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(p,q)-gamma operators which preserve x^2

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Abstract

In this paper, we introduce (p,q)-gamma operators which preserve x^2, we estimate the moments of these operators, and establish direct and local approximation theorems of these operators. Then two approximation theorems about Lipschitz functions are obtained. The estimates on the rate of convergence and some weighted approximation theorems of the operators are also obtained. Furthermore, the Voronovskaja-type asymptotic formula is also presented.

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1 Introduction

With the rapid development of the approximation theory about the operators since the last century, lots of operators, such as Bernstein operators [4], Szász–Mirakjan operators [32, 37], Baskakov operators [3], Bleimann–Butzer–Hann operators [5], and Meyer–König–Zeller operators [31], have been proposed and constructed by several researchers due to Weierstrass and the important convergence theorem of Korovkin [26], see also [17]. In [23], Karsli considered gamma operators and studied the rate of convergence of these operators for the functions with derivative of bounded variation

$$L_n(f;x) = \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t)\,\mathrm{d}t, \quad x > 0. \tag{1}$$

In [25], Karsli and Ozarslan established some local and global approximation results for the operators L_n.

In recent years, with the rapid development of q-calculus [22], the study of new polynomials and operators constructed with q-integer has attracted more and more attention. Lupas first introduced q-Bernstein polynomials [27], and Phillips [36] proposed other q-analogue of Bernstein polynomials. Later, many researchers have performed studies in this field, and the q-analogue of classical operators and modified operators, such as q-Szász–Mirakjan operators [28], q-Baskakov operators [13], q-Meyer–König–Zeller operators [12], q-Bleimann–Butzer–Hann operators [11] q-Phillips operators [29], q-Baskakov–Kantorovich operators [20], q-Baskakov–Durrmeyer operators [19], q-Szász-beta operators [18], and q-Meyer–König–Zeller–Durrmeyer operators [15], has been constructed;

see also [2]. In [6], Cai and Zeng defined q-gamma operators

$$G_{n,q}(f;x) = \frac{[2n+3]!(q^{n+\frac{3}{2}}x)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!}\int_0^\infty \frac{t^n}{(q^{n+\frac{3}{2}}x+t)_q^{2n+4}}f(t)\,\mathrm{d}_qt, \quad x>0 \tag{2}$$

and gave their approximation properties.

Then many operators have been constructed with two parameters (p,q)-integer based on post-quantum calculus ((p,q)-calculus) which has been used efficiently in many areas of sciences such as Lie group, different equations, hypergeometric series, physical sciences, and so on. Recently, approximation by sequences of linear positive operators has been transferred to operators with (p,q)-integer. Let us review some useful notations and definitions about (p,q)-calculus in [2, 17, 21].

Let $0<q<p\le 1$. For each nonnegative integer n, the (p,q)-integer $[n]_{p,q}$, (p,q)-factorial $[n]_{p,q}!$ are defined by

$$[n]_{p,q} = \frac{p^n-q^n}{p-q}, \quad n=0,1,2,\ldots$$

and

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q}\cdots[n]_{p,q}, & n\ge 1; \\ 1, & n=0. \end{cases}$$

Further, the (p,q)-power basis is defined by

$$(x\oplus y)_{p,q}^n = (x+y)(px+qy)\bigl(p^2x+q^2y\bigr)\cdots\bigl(p^{n-1}x+q^{n-1}y\bigr)$$

and

$$(x\ominus y)_{p,q}^n = (x-y)(px-qy)\bigl(p^2x-q^2y\bigr)\cdots\bigl(p^{n-1}x-q^{n-1}y\bigr).$$

Let n be a non-negative integer, the (p,q)-gamma function is defined as

$$\varGamma_{p,q}(n+1) = \frac{(p\ominus q)_{p,q}^n}{(p-q)^n} = [n]_{p,q}!, \quad 0<q<p\le 1.$$

Aral and Gupta [1] proposed a (p,q)-beta function of the second kind for $m,n\in\mathbb{N}$ as follows:

$$B_{p,q}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1\oplus px)_{p,q}^{m+n}}\,\mathrm{d}_{p,q}x$$

and gave the relation of the (p,q)-analogues of beta and gamma functions:

$$B_{p,q}(m,n) = \frac{q\varGamma_{p,q}(m)\varGamma_{p,q}(n)}{(p^{m+1}q^{m-1})^{\frac{m}{2}}\varGamma_{p,q}(m+n)}.$$

As a special case, if $p = q = 1$, $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. It is obvious that order is important for (p,q)-setting, which is the reason why a (p,q)-variant of beta function does not satisfy commutativity property, i.e., $B_{p,q}(m,n) \neq B_{p,q}(n,m)$.

Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions f on the interval $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} \bigl|f(x)\bigr|.$$

Let $\delta > 0$ and $C_B^2[0,\infty) = \{g : g', g'' \in C_B[0,\infty)\}$, the following K-functional is defined:

$$K(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \bigl\{\|f-g\| + \delta\bigl\|g''\bigr\|\bigr\}.$$

Using DeVore–Lorentz theorem (see [10]), there exists a constant $C > 0$ such that

$$K(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{3}$$

where

$$\omega_2(f;\delta) = \sup_{0<|t|\leq\delta} \sup_{x\in[0,\infty)} \bigl|f(x+2t) - 2f(x+t) + f(x)\bigr|$$

is the second order modulus of smoothness of f. Also, by $\omega(f;\delta)$ we denote the usual modulus of continuity of $f \in C_B[0,\infty)$ defined as

$$\omega(f;\delta) = \sup_{0<|t|\leq\delta} \sup_{x\in[0,\infty)} \bigl|f(x+t) - f(x)\bigr|.$$

Let $B_{x^2}[0,\infty)$ denote the function space of all functions f such that $|f(x)| \leq C_f(1+x^2)$, where C_f is a positive constant depending on f. By $C_{x^2}[0,\infty)$ we denote the subspace of all continuous functions in the function space $B_{x^2}[0,\infty)$. By $C_{x^2}^0[0,\infty)$ we denote the subspace of all functions $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty} \frac{|f(x)|}{1+x^2}$ is endowed with the norm

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

For $a > 0$, the modulus of continuity of f on $[0,a]$ is defined as follows:

$$\omega_a(f;\delta) = \sup_{|y-x|<\delta} \sup_{0\leq x,y\leq a} \bigl|f(y) - f(x)\bigr|.$$

As is known, if f is not uniformly continuous on $[0,\infty)$, we cannot get $\omega(f;\delta) \to 0$ as $\delta \to 0$. In [38], Yuksel and Ispir defined the weighted modulus of continuity $\Omega(f;\delta) = \sup_{0<h\leq\delta, x\geq 0} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}$ while $f \in C_{x^2}^0[0,\infty)$ and proved the properties of monotone increasing about $\Omega(f;\delta)$ as $\delta > 0$ and the inequality $\Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$ while $\lambda > 0$ and $f \in C_{x^2}^0[0,\infty)$.

Let $f \in C_B[0,\infty)$, $M > 0$, and $\gamma \in (0,1]$. We recall that $f \in \mathrm{Lip}_M(\gamma)$ if the following inequality

$$\bigl|f(x) - f(y)\bigr| \leq M|x-y|^\gamma, \quad x,y \in [0,\infty)$$

is satisfied. Let F be a subset of the interval $[0,\infty)$, we define that $f \in \mathrm{Lip}_M(\gamma,F)$ if the following inequality

$$\left|f(x)-f(y)\right| \leq M|x-y|^{\gamma}, \quad x \in F \text{ and } y \in [0,\infty)$$

holds.

Recently, Mursaleen first applied (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators [33], (p,q)-Bernstein–Stancu operators [34], (p,q)-Bernstein–Schurer operators [35] and investigated their approximation properties. In addition, many well-known approximation operators with (p,q)-integer, such as (p,q)-Bernstein–Stancu–Schurer–Kantorovich operators [8], (p,q)-Szász–Baskakov operators [16], (p,q)-Baskakov-beta operators [30] have been introduced. All this achievement motivates us to construct the (p,q)-analogue of the gamma operator (1), as we know that many researchers have studied approximation properties of the gamma operators and their modifications (see [7, 9, 24, 39]). The rest of the paper is organized as follows. In Sect. 2, we define the (p,q)-gamma operators and obtain the moments and the central moments of them. In Sect. 3, we study the properties of the (p,q)-gamma operators about Lipschitz condition. Then some direct theorems about local approximation, rate of convergence, weighted approximation, and Voronovskaja-type approximation are obtained.

2 (p,q)-gamma operators and moments

We first define the analogue of gamma operators via (p,q)-calculus as follows.

