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Approximation on parametric extension of Baskakov–Durrmeyer operators on weighted spaces

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Abstract

In the present manuscript, we define a non-negative parametric variant of Baskakov–Durrmeyer operators to study the convergence of Lebesgue measurable functions and introduce these as α-Baskakov–Durrmeyer operators. We study the uniform convergence of these operators in weighted spaces.

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1 Introduction

In the field of mathematical analysis, Karl Weierstrass established an elegant theorem, the first Weierstrass approximation theorem, in 1885. This theorem has specially a big role in polynomial interpolation corresponding to every continuous function $f(x)$ on interval $[a,b]$. The proof given by Weierstrass was rigorous and difficult to understand. In 1912, Bernstein [1] gave a simple proof of this theorem by introducing the Bernstein polynomials with the aid of the binomial distribution, hence for $f \in C[0,1]$, we have

$$B_n(f;x) = \sum_{k=0}^{n} \mathcal{S}_{n,k}(x) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, 0 \le x \le 1, \tag{1.1}$$

where $\mathcal{S}_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. Many mathematicians researched in this direction and studied various modifications in several functional spaces using different error optimization techniques, i.e., Acar et al. [2–7], Acu et al. [8, 9], Barbosu [10], Agrawal et al. [11], Aral [12], Mursaleen et al. [13–17], Srivastava et al. [18–20]; for more details see also the references therein and [21–30].

2 Construction of the α-Baskakov–Durrmeyer operators and estimation of their moments

Recently, Cai, Lian and Zhou [31] presented a new sequence of α-Bernstein operators with $\alpha \in [-1,1]$. Later, Ali Aral et al. [32] gave a sequence of α-Bernstein operators as follows:

$$L_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\mathcal{S}_{n,k}^{(\alpha)}(x), \quad n \in \mathbb{N}, x \in [0,\infty), \tag{2.1}$$

where $f \in C_B[0,\infty)$ which denotes the set of all continuous and bounded functions and

$$\begin{aligned}\mathcal{S}_{n,k}^{(\alpha)}(x) = \frac{x^{k-1}}{(1+x)^{n+k-1}}\left\{\frac{\alpha x}{1+x}\binom{n+k-1}{k} - (1-\alpha)(1+x)\binom{n+k-3}{k-2}\right. \\ \left. + (1-\alpha)y\binom{n+k-1}{k}\right\}\end{aligned}$$

with

$$\binom{n-3}{-2} = \binom{n-2}{-1} = 0.$$

The operators defined by (2.1) are restricted for continuous functions only. To approximate the functions in Lebesgue measurable space, we design a new sequence of operators:

$$L_{n,\alpha}^{*}(f;x) = \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \int_0^{\infty} \mathcal{Q}_{n,k}(t) f(t)\,dt, \tag{2.2}$$

where $\mathcal{Q}_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{(n+k+1)}}$. Note that, simply in the case of $\alpha = 1$, the operators reduced to Baskakov–Durrmeyer type operators; for details see [33].

For $r \in \{0,1,2,3,4\}$, we consider the test functions and central moments,

$$e_r = t^r \quad \text{and} \quad \psi_y^r(t;x) = (t-x)^r. \tag{2.3}$$

Lemma 2.1 ([31]) *We have*

$$\begin{aligned} L_{n,\alpha}(e_0;x) &= 1, \\ L_{n,\alpha}(e_1;x) &= x + \frac{2}{n}(\alpha - 1), \\ L_{n,\alpha}(e_2;x) &= x^2 + \frac{4\alpha - 3}{n}x + \frac{1}{n^2}(n + 4\alpha - 4). \end{aligned}$$

Lemma 2.2 *Let the test functions e_r defined by* (2.3), *then, for all $L_{n,\alpha}^{*}$, we have*

$$\begin{aligned} L_{n,\alpha}^{*}(e_0;x) &= 1, \\ L_{n,\alpha}^{*}(e_1;x) &= \left(\frac{n}{n-1} + \frac{2(\alpha-1)}{n-1}\right)x + \frac{1}{n-1}, \\ L_{n,\alpha}^{*}(e_2;x) &= \left(\frac{n^2}{(n-2)(n-1)} + \frac{n(4\alpha-3)}{(n-2)(n-1)}\right)x^2 + \frac{(4n+10\alpha-10)}{(n-2)(n-1)}x + \frac{2}{(n-2)(n-1)}. \end{aligned}$$

