2}}\left\|\left[2^j\left(x-\frac{1}{2}+\frac{1}{2}\right)\right]^u g^{(u)}\left[2^j\left(x-\frac{1}{2}\right)\right]\right\|_{B^{s+1}_{r,q}}\\ &\le 2^{(u+\frac{1}{2})j}\sum_{\tau=0}^{u}C_u^\tau\left\|\left[2^j\left(x-\frac{1}{2}\right)\right]^{u-\tau}g^{(u)}\left[2^j\left(x-\frac{1}{2}\right)\right]\right\|_{B^{s+1}_{r,q}}\\ &\le 2^{j(s-\frac{1}{r}+\frac{1}{2}+u+1)}\sum_{\tau=0}^{u}C_u^\tau\left\|x^{u-\tau}g^{(u)}(x)\right\|_{B^{s+1}_{r,q}}. \end{aligned} \tag{38}$$

Denote $M'=\max\{C_u, C_u^\tau : \tau=0,\dots,u; u=1,\dots,\nu\}$. Then there exists a constant $C>0$ such that

$$x^{u-\tau}g^{(u)} \in B^{s+1}_{r,q}\left(\frac{M}{2\nu^2M'^2}\right)$$

thanks to $g(x)=Cm(x)$ and $m\in C_0^\infty\subseteq B^{s+1}_{r,q}$. This with (37) and (38) leads to

$$\|h_1\|_{B^{s+1}_{r,q}} \le \|h_0\|_{B^{s+1}_{r,q}} + a_j\sum_{u=1}^{\nu}uM'^2 2^{j(s-\frac{1}{r}+\frac{1}{2}+u+1)}\frac{M}{2\nu^2M'^2}\le\frac{M}{2}+\frac{M}{2}=M,$$

because $h_0\in B^{s+1}_{r,q}(\frac{M}{2})$ and $a_j=2^{-j(s-\frac{1}{r}+\frac{1}{2}+\nu+1)}$. Therefore, $h_1\in B^{s+1}_{r,q}(M)$ and $h_1'\in B^s_{r,q}(M)$.

According to (35),

$$\left[G_\nu(g_{j,l})(x)\right]'=(-1)^\nu\sum_{u=0}^{\nu}C_u'x^u(g_{j,l})^{(u+1)}(x),$$

where $C_u'>0$ is a constant and $l=2^{j-1}$. Hence,

$$\begin{aligned}\left\|h_1'-h_0'\right\|_p &= a_j\left\|\left[G_\nu(g_{j,l})\right]'\right\|_p\\ &= a_j2^{\frac{3j}{2}}\left\|\sum_{u=0}^{\nu}C_u'\left[2^j\left(x-\frac{1}{2}+\frac{1}{2}\right)\right]^u g^{(u+1)}\left[2^j\left(x-\frac{1}{2}\right)\right]\right\|_p. \end{aligned}\tag{39}$$

On the other hand, by using $[2^j(x-\frac{1}{2}+\frac{1}{2})]^u=\sum_{\tau=0}^{u}C_u^\tau[2^j(x-\frac{1}{2})]^{u-\tau}2^{-\tau}2^{\tau j}$, one concludes that

$$\begin{aligned}&\left\|\sum_{u=0}^{\nu}C_u'\left[2^j\left(x-\frac{1}{2}+\frac{1}{2}\right)\right]^u g^{(u+1)}\left[2^j\left(x-\frac{1}{2}\right)\right]\right\|_p\\ &\quad\ge\left\|\sum_{u=0}^{\nu}C_u'2^{-u}2^{uj}g^{(u+1)}\left[2^j\left(x-\frac{1}{2}\right)\right]\right\|_p\end{aligned}$$

$$- \left\| \sum_{u=0}^{\nu} C'_u \left\{ \sum_{\tau=0}^{u-1} C_u^\tau \left[2^j \left(x - \frac{1}{2} \right) \right]^{u-\tau} 2^{-\tau} 2^{\tau j} \right\} g^{(u+1)} \left[2^j \left(x - \frac{1}{2} \right) \right] \right\|_p \tag{40}$$

and

$$\begin{aligned} &\left\| \sum_{u=0}^{\nu} C'_u 2^{-u} 2^{uj} g^{(u+1)} \left[2^j \left(x - \frac{1}{2} \right) \right] \right\|_p \\ &\quad \geq \left\| C'_\nu 2^{-\nu} 2^{\nu j} g^{(\nu+1)} \left[2^j \left(x - \frac{1}{2} \right) \right] \right\|_p \\ &\qquad - \left\| \sum_{u=0}^{\nu-1} C'_u 2^{-u} 2^{uj} g^{(u+1)} \left[2^j \left(x - \frac{1}{2} \right) \right] \right\|_p . \end{aligned} \tag{41}$$