Definition 2.1 *For $n \in \mathbb{N}$, $x \in (0,\infty)$ and $0 < q < p \leq 1$, the (p,q)-gamma operators can be defined as follows*:

$$G_n^{p,q}(f;x) = \frac{x^{n+3}(q^{n+\frac{3}{2}})^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}}{B_{p,q}(n+1,n+3)} \int_0^{\infty} \frac{t^n}{((pq)^{n+\frac{3}{2}}x \oplus t)_{p,q}^{2n+4}} f(t)\,\mathrm{d}_{p,q}t.$$

Operators $G_n^{p,q}$ are linear and positive. For $p = 1$, they turn out to be the q-gamma operators defined in (2). We will derive the moments $G_n^{p,q}(t^k;x)$ and the central moments $G_n^{p,q}((t-x)^k;x)$ for $k = 0,1,2,3,4$.

Lemma 2.1 *For $x \in (0,\infty)$, $0 < q < p \leq 1$, and $k = 0,1,\ldots,n+2$, we have*

$$G_n^{p,q}\big(t^k;x\big) = \frac{x^k(pq)^{k-\frac{k^2}{2}}[n+k]_{p,q}![n-k+2]_{p,q}!}{[n]_{p,q}![n+2]_{p,q}!}. \tag{4}$$

Proof Using the properties of (p,q)-beta function and (p,q)-gamma function, we have

$$\begin{aligned} G_n^{p,q}\big(t^k;x\big) &= \frac{x^{n+3}(q^{n+\frac{3}{2}})^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}}{B_{p,q}(n+1,n+3)} \int_0^{\infty} \frac{t^{n+k}}{((pq)^{n+\frac{3}{2}}x \oplus t)_{p,q}^{2n+4}}\,\mathrm{d}_{p,q}t \\ &= \frac{x^{n+3}(q^{n+\frac{3}{2}})^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}}{B_{p,q}(n+1,n+3)} \int_0^{\infty} \frac{1}{(pq)^{(2n+3)(n+2)}x^{2n+4}} \\ &\quad \times \frac{t^{n+k}}{(1 \oplus \frac{pt}{xq^{n+\frac{3}{2}}p^{n+\frac{5}{2}}})_{p,q}^{2n+4}}\,\mathrm{d}_{p,q}t \end{aligned}$$

$$\begin{aligned}
&= \frac{x^{n+3}(q^{n+\frac{3}{2}})^{n+3}p^{n^2+\frac{7}{2}n+\frac{7}{2}}}{B_{p,q}(n+1,n+3)}\int_0^\infty \frac{(xq^{n+\frac{3}{2}}p^{n+\frac{5}{2}})^{n+k+1}}{(pq)^{(2n+3)(n+2)}x^{2n+4}} \\
&\quad\times \frac{(\frac{t}{xq^{n+\frac{3}{2}}p^{n+\frac{5}{2}}})^{n+k}}{(1\oplus \frac{pt}{xq^{n+\frac{3}{2}}p^{n+\frac{5}{2}}})_{p,q}^{2n+4}}\,\mathrm{d}_{p,q}\left(\frac{t}{xq^{n+\frac{3}{2}}p^{n+\frac{5}{2}}}\right) \\
&= \frac{x^k p^{kn+\frac{5}{2}k}q^{kn+\frac{3}{2}k}B_{p,q}(n+k+1,n-k+3)}{B_{p,q}(n+1,n+3)} \\
&= \frac{x^k(pq)^{k-\frac{k^2}{2}}[n+k]_{p,q}![n-k+2]_{p,q}!}{[n]_{p,q}![n+2]_{p,q}!}.
\end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.2 *For $x \in (0,\infty)$, $0 < q < p \le 1$, the following equalities hold*:

1. $G_n^{p,q}(1;x) = 1$;
2. $G_n^{p,q}(t;x) = \sqrt{\frac{p}{q}}(1 - \frac{p^{n+1}}{[n+2]_{p,q}})x$;
3. $G_n^{p,q}(t^2;x) = x^2$;
4. $G_n^{p,q}(t^3;x) = \frac{[n+3]_{p,q}x^3}{(pq)^{\frac{3}{2}}[n]_{p,q}}$;
5. $G_n^{p,q}(t^4;x) = \frac{[n+3]_{p,q}[n+4]_{p,q}x^4}{(pq)^4[n]_{p,q}[n-1]_{p,q}}$ *for* $n > 1$.