Proof Take $f = e_0$, then from Lemma 2.1, we have

$$\begin{aligned}
L_{n,\alpha}^{*}(e_0;x) &= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \int_0^{\infty} \mathcal{Q}_{n,k}(t)\,dt \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \frac{B(k+1,n)}{B(k+1,n)} \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \\
&= 1.
\end{aligned}$$

For $r = 1$

$$\begin{aligned}
L_{n,\alpha}^{*}(e_1;x) &= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \int_0^{\infty} t\mathcal{Q}_{n,k}(t)\,dt \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \frac{B(k+2,n-1)}{B(k+1,n)} \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \frac{(k+1)B(k+1,n)}{(n-1)B(k+1,n)} \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \frac{(k+1)}{(n-1)} \\
&= \left(\frac{n}{n-1} + \frac{2(\alpha-1)}{n-1}\right)x + \frac{1}{n-1}.
\end{aligned}$$

For $r = 2$

$$\begin{aligned}
L_{n,\alpha}^{*}(e_2;x) &= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \int_0^{\infty} t^2\mathcal{Q}_{n,k}(t)\,dt \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \frac{B(k+3,n-2)}{B(k+1,n)} \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \frac{(k+2)(k+1)B(k+1,n)}{(n-2)(n-1)B(k+1,n)} \\
&= \sum_{k=0}^{\infty} \mathcal{S}_{n,k}^{(\alpha)}(x) \frac{(k+2)(k+1)}{(n-2)(n-1)} \\
&= \frac{n^2 + n(4\alpha-3)}{(n-2)(n-1)}x^2 + \frac{(4n+10\alpha-10)}{(n-2)(n-1)}x + \frac{2}{(n-2)(n-1)}.
\end{aligned}$$

□

Lemma 2.3 *Let the operators given by* (2.2). *Then we have*

$$\begin{aligned}
&L_{n,\alpha}^{*}\left(\psi_x^0;x\right) = 1, \\
&L_{n,\alpha}^{*}\left(\psi_x^1;x\right) = \frac{2\alpha-1}{n-1}x + \frac{1}{n-1},
\end{aligned}$$

$$L^*_{n,\alpha}(\psi_x^2;x) = \frac{2n+2(4\alpha-3)}{(n-2)(n-1)}x^2 + \frac{2n+2(5\alpha-3)}{(n-2)(n-1)}x + \frac{2}{(n-2)(n-1)}.$$

Proof In view of Lemmas 2.1 and 2.2 we can apply the linearity and easily complete the proof. □

3 Approximation in Korovkin and weighted Korovkin spaces

Take $C_B(\mathbb{R}^+)$ be the space of all bounded and continuous functions defined on the set $\mathbb{R}^+$, where $\mathbb{R}^+ = [0,\infty)$ and a normed defined on C_B as

$$\|f\|_{C_B} = \sup_{x\geq 0}\left|f(x)\right|.$$

Let

$$E := \left\{f : x \in \mathbb{R}^+ \text{ and } \lim_{x\to\infty}\left(\frac{f(x)}{1+x^2}\right) < \infty\right\}.$$

Lemma 3.1 *For every $f \in C[0,\infty)\cap E$ the operators $L^*_{n,\alpha}$ given in (2.2) are uniformly convergent to f on each compact subset of $[0,A]$, whenever $A \in (0,\infty)$.*

Proof In the view of Korovkin-type property, it is enough to show that

$$L^*_{n,\alpha}(e_s;x) \to e_s(x), \quad \text{for } s = 0,1,2.$$

From Lemma 2.2, obviously $L^*_{n,\alpha}(e_0;y) \to e_0(x)$ as $n\to\infty$ and for $s = 1$

$$\lim_{n\to\infty} L^*_{n,\alpha}(e_1;x) = \lim_{n\to\infty}\left(\frac{n+2(\alpha-1)}{n-1}x + \frac{1}{n-1}\right) = e_1(x).$$

Similarly, we can prove for $s = 2$ that $L^*_{n,\alpha}(e_2;x) \to e_2$, which proves Proposition 3.1. □