Let $x' = 2^j(x - \frac{1}{2})$. Then there exists a constant $C' > 0$ such that

$$\begin{aligned} \left\| h'_1 - h'_0 \right\|_p &\geq a_j 2^{\frac{3}{2}j} 2^{-\frac{j}{p}} \left\{ C'_\nu 2^{-\nu} 2^{\nu j} \left\| g^{(\nu+1)} \right\|_p - 2^{(\nu-1)j} \sum_{u=0}^{\nu-1} C'_u 2^{-u} \left\| g^{(u+1)} \right\|_p \right. \\ &\qquad \left. - 2^{(\nu-1)j} \sum_{u=0}^{\nu} C'_u \sum_{\tau=0}^{u-1} C_u^\tau 2^{-\tau} \left\| \left(x' \right)^{u-\tau} g^{(u+1)} \right\|_p \right\} \\ &\geq C' a_j 2^{\frac{3}{2}j} 2^{-\frac{j}{p}} 2^{j\nu} = C' 2^{-j(s - \frac{1}{r} + \frac{1}{p})} := \delta_j \end{aligned}$$

thanks to (39)–(41), $g \in C_0^\infty$ and $a_j = 2^{-j(s - \frac{1}{r} + \frac{1}{2} + \nu + 1)}$.

Define $A_k = \{ \| \widehat{f'_{\sigma^2}} - h'_k \|_p < \frac{\delta_j}{2} \}$ ($k \in \{0, 1\}$). Then $A_0 \cap A_1 = \emptyset$. According to Lemma 4.5,

$$\sup_{k \in \{0,1\}} P^n_{f_{s_k}} \left(A_k^c \right) \geq \min \left\{ \frac{1}{2}, \exp\left(-3e^{-1} - \kappa_1 \right) \right\}, \tag{42}$$

where P_f^n stands for the probability measure corresponding to the density function $f^n(x) := f(x_1) f(x_2) \cdots f(x_n)$. Hence,

$$E \left\| \widehat{f'_{\sigma^2}} - h'_k \right\|_p^p \geq \left(\frac{\delta_j}{2} \right)^p P^n_{f_{s_k}} \left(\left\| \widehat{f'_{\sigma^2}} - h'_k \right\|_p \geq \frac{\delta_j}{2} \right) = \left(\frac{\delta_j}{2} \right)^p P^n_{f_{s_k}} \left(A_k^c \right).$$

This with (42) implies

$$\sup_{k \in \{0,1\}} E \left\| \widehat{f'_{\sigma^2}} - h'_k \right\|_p^p \geq \sup_{k \in \{0,1\}} \left(\frac{\delta_j}{2} \right)^p P^n_{f_{s_k}} \left(A_k^c \right) \geq \left(\frac{\delta_j}{2} \right)^p \min \left\{ \frac{1}{2}, \exp\left(-3e^{-1} - \kappa_1 \right) \right\}. \tag{43}$$

Next, one shows $\kappa_1 \leq C_0 n a_j^2$. Recall that

$$\kappa_1 = \inf_{0 \leq u \leq 1} \sum_{k \neq u} K\left(P^n_{f_{s_k}}, P^n_{f_{s_u}} \right) \leq K\left(P^n_{f_{s_1}}, P^n_{f_{s_0}} \right). \tag{44}$$