Proof The proof of this lemma is an immediate consequence of Lemma 2.1. Hence the details are omitted. □

Lemma 2.3 *Let $n > 1$ and $x \in (0,\infty)$, then for $0 < q < p \le 1$, we have the central moments as follows*:

1. $A(x) := G_n^{p,q}(t-x;x) = ((\sqrt{\frac{p}{q}} - 1) - \sqrt{\frac{p}{q}}\frac{p^{n+1}}{[n+2]_{p,q}})x$;
2. $B(x) := G_n^{p,q}((t-x)^2;x) = -2((\sqrt{\frac{p}{q}} - 1) - \sqrt{\frac{p}{q}}\frac{p^{n+1}}{[n+2]_{p,q}})x^2$;
3. $G_n^{p,q}((t-x)^4;x) = (\frac{[n+2]_{p,q}[n+3]_{p,q}[n+4]_{p,q}-4(pq)^{\frac{5}{2}}[n-1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}{(pq)^4[n-1]_{p,q}[n]_{p,q}[n+2]_{p,q}} + \frac{-4(pq)^{\frac{9}{2}}[n-1]_{p,q}[n]_{p,q}[n+1]_{p,q}+7(pq)^4[n-1]_{p,q}[n]_{p,q}[n+2]_{p,q}}{(pq)^4[n-1]_{p,q}[n]_{p,q}[n+2]_{p,q}})x^4$.

Proof Because $G_n^{p,q}(t-x;x) = G_n^{p,q}(t;x) - x$, $G_n^{p,q}((t-x)^2;x) = G_n^{p,q}(t^2;x) - 2xG_n^{p,q}(t;x) + x^2$, and $G_n^{p,q}((t-x)^4;x) = G_n^{p,q}(t^4;x) - 4xG_n^{p,q}(t^3;x) + 6x^2G_n^{p,q}(t^2;x) - 4x^3G_n^{p,q}(t;x) + x^4$, and from Lemma 2.2, we obtain Lemma 2.3 easily. □

Lemma 2.4 *The sequences (p_n), (q_n) satisfy $0 < q_n < p_n \le 1$ such that $p_n \to 1$, $q_n \to 1$ and $p_n^n \to \alpha$, $q_n^n \to \beta$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty}[n-1]_{p_n,q_n}G_n^{p_n,q_n}(t-x;x) = -\frac{\alpha+\beta}{2}x; \tag{5}$$

$$\lim_{n\to\infty}[n-1]_{p_n,q_n}G_n^{p_n,q_n}\bigl((t-x)^2;x\bigr) = (\alpha+\beta)x^2; \tag{6}$$

$$\lim_{n\to\infty}[n-1]_{p_n,q_n}G_n^{p_n,q_n}\bigl((t-x)^4;x\bigr) = 0. \tag{7}$$

Proof Using

$$\begin{aligned}
&\lim_{n\to\infty}[n-1]_{p_n,q_n}\left(\left(\sqrt{\frac{p_n}{q_n}}-1\right)-\sqrt{\frac{p_n}{q_n}}\frac{p_n^{n+1}}{[n+2]_{p_n,q_n}}\right)\\
&\quad=\lim_{n\to\infty}[n+2]_{p_n,q_n}\left(\left(\sqrt{\frac{p_n}{q_n}}-1\right)-\sqrt{\frac{p_n}{q_n}}\frac{p_n^{n+1}}{[n+2]_{p_n,q_n}}\right)\\
&\quad=\lim_{n\to\infty}\left(\frac{p_n^{n+2}-q_n^{n+2}}{p_n-q_n}\frac{\sqrt{p_n}-\sqrt{q_n}}{\sqrt{q_n}}-\sqrt{\frac{p_n}{q_n}}p_n^{n+1}\right)\\
&\quad=\frac{\alpha-\beta}{2}-\alpha=-\frac{\alpha+\beta}{2},
\end{aligned}$$