Suppose $C[0,\infty)$ is the set of all continuous functions and $f \in C[0,\infty)$ with the weight function $\sigma(x) = 1 + x^2$,

$$\mathfrak{P}_\sigma(x) = \left\{f : \left|f(x)\right| \leq \mathcal{M}_f\sigma(x), x \in [0,\infty)\right\},$$

$$\mathfrak{Q}_\sigma(x) = \left\{f : f \in C[0,\infty)\cap\mathfrak{P}_\sigma(x), x \in [0,\infty)\right\},$$

$$\mathfrak{Q}^m_\sigma(x) = \left\{f : f \in \mathfrak{Q}_\sigma(x), \lim_{x\to\infty}\frac{f(x)}{\sigma(x)} = m, x \in [0,\infty)\right\},$$

where the norm defined on weight function σ such as $\|f\|_\sigma = \sup_{x\in[0,\infty)}\frac{|f(x)|}{\sigma(x)}$ and the constant $\mathcal{M}_f$ depends only on f.

Theorem 3.2 *For all $f \in \mathfrak{Q}^m_\sigma(x)$ the operators $L^*_{n,\alpha}(\cdot\,;\cdot)$ defined by (2.2) satisfy*

$$\lim_{n\to\infty}\left\|L^*_{n,\alpha}(f;x) - f\right\|_\sigma = 0.$$

Proof Take $f(t) \in \mathfrak{Q}_\sigma^m(x)$ with $x \in [0,\infty)$ and $f(t) = e_\nu$ for $\nu = 0,1,2$. Then from the well-known Korovkin theorem $L_{n,\alpha}^*(e_\nu;x) \to x^\nu$, satisfying the properties of uniformly behaving as $n \to \infty$. Since for $\nu = 0$, from Lemma 2.2 $L_{n,\alpha}^*(e_0;x) = 1$, thus we have

$$\left\| L_{n,\alpha}^*(e_0;x) - 1 \right\|_\sigma = 0. \tag{3.1}$$

For $\nu = 1$, we have

$$\begin{aligned}\left\| L_{n,\alpha}^*(e_1;x) - x \right\|_\sigma &= \sup_{x\in[0,\infty)} \frac{|L_{n,\alpha}^*(e_1;x) - x|}{1+x^2} \\ &= \left(\frac{n+2(\alpha-1)}{n-1} - 1 \right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{1}{(n-1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}.\end{aligned}$$

As $n \to \infty$,

$$\left\| L_{n,\alpha}^*(e_1;x) - x \right\|_\sigma = 0. \tag{3.2}$$

In a similar way for $\nu = 2$,

$$\begin{aligned}&\left\| L_{n,\alpha}^*(e_2;x) - x^2 \right\|_\sigma \\ &\quad = \sup_{y\in[0,\infty)} \frac{|L_{n,\alpha}^*(e_2;x) - x^2|}{1+x^2} \\ &\quad = \left(\frac{n^2+n(4\alpha-3)}{(n-2)(n-1)} - 1 \right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\ &\qquad + \left(\frac{4n+10\alpha-10}{(n-2)(n-1)} \right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{2}{(n-2)(n-1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}, \\ &\left\| L_{n,\alpha}^*(e_2;x) - x^2 \right\|_\sigma = 0 \quad \text{when } n \to \infty. \end{aligned} \tag{3.3}$$

This completes the proof. □

4 Pointwise approximation properties by $L_{n,\alpha}^*$

Here, we study the order of approximation of a function f with the aid of positive linear operators $L_{n,\alpha}^*(f;x)$ defined by (2.2) in terms of the classical modulus of continuity, the second-order modulus of continuity, Peetres K-functional and the Lipschitz class. A well-known property is the modulus of continuity of order one and of order two defined as follows. For $\delta > 0$ and $f \in C[a,b]$ the classical modulus of continuity of order one is given by

$$\omega(f;\delta) = \sup_{x_1,x_2\in[a,b],|x_1-x_2|\leq\delta} \left| f(x_1) - f(x_2) \right|,$$

and of order two it is given by

$$\omega_2\left(f;\delta^{\frac{1}{2}}\right) = \sup_{0<h<\delta^{\frac{1}{2}}} \sup_{x\in\mathbb{R}^+} \left| f(x) - 2f(x+h) + f(x+2h) \right|. \tag{4.1}$$