Then

$$K\left(P^n_{f_{s_1}}, P^n_{f_{s_0}} \right) \leq \sum_{i=1}^n \int f_{s_1}(x_i) \ln \frac{f_{s_1}(x_i)}{f_{s_0}(x_i)} \, dx_i = n \int f_{s_1}(x) \ln \frac{f_{s_1}(x)}{f_{s_0}(x)} \, dx$$

due to $f_{s_0}^n(x) = \prod_{j=1}^n f_{s_0}(x_j)$ and $f_{s_1}^n(x) = \prod_{j=1}^n f_{s_1}(x_j)$. Since $\ln u \le u - 1$ holds for $u > 0$, one knows

$$K\big(P_{f_{s_1}}^n, P_{f_{s_0}}^n\big) \le n \int f_{s_1}(x)\bigg(\frac{f_{s_1}(x)}{f_{s_0}(x)} - 1\bigg)\,dx$$
$$= n \int f_{s_0}^{-1}(x)\big|f_{s_1}(x) - f_{s_0}(x)\big|^2\,dx. \tag{45}$$

According to Chesneau's work in Ref. [8], $f_{s_k}(x) = \frac{1}{(\nu-1)!}\int_x^1 (\ln y - \ln x)^{\nu-1} h_k(y)\frac{1}{y}\,dy$. Then

$$f_{s_1}(x) - f_{s_0}(x) = \frac{a_j}{(\nu-1)!}\int_x^{\frac{1}{2}+2^{-j}} (\ln y - \ln x)^{\nu-1} G_\nu(g_{j,l})(y)\frac{1}{y}\,dy$$
$$= -\frac{a_j}{(\nu-1)!}\int_x^{\frac{1}{2}+2^{-j}} (\ln y - \ln x)^{\nu-1}\big[G_{\nu-1}(g_{j,l})(y)\big]'\,dy$$

because of (34) and $G_\nu(g_{j,l})(x) = -x[G_{\nu-1}(g_{j,l})(x)]'$.

By the formula of integration by parts,

$$f_{s_1}(x) - f_{s_0}(x) = -\frac{a_j}{(\nu-2)!}\int_x^{\frac{1}{2}+2^{-j}} (\ln y - \ln x)^{\nu-2}\big[G_{\nu-2}(g_{j,l})(y)\big]'\,dy = \cdots$$
$$= -\frac{a_j}{(\nu-m)!}\int_x^{\frac{1}{2}+2^{-j}} (\ln y - \ln x)^{\nu-m}\big[G_{\nu-m}(g_{j,l})(y)\big]'\,dy = \cdots$$
$$= -a_j\int_x^{\frac{1}{2}+2^{-j}} (g_{j,l})'(y)\,dy = a_j g_{j,l}(x), \tag{46}$$

because $l = 2^{j-1}$ and $(\ln y - \ln x)^{\nu-m} G_{\nu-m}(g_{j,l})(y)|_x^{\frac{1}{2}+2^{-j}} = 0$ for any $m \in \{1, \ldots, \nu-1\}$. On the other hand, for each $x \in [\frac{1}{2}, \frac{1}{2} + 2^{-j}]$ and large j,

$$f_{s_0}(x) \ge \frac{M_1}{(\nu-1)!}\int_x^{\frac{3}{4}} (\ln y - \ln x)^{\nu-1}\frac{1}{y}\,dy = \frac{M_1}{\nu!}\bigg(\ln\frac{3}{4} - \ln x\bigg)^\nu$$
$$\ge \frac{M_1}{\nu!}\bigg[\ln\frac{3}{4} - \ln\bigg(\frac{1}{2} + 2^{-j}\bigg)\bigg]^\nu \ge M_2 > 0 \tag{47}$$

thanks to $f_{s_0}(x) = \frac{1}{(\nu-1)!}\int_x^1 (\ln y - \ln x)^{\nu-1} h_0(y)\frac{1}{y}\,dy$ and $h_0(x)|_{[\frac{1}{2},\frac{3}{4}]} \ge M_1$. Combining with (44)–(47), one obtains

$$\kappa_1 \le M_2^{-1} n \int \big|a_j g_{j,l}(x)\big|^2\,dx \le C_0 n a_j^2,$$

where $C_0 > 0$ is a constant.