we get (5) and (6) easily. Let $k=n-2$, we have

$$\begin{aligned}
&[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}[n+4]_{p_n,q_n}\\
&\quad=\left(q_n^3[k]_{p_n,q_n}+p_n^k[3]_{p_n,q_n}\right)\left(q_n^4[k]_{p_n,q_n}+p_n^k[4]_{p_n,q_n}\right)\left(q_n^5[k]_{p_n,q_n}+p_n^k[5]_{p_n,q_n}\right)\\
&\quad\sim q_n^{12}[k]_{p_n,q_n}^3+p_n^k\left(q_n^7[5]_{p_n,q_n}+q_n^8[4]_{p_n,q_n}+q_n^9[3]_{p_n,q_n}\right)[k]_{p_n,q_n}^2.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&[n-1]_{p_n,q_n}[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}\sim q_n^7[k]_{p_n,q_n}^3+p_n^k\left(q_n^3[4]_{p_n,q_n}+q_n^4[3]_{p_n,q_n}\right)[k]_{p_n,q_n}^2,\\
&[n-1]_{p_n,q_n}[n]_{p_n,q_n}[n+2]_{p_n,q_n}\sim q_n^4[k]_{p_n,q_n}^3+p_n^k\left(q_n^3+q_n[3]_{p_n,q_n}\right)[k]_{p_n,q_n}^2,\\
&[n-1]_{p_n,q_n}[n]_{p_n,q_n}[n+1]_{p_n,q_n}\sim q_n^3[k]_{p_n,q_n}^3+p_n^k\left(q_n^2+q_n[2]_{p_n,q_n}\right)[k]_{p_n,q_n}^2.
\end{aligned}$$

By Lemma 2.3, we can have

$$G_n^{p_n,q_n}\left((t-x)^4;x\right)\sim\left(A_n+\frac{1}{[k]_{p_n,q_n}}B_n\right)x^4,$$

where $A_n=q_n^{12}-4p_n^{\frac{5}{2}}q_n^{\frac{19}{2}}-4p_n^{\frac{9}{2}}q_n^{\frac{15}{2}}+7p_n^4q_n^8$ and

$$\begin{aligned}
B_n={}&p_n^k\Big(q_n^7[5]_{p_n,q_n}+q_n^8[4]_{p_n,q_n}+q_n^9[3]_{p_n,q_n}-4(p_nq_n)^{\frac{5}{2}}\left(q_n^3[4]_{p_n,q_n}+q_n^4[3]_{p_n,q_n}\right)\\
&-4(p_nq_n)^{\frac{9}{2}}\left(q_n^2+q_n[2]_{p_n,q_n}\right)+7(p_nq_n)^4\left(q_n^3+q_n[3]_{p_n,q_n}\right)\Big).
\end{aligned}$$

Set $P=\sqrt{p_n}$, $Q=\sqrt{q_n}$, by

$$\begin{aligned}
A_n&=P^{24}-4P^5Q^{19}-4P^9Q^{15}+7P^8Q^{16}\\
&\sim P^9-4P^5Q^4-4P^9+7P^8Q\\
&=3P^5\left(P^4-Q^4\right)-Q^4\left(P^5-Q^5\right)-7P^8(P-Q)\\
&=(P-Q)\left(3P^5\sum_{i=0}^{3}P^iQ^{3-i}-Q^4\sum_{i=0}^{4}P^iQ^{4-i}-7P^8\right),
\end{aligned}$$

we easily obtain

$$\begin{aligned}
[n-1]_{p_n,q_n}A_n &\sim [n]_{p_n,q_n}(P-Q)\left(3P^5\sum_{i=0}^{3}P^iQ^{3-i}-Q^4\sum_{i=0}^{4}P^iQ^{4-i}-7P^8\right)\\
&\sim \frac{p_n^n-q_n^n}{p_n-q_n}\frac{p_n-q_n}{\sqrt{p_n}+\sqrt{q_n}}\left(3P^5\sum_{i=0}^{3}P^iQ^{3-i}-Q^4\sum_{i=0}^{4}P^iQ^{4-i}-7P^8\right)\\
&\sim \frac{a-b}{2}(3\times 4-5-7)=0.
\end{aligned}$$

Similarly, $B_n \sim 5+4+3-4\times(4+3)-4\times(1+2)+7\times(1+3)=0$, we obtain (7). □

3 Approximation properties of (p,q)-gamma operators

In this section, we research the approximation properties of (p,q)-gamma operators. The following two theorems show approximation properties about Lipschitz functions.

Theorem 3.1 *Let $0<q<p\leq 1$ and F be any bounded subset of the interval $[0,\infty)$. If $f\in C_B[0,\infty)\cap \mathrm{Lip}_M(\gamma,F)$, then, for all $x\in(0,\infty)$, we have*

$$\left|G_n^{p,q}(f;x)-f(x)\right|\leq M\left(\left(B(x)\right)^{\frac{\gamma}{2}}+2d^{\gamma}(x;F)\right),$$

where $d(x;F)$ is the distance between x and F defined by $d(x;F)=\inf\{|x-y|:y\in F\}$.