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $[0,\infty)$ and

$$C_B^2[0,\infty)=\bigl\{\psi\in C_B[0,\infty):\psi',\psi''\in C_B[0,\infty)\bigr\},\tag{4.2}$$

with the norm

$$\|\psi\|_{C_B^2[0,\infty)}=\|\psi\|_{C_B[0,\infty)}+\bigl\|\psi'\bigr\|_{C_B[0,\infty)}+\bigl\|\psi''\bigr\|_{C_B[0,\infty)},\tag{4.3}$$

also

$$\|\psi\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}\bigl|\psi(x)\bigr|.\tag{4.4}$$

Lemma 4.1 ([31]) *Let $\{P_n\}_{n\geq 1}$ be the sequence for the positive integer n with $P_n(1;x)=1$. Then for every $\psi\in C_B^2[0,\infty)$*

$$\bigl|P_n(\psi;x)-\psi(x)\bigr|\leq\bigl\|g'\bigr\|\sqrt{P_n\bigl((s-x)^2;x\bigr)}+\frac{1}{2}\bigl\|\psi''\bigr\|P_n\bigl((s-x)^2;x\bigr).$$

Lemma 4.2 ([31]) *For all $f\in C[a,b]$ and $h\in(0,\frac{b-a}{2})$, we have the following inequalities*:

(i) $\|f_h-f\|\leq\dfrac{3}{4}\omega_2(f,h)$,

(ii) $\bigl\|f_h''\bigr\|\leq\dfrac{3}{2h^2}\omega_2(f,h)$,

where f_h denotes the second-order Steklov function.

Theorem 4.3 *For all $f\in C_B[0,\infty)$ and $x\in[0,a]$, $a>0$ we have*

$$\bigl|L_{n,\alpha}^*(f;x)-f(x)\bigr|\leq 2\omega\bigl(f;\sqrt{\Theta_n(x)}\bigr),$$

where $\Theta_n(x)=L_{n,\alpha}^*(\psi_x^2;x)$ and $L_{n,\alpha}^*(\psi_x^2;x)$ is defined by Lemma 2.3.

Proof In view of the classical modulus of continuity, we have

$$\begin{aligned}\bigl|L_{n,\alpha}^*(f;x)-f(x)\bigr|&\leq\sum_{k=0}^{\infty}\mathcal{S}_{n,k}^{(\alpha)}(x)\int_0^\infty\mathcal{Q}_{n,k}(t)\bigl|f(t)-f(x)\bigr|\,dt\\&\leq\left\{1+\frac{1}{\delta}\sum_{k=0}^{\infty}\mathcal{S}_{n,k}^{(\alpha)}(x)\int_0^\infty\mathcal{Q}_{n,k}(t)|t-x|\,dt\right\}\omega(f;\delta).\end{aligned}$$

In the light of the Cauchy–Schwartz inequality, we get

$$\begin{aligned}\bigl|L_{n,\alpha}^*(f;x)-f(x)\bigr|&\leq\left\{1+\frac{1}{\delta}\left(\sum_{k=0}^{\infty}\mathcal{S}_{n,k}^{(\alpha)}(x)\int_0^\infty\mathcal{Q}_{n,k}(t)(t-x)^2\,dt\right)^{\frac{1}{2}}\right\}\omega(f;\delta)\\&=\left\{1+\frac{1}{\delta}\sqrt{L_{n,\alpha}^*\bigl(\psi_x^2;x\bigr)}\right\}\omega(f;\delta).\end{aligned}$$

Choosing $\delta=(\Theta_n(x))^{\frac{1}{2}}=\sqrt{L_{n,\alpha}^*(\psi_x^2;x)}$, we arrive at the desired result. □

Theorem 4.4 *For every $f \in C[0,a]$, $a > 0$ the operators $L^*_{n,\alpha}(\cdot\,;\cdot)$ defined by* (2.2) *satisfy*

$$\left|L^*_{n,\alpha}(f;x) - f(x)\right| \le \frac{2}{a}\|f\|\delta^2 + \frac{3}{4}\left(a + 2 + h^2\right)\omega_2(f;\delta),$$

where $\delta = (\Theta_n(x))^{\frac{1}{2}}$ is defined by Theorem 4.3 *and $\omega_2(f;\delta)$ is by* (4.1) *equipped with the norm $\|f\| = \max_{x\in[a,b]}|f(x)|$.*