Choose $2^j \sim n^{\frac{1}{2(s-\frac{1}{r})+2(\nu+1)+1}}$ and recall $a_j = 2^{-j(s-\frac{1}{r}+\frac{1}{2}+\nu+1)}$. Then

$$\kappa_1 \lesssim n a_j^2 = n 2^{-j[2(s-\frac{1}{r})+2(\nu+1)+1]} \sim 1 \quad \text{and} \quad e^{-\kappa_1} \gtrsim 1.$$

Substituting $\delta_j \sim 2^{-j(s-\frac{1}{r}+\frac{1}{p})}$, $2^j \sim n^{\frac{1}{2(s-\frac{1}{r})+2(\nu+1)+1}}$ into (43), one obtains

$$\sup_{k\in\{0,1\}} E\big\|\widehat{f}'_{\sigma^2} - h'_k\big\|_p^p \gtrsim \delta_j^p \gtrsim n^{-\frac{(s-\frac{1}{r}+\frac{1}{p})p}{2(s-\frac{1}{r})+2(\nu+1)+1}},$$

which is the desired conclusion. □

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References

1. Abbaszadeh, M., Chesneau, C., Doosti, H.: Multiplicative censoring: estimation of a density and its derivatives under the L^p-risk. REVSTAT **11**(3), 255–276 (2013)
2. Andersen, K., Hansen, M.: Multiplicative censoring: density estimation by a series expansion approach. J. Stat. Plan. Inference **98**, 137–155 (2001)
3. Asgharian, M., Carone, M., Fakoor, V.: Large-sample study of the kernel density estimators under multiplicative censoring. Ann. Stat. **40**, 159–187 (2012)
4. Blumensath, T., Davies, M.: Iterative hard thresholding for compressed sensing. Appl. Comput. Harmon. Anal. **27**, 265–274 (2009)
5. Cai, T.T.: Rates of convergence and adaption over Besov spaces under pointwise risk. Stat. Sin. **13**, 881–902 (2003)
6. Chaubey, Y.P., Chesneau, C., Doosti, H.: On linear wavelet density estimation: some recent developments. J. Indian Soc. Agric. Stat. **65**, 169–179 (2011)
7. Chaubey, Y.P., Chesneau, C., Doosti, H.: Adaptive wavelet estimation of a density from mixtures under multiplicative censoring. Statistics **49**(3), 638–659 (2015)
8. Chesneau, C.: Wavelet estimation of a density in a GARCH-type model. Commun. Stat., Theory Methods **42**, 98–117 (2013)
9. Chesneau, C., Doosti, H.: Wavelet linear density estimation for a GARCH model under various dependence structures. J. Iran. Stat. Soc. **11**, 1–21 (2012)
10. Devore, R., Kerkyacharian, G., Picard, D., Temlyakov, V.: On mathematical methods of learning. In: Found. Comput. Math., vol. 6, pp. 3–58 (2006). Special Issue for D. Smale
11. Donoho, D.L., Johnstone, M.I., Kerkyacharian, G., Picard, D.: Density estimation by wavelet thresholding. Ann. Stat. **24**(2), 508–539 (1996)
12. Guo, H.J., Liu, Y.M.: Convergence rates of multivariate regression estimators with errors-in-variables. Numer. Funct. Anal. Optim. **38**(12), 1564–1588 (2017)
13. Härdle, W., Kerkyacharian, G., Picard, D., Tsybakov, A.: Wavelet, Approximation and Statistical Applications. Springer, New York (1998)
14. Li, R., Liu, Y.M.: Wavelet optimal estimations for a density with some additive noises. Appl. Comput. Harmon. Anal. **36**, 416–433 (2014)
15. Prakasa Rao, B.L.S.: Nonparametric Functional Estimation. Academic Press, Orlando (1983)
16. Prakasa Rao, B.L.S.: Nonparametric function estimation: an overview. In: Ghosh, S. (ed.) Asymptotics, Nonparametrics and Time Series, pp. 461–509. Marcel Dekker, New York (1999)
17. Prakasa Rao, B.L.S.: Wavelet estimation for derivative of a density in a GARCH-type model. Commun. Stat., Theory Methods **46**, 2396–2410 (2017)
18. Vardi, Y.: Multiplicative censoring, renewal process, deconvolution and decreasing density: nonparametric estimation. Biometrika **76**, 751–761 (1989)
19. Vardi, Y., Zhang, C.H.: Large sample study of empirical distributions in a random multiplicative censoring model. Ann. Stat. **20**, 1022–1039 (1992)