Proof Let $\overline{F}$ be the closure of F in $[0,\infty)$. Using the properties of infimum, there is at least a point $y_0\in\overline{F}$ such that $d(x;F)=|x-y_0|$. By the triangle inequality, we can obtain

$$\begin{aligned}
\left|G_n^{p,q}(f;x)-f(x)\right| &\leq G_n^{p,q}\left(\left|f(x)-f(t)\right|;x\right)\\
&\leq G_n^{p,q}\left(\left|f(x)-f(y_0)\right|;x\right)+G_n^{p,q}\left(\left|f(t)-f(y_0)\right|;x\right)\\
&\leq M\left(G_n^{p,q}\left(|t-y_0|^{\gamma};x\right)+G_n^{p,q}\left(|x-y_0|^{\gamma};x\right)\right)\\
&\leq M\left(G_n^{p,q}\left(|x-t|^{\gamma};x\right)+2d^{\gamma}(x;F)\right).
\end{aligned}$$

Choosing $k_1=\frac{2}{\gamma}$ and $k_2=\frac{2}{2-\gamma}$ and using the well-known Hölder inequality, we have

$$\begin{aligned}
\left|G_n^{p,q}(f;x)-f(x)\right| &\leq M\left(\left(G_n^{p,q}\left(|x-t|^{k_1\gamma};x\right)\right)^{\frac{1}{k_1}}\left(G_n^{p,q}\left(1^{k_2};x\right)\right)^{\frac{1}{k_2}}+2d^{\gamma}(x;F)\right)\\
&\leq M\left(G_n^{p,q}\left((x-t)^2;x\right)^{\frac{\gamma}{2}}+2d^{\gamma}(x;F)\right)\\
&= M\left(\left(B(x)\right)^{\frac{\gamma}{2}}+2d^{\gamma}(x;F)\right).
\end{aligned}$$

This completes the proof. □

Theorem 3.2 *Let $0<q<p\leq 1$. Then, for all $f\in \mathrm{Lip}_M(\gamma)$, we have*

$$\left|G_n^{p,q}(f;x)-f(x)\right|\leq MB^{\frac{\gamma}{2}}(x).$$

Proof Using the monotonicity of the operators $G_n^{p,q}$ and the Hölder inequality, we can obtain

$$\begin{aligned}\left|G_n^{p,q}(f;x)-f(x)\right| &\le G_n^{p,q}\left(\left|f(t)-f(x)\right|;x\right) \le MG_n^{p,q}\left(|t-x|^{\gamma};x\right)\\ &= MG_n^{p,q}\left(\left(|t-x|^2\right)^{\frac{\gamma}{2}};x\right) \le M\left(G_n^{p,q}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}} = MB^{\frac{\gamma}{2}}(x). \qquad \square\end{aligned}$$

The third theorem is a direct local approximation theorem for the operators $G_n^{p,q}(f;x)$.

Theorem 3.3 *Let $0 < q < p \le 1$, $f \in C_B[0,\infty)$. Then, for every $x \in (0,\infty)$, there exists a positive constant C_1 such that*

$$\left|G_n^{p,q}(f;x)-f(x)\right| \le C_1\omega_2\left(f;\sqrt{B(x)+A^2(x)}\right)+\omega\left(f;\left|A(x)\right|\right).$$

Proof For $x \in (0,\infty)$, we consider new operators $H_n^{p,q}(f;x)$ defined by

$$H_n^{p,q}(f;x) = G_n^{p,q}(f;x)+f(x)-f\left(A(x)+x\right).$$

Using the operator above and Lemma 2.3, we have

$$H_n^{p,q}(t-x;x) = G_n^{p,q}(t-x;x)-A(x) = 0.$$

Let $x,t \in (0,\infty)$ and $g \in C_B^2[0,\infty)$. Using Taylor's expansion, we can obtain