Proof Consider f_h is the Steklov function define in Lemma 4.2. Using Lemma 2.2, we obtain

$$\begin{aligned}\left|L^*_{n,\alpha}(f;x) - f(x)\right| &\le \left|L^*_{n,\alpha}(f - f_h;x)\right| + \left|f_h - f(x)\right| + \left|L^*_{n,\alpha}(f_h;x) - f_h(x)\right| \\ &\le 2\|f_h - f\| + \left|L^*_{n,\alpha}(f_h;x) - f_h(x)\right|.\end{aligned}$$

In view of the fact that $f_h \in C^2[0,a]$ and using Lemma 4.1, we obtain

$$\left|L^*_{n,\alpha}(f;x) - f(x)\right| \le \|f_h'\|\sqrt{L^*_{n,\alpha}\left((e_1 - x)^2;x\right)} + \frac{1}{2}\|f_h''\|L^*_{n,\alpha}\left((e_1 - x)^2;x\right). \tag{4.5}$$

From the Landau inequality and Lemma 4.2, we have

$$\begin{aligned}\|f_h\| &\le \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\| \\ &\le \frac{2}{a}\|f_h\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h).\end{aligned}$$

On choosing $\delta = (\Theta_n(x))^{\frac{1}{4}}$, one has

$$\left|L^*_{n,\alpha}(f_h;x) - f_h(x)\right| \le \frac{2}{a}\|f\|h^2 + \frac{3a}{4}\omega_2(f;h) + \frac{3}{4}h^2\omega_2(f;h). \tag{4.6}$$

Combining (4.6), (4.5) and Lemma 4.2, we obtain the required result. □

Theorem 4.5 *Let $L^*_{n,\alpha}(\cdot\,;\cdot)$ be the operators defined by* (2.2). *Then, for every $f \in C_B^2[0,\infty)$,*

$$\lim_{n\to\infty}(n-1)\left(L^*_{n,\alpha}(f;x) - f(x)\right) = \left(1 + 2\alpha x - x^2\right)f'(x) + 2\left(x + x^2\right)f''(x),$$

uniformly for $0 \le x \le a$, $a > 0$.

Proof Let $x_0 \in [0,\infty)$ be a fixed number; all $x \in [0,\infty)$. Then using Taylor's series, we have

$$f(x) - f(x_0) = (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \varphi(x,x_0)(x - x_0)^2, \tag{4.7}$$

where $\varphi(x,x_0) \in C_B[0,\infty)$ and $\lim_{x\to x_0}\varphi(x,x_0) = 0$.

By applying the operators $L^*_{n,\alpha}$ on (4.7), we deduce

$$\begin{aligned}L^*_{n,\alpha}(f;x_0) - f(x_0) = {}& f'(x_0)L^*_{n,\alpha}(e_1 - x_0;x_0) + \frac{1}{2}L^*_{n,\alpha}\left((x - x_0)^2;x_0\right)f''(x_0) \\ &+ L^*_{n,\alpha}\left(\varphi(x,x_0)(x - x_0)^2\right).\end{aligned} \tag{4.8}$$

In view of the Cauchy–Schwartz inequality for the last term of Eq. (4.8), we get

$$(n-1)L^{*}_{n,\alpha}\big(\varphi(x,x_0)(t-x_0)^2\big) \le (n-1)^2\sqrt{L^{*}_{n,\alpha}\big((e_1-x_0)^2\big)L^{*}_{n,\alpha}\big(\varphi^2(x,x_0)\big)}. \tag{4.9}$$

We have

$$\begin{aligned}
&\lim_{n\to\infty}(n-1)\big(L^{*}_{n,\alpha}(e_0-x_0;x)\big)=\big(1+2\alpha x-x^2\big)f'(x),\\
&\lim_{n\to\infty}(n-1)\big(L^{*}_{n,\alpha}\big((e_0-x_0)^2;x\big)\big)=2\big(x+x^2\big)f''(x),\\
&\lim_{n\to\infty}\big(L^{*}_{n,\alpha}\big((e_0-x_0)^4;x\big)\big)=0.
\end{aligned}$$

This completes the proof. □

Now here we estimate the rate of convergence in terms of the usual Lipschitz class $\mathrm{Lip}_M(\nu)$. Let $f\in C[0,a)$, $a>0$ and M be a positive constant, and, for any $\nu\in(0,1]$, the Lipschitz class $\mathrm{Lip}_M(\nu)$ is as follows:

$$\mathrm{Lip}_M(\nu)=\big\{f:\big|f(\varsigma_1)-f(\varsigma_2)\big|\le M|\varsigma_1-\varsigma_2|^{\nu}\ \big(\varsigma_1,\varsigma_2\in[0,\infty)\big)\big\}. \tag{4.10}$$