$$g(t) = g(x)+g'(x)(t-x)+\int_x^t g''(u)(t-u)\,\mathrm{d}u.$$

Hence,

$$\begin{aligned}\left|H_n^{p,q}(g;x)-g(x)\right| &= \left|g'(x)H_n^{p,q}\left((t-x);x\right)+H_n^{p,q}\left(\int_x^t g''(u)(t-u)\,\mathrm{d}u;x\right)\right|\\ &\le \left|H_n^{p,q}\left(\int_x^t g''(u)(t-u)\,\mathrm{d}u;x\right)\right|\\ &\le \left|G_n^{p,q}\left(\int_x^t g''(u)(t-u)\,\mathrm{d}u;x\right)-\int_x^{A(x)+x} g''(u)\left(A(x)+x-u\right)\mathrm{d}u\right|\\ &\le G_n^{p,q}\left(\int_x^t \left|g''(u)\right|(t-u)\,\mathrm{d}u;x\right)+\left|\int_x^{A(x)+x}\left|g''(u)\right|\left(A(x)+x-u\right)\mathrm{d}u\right|\\ &\le \left(B(x)+A^2(x)\right)\left\|g''\right\|.\end{aligned}$$

Using $|G_n^{p,q}(f;x)| \le \|f\|$, we have

$$\begin{aligned}&\left|G_n^{p,q}(f;x)-f(x)\right|\\ &\quad= \left|H_n^{p,q}(f;x)+f\left(A(x)+x\right)-2f(x)\right|\\ &\quad\le \left|H_n^{p,q}(f-g;x)-(f-g)(x)\right|+\left|H_n^{p,q}(g;x)-g(x)\right|+\left|f\left(A(x)+x\right)-f(x)\right|\\ &\quad\le 4\|f-g\|+\left(B(x)+A^2(x)\right)\left\|g''\right\|+\omega\left(f;\left|A(x)\right|\right).\end{aligned}$$

Taking infimum over all $g \in C_B^2[0,\infty)$ and using (3), we can obtain the desired assertion. □

The fourth theorem is a result about the rate of convergence for the operators $G_n^{p,q}(f;x)$:

Theorem 3.4 *Let $f \in C_{x^2}[0,\infty)$, $0 < q < p \leq 1$, and $a > 0$, we have*

$$\left\| G_n^{p,q}(f;x) - f(x) \right\|_{C(0,a]} \leq 4C_f\left(1+a^2\right)B(a) + 2\omega_{a+1}\left(f;\sqrt{B(a)}\right).$$

Proof For all $x \in (0,a]$ and $t > a+1$, we easily have $(t-x)^2 \geq (t-a)^2 \geq 1$, therefore,

$$\begin{aligned} \left|f(t)-f(x)\right| &\leq \left|f(t)\right| + \left|f(x)\right| \leq C_f\left(2+x^2+t^2\right) \\ &= C_f\left(2+x^2+(x-t-x)^2\right) \leq C_f\left(2+3x^2+2(x-t)^2\right) \\ &\leq C_f\left(4+3x^2\right)(t-x)^2 \leq 4C_f\left(1+a^2\right)(t-x)^2, \end{aligned} \tag{8}$$

and for all $x \in (0,a]$, $t \in (0,a+1]$, and $\delta > 0$, we have

$$\left|f(t)-f(x)\right| \leq \omega_{a+1}\left(f,|t-x|\right) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{9}$$

From (8) and (9), we get

$$\left|f(t)-f(x)\right| \leq 4C_f\left(1+a^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta).$$

By Schwarz's inequality and Lemma 2.3, we have

$$\begin{aligned} &\left|G_n^{p,q}(f;x)-f(x)\right| \\ &\quad\leq G_n^{p,q}\left(\left|f(t)-f(x)\right|;x\right) \\ &\quad\leq 4C_f\left(1+a^2\right)G_n^{p,q}\left((t-x)^2;x\right) + G_n^{p,q}\left(\left(1+\frac{|t-x|}{\delta}\right);x\right)\omega_{a+1}(f;\delta) \\ &\quad\leq 4C_f\left(1+a^2\right)G_n^{p,q}\left((t-x)^2;x\right) + \omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{G_n^{p,q}\left((t-x)^2;x\right)}\right) \\ &\quad\leq 4C_f\left(1+a^2\right)B(x) + \omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{B(x)}\right) \\ &\quad\leq 4C_f\left(1+a^2\right)B(a) + \omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{B(a)}\right). \end{aligned}$$

By taking $\delta = \sqrt{B(a)}$ and supremum over all $x \in (0,a]$, we accomplish the proof of Theorem 3.4. □

The following three results are theorems about weighted approximation for the operators $G_n^{p,q}(f;x)$.