Theorem 4.6 *Let $f\in\mathrm{Lip}_M(\nu)$ with $M>0$ and $0<\nu\le1$. Then the operators $L^{*}_{n,\alpha}(\cdot\,;\cdot)$ satisfy*

$$\big|L^{*}_{n,\alpha}(f;x)-f(x)\big|\le M\big(\Theta_n(x)\big)^{\frac{\nu}{2}},$$

where $n>2$ and $\Theta_n(x)$ defined by Theorem 4.3.

Proof From the Hölder inequality and (4.10), we conclude

$$\begin{aligned}
\big|L^{*}_{n,\alpha}(f;x)-f(x)\big| &\le \big|L^{*}_{n,\alpha}\big(f(t)-f(x);x\big)\big|\\
&\le L^{*}_{n,\alpha}\big(\big|f(t)-f(x)\big|;x\big)\\
&\le ML^{*}_{n,\alpha}\big(|t-x|^{\nu};x\big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\big|L^{*}_{n,\alpha}(f;x)-f(x)\big|\\
&\quad\le M\sum_{k=0}^{\infty}\mathcal{S}^{(\alpha)}_{n,k}(x)\int_0^{\infty}\mathcal{Q}_{n,k}(t)|t-x|^{\nu}\,dt\\
&\quad\le M\sum_{k=0}^{\infty}\big(\mathcal{S}^{(\alpha)}_{n,k}(x)\big)^{\frac{2-\nu}{2}}\\
&\qquad\times\big(\mathcal{S}^{(\alpha)}_{n,k}(x)\big)^{\frac{\nu}{2}}\int_0^{\infty}\mathcal{Q}_{n,k}(t)|t-x|^{\nu}\,dt\\
&\quad\le M\Bigg(\sum_{k=0}^{\infty}\mathcal{S}^{(\alpha)}_{n,k}(x)\int_0^{\infty}\mathcal{Q}_{n,k}(t)\,dt\Bigg)^{\frac{2-\nu}{2}}
\end{aligned}$$

$$\times\left(\sum_{k=0}^{\infty}\mathcal{S}_{n,k}^{(\alpha)}(x)\int_0^{\infty}\mathcal{Q}_{n,k}(t)|t-x|^2\,dt\right)^{\frac{\nu}{2}}$$
$$=M\big(L_{n,\alpha}^{*}\big(\psi_x^2;x\big)\big)^{\frac{\nu}{2}}.$$

This completes the proof. □

Theorem 4.7 *For all $\psi\in C_B^2[0,\infty)$ and $n>2$,*

$$\big|L_{n,\alpha}^{*}(\psi;x)-\psi(x)\big|\le\left(\Delta_n(x)+\frac{\Theta_n(x)}{2}\right)\|\psi\|_{C_B^2[0,\infty)},$$

where $\Delta_n(x)=(\frac{2\alpha-1}{n-1}x+\frac{1}{n-1})$ and $\Theta_n(x)$ is defined by Theorem 4.3.

Proof Let $\psi\in C_B^2(\mathbb{R}^+)$; for all $\varphi\in(x,t)$ a Taylor series expansion is

$$\psi(t)=\frac{(t-x)^2}{2}\psi''(\varphi)+(t-x)\psi'(x)+\psi(x).$$

On applying $L_{n,\alpha}^{*}$, using linearity,

$$L_{n,\alpha}^{*}(\psi;x)-\psi(x)=\psi'(x)L_{n,\alpha}^{*}\big((t-x);x\big)+\frac{\psi''(\varphi)}{2}L_{n,\alpha}^{*}\big((t-x)^2;x\big),$$

which implies that

$$\begin{aligned}&\big|L_{n,\alpha}^{*}(\psi;x)-\psi(x)\big|\\&\quad\le\left(\frac{2\alpha-1}{n-1}x+\frac{1}{n-1}\right)\|\psi'\|_{C_B[0,\infty)}\\&\qquad+\left\{\frac{2n+2(4\alpha-3)}{(n-2)(n-1)}x^2+\frac{2n+2(5\alpha-3)}{(n-2)(n-1)}x+\frac{2}{(n-2)(n-1)}\right\}\frac{\|\psi''\|_{C_B[0,\infty)}}{2}.\end{aligned}$$

From (4.3) we have $\|\psi'\|_{C_B[0,\infty)}\le\|\psi\|_{C_B^2[0,\infty)}$, $\|\psi''\|_{C_B[0,\infty)}\le\|\psi\|_{C_B^2[0,\infty)}$.

$$\begin{aligned}&\big|L_{n,\alpha}^{*}(\psi;x)-\psi(x)\big|\\&\quad\le\left(\frac{2\alpha-1}{n-1}x+\frac{1}{n-1}\right)\|\psi\|_{C_B^2[0,\infty)}\\&\qquad+\left\{\frac{2n+2(4\alpha-3)}{(n-2)(n-1)}x^2+\frac{2n+2(5\alpha-3)}{(n-2)(n-1)}x+\frac{2}{(n-2)(n-1)}\right\}\frac{\|\psi\|_{[0,\infty)}}{2}.\end{aligned}$$

This completes the proof. □

In 1968 [34] for investigating the interpolation between two Banach spaces Peetre introduced the K-functional by

$$K_2(f;\delta)=\inf_{C_B^2[0,\infty)}\big\{\big(\|f-\psi\|_{C_B[0,\infty)}+\delta\|\psi\|_{C_B^2[0,\infty)}\big):\psi\in C_B^2[0,\infty)\big\}\tag{4.11}$$

and a positive constant $\mathfrak{D}$ exists such that $K_2(f;\delta)\le\mathfrak{D}\omega_2(f;\delta^{\frac{1}{2}})$ with $\delta>0$ and $\omega_2(f;\delta)$ is the second-order modulus of continuity.

Theorem 4.8 *Suppose $C_B[0,\infty)$ is the set of all bounded and continuous functions on $[0,\infty)$. Then for every $f \in C_B[0,\infty)$*

$$\left|L^*_{n,\alpha}(f;x) - f(x)\right| \leq 2\mathfrak{D}\left\{\omega_2\left(f;\sqrt{\mathfrak{K}_n(x)}\right) + \min\left(1,\mathfrak{K}_n(x)\right)\|f\|_{C_B[0,\infty)}\right\},$$

where $\mathfrak{K}_n(x) = \frac{2\Delta_n(x)+\Theta_n(x)}{4}$ is defined by Theorem 4.7.

Proof In the light of results obtained by Theorem 4.7, we prove the desired theorem; hence

$$\begin{aligned}\left|L^*_{n,\alpha}(f;x) - f(x)\right| &\leq \left|L^*_{n,\alpha}(f-\psi;x)\right| + \left|f(x)-\psi(x)\right| + \left|L^*_{n,\alpha}(\psi;x)-\psi(x)\right| \\ &\leq 2\|f-\psi\|_{C_B[0,\infty)} + \left(\frac{\Theta_n(x)}{2} + \Delta_n(x)\right)\|\psi\|_{C^2_B[0,\infty)} \\ &= 2\left(\|f-\psi\|_{C_B[0,\infty)} + \left(\frac{\Theta_n(x)}{4} + \frac{\Delta_n(x)}{2}\right)\|\psi\|_{C^2_B[0,\infty)}\right).\end{aligned}$$

If we take the infimum over all $\psi \in C^2_B[0,\infty)$ and we use (4.11), we get

$$\left|L^*_{n,\alpha}(f;x) - f(x)\right| \leq 2K_2\left(f;\left(\frac{\Theta_n(x)}{4} + \frac{\Delta_n(x)}{2}\right)\right).$$

Now from [35] we use the relation for an absolute constant $\mathfrak{D} > 0$

$$K_2(f;\delta) \leq \mathfrak{D}\left\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|\right\}.$$

This completes the proof. □

5 Conclusion and observations

The manuscript parametric variant of Baskakov–Durrmeyer operators is a new extension of Baskakov Durrmeyer type operators. In the present investigation in our manuscript in order to get uniform convergence for the operators of the α-type extended version we study the order of approximation, the rate of convergence, the Korovkin-type, the weighted Korovkin-type approximation theorems, Peetres K-functional, Lipschitz functions and a set of direct theorems. It must be noted that we have more modeling flexibility when adding the parameter α to the Baskakov–Durrmeyer operators.

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