Theorem 3.5 *Let $f \in C_{x^2}^0[0,\infty)$ and the sequences (p_n), (q_n) satisfy $0 < q_n < p_n \leq 1$ such that $p_n^n \to 1$, $q_n^n \to 1$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$, then there exists a positive integer $N \in \mathbb{N}_+$*

such that, for all $n > N$ and $\nu > 0$, the inequality

$$\sup_{x\in(0,\infty)} \frac{|G_n^{p_n,q_n}(f;x) - f(x)|}{(1+x^2)^{\frac{3}{2}+\nu}} \le 4\sqrt{2}\Omega\left(f;\frac{1}{\sqrt{[n-1]_{p_n,q_n}}}\right) \tag{10}$$

holds.

Proof For $t > 0$, $x \in (0,\infty)$ and $\delta > 0$, by the definition and properties of $\Omega(f;\delta)$, we get

$$\begin{aligned} |f(t) - f(x)| &\le \bigl(1 + (x + |x-t|)\bigr)^2 \Omega\bigl(f;|t-x|\bigr) \\ &\le 2\bigl(1+x^2\bigr)\bigl(1+(t-x)^2\bigr)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta). \end{aligned}$$

Using $p_n^n \to 1$, $q_n^n \to 1$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$ and Lemma 2.4, there exists a positive integer $N \in \mathbb{N}_+$ such that, for all $n > N$,

$$G_n^{p_n,q_n}\bigl((t-x)^2;x\bigr) \le \frac{2(1+x^2)}{[n-1]_{p_n,q_n}}, \tag{11}$$

$$G_n^{p_n,q_n}\bigl((t-x)^4;x\bigr) \le 1. \tag{12}$$

Since $G_n^{p_n,q_n}$ is linear and positive, we have

$$\begin{aligned} \bigl|G_n^{p_n,q_n}(f;x) - f(x)\bigr| \le 2\bigl(1+x^2\bigr)\Omega(f;\delta)\Bigl\{1 + G_n^{p_n,q_n}\bigl((t-x)^2;x\bigr) \\ + G_n^{p_n,q_n}\left(\bigl(1+(t-x)^2\bigr)\frac{|t-x|}{\delta};x\right)\Bigr\}. \end{aligned} \tag{13}$$

To estimate the second term of (13), applying the Cauchy–Schwarz inequality and $(x+y)^2 \le 2(x^2+y^2)$, we have

$$G_n^{p_n,q_n}\left(\bigl(1+(t-x)^2\bigr)\frac{|t-x|}{\delta};x\right) \le \sqrt{2}\bigl(G_n^{p_n,q_n}\bigl(1+(t-x)^4;x\bigr)\bigr)^{\frac{1}{2}}\left(G_n^{p_n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac{1}{2}}.$$

By (11) and (12),

$$G_n^{p_n,q_n}\left(\bigl(1+(t-x)^2\bigr)\frac{|t-x|}{\delta};x\right) \le \frac{2\sqrt{2}(1+x^2)^{\frac{1}{2}}}{\delta[n-1]_{p_n,q_n}}.$$

Taking $\delta = \frac{1}{\sqrt{[n-1]_{p_n,q_n}}}$, we can obtain

$$\bigl|G_n^{p_n,q_n}(f;x) - f(x)\bigr| \le 4\sqrt{2}\bigl(1+x^2\bigr)^{\frac{3}{2}}\Omega\left(f;\frac{1}{\sqrt{[n-1]_{p_n,q_n}}}\right).$$

The proof is completed. □

Theorem 3.6 *Let the sequences (p_n), (q_n) satisfy $0 < q_n < p_n \le 1$ such that $p_n \to 1$, $q_n \to 1$, and $p_n^n \to \alpha$, $q_n^n \to \beta$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$. Then, for $f \in C_{x^2}^0[0,\infty)$, we have*

$$\lim_{n\to\infty}\bigl\|G_n^{p_n,q_n}(f;x) - f(x)\bigr\|_{x^2} = 0. \tag{14}$$

Proof By the Korovkin theorem in [14], we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\| G_n^{p_n,q_n}\left(t^k;x\right)-x^k\right\|_{x^2}=0,\quad k=0,1,2. \tag{15}$$

Since $G_n^{p_n,q_n}(1;x)=1$, $G_n^{p_n,q_n}(t^2;x)=x^2$, then (15) holds true for $k=0,2$. By Lemma 2.2, we can get

$$\begin{aligned}\left\| G_n^{p_n,q_n}(t;x)-x\right\|_{x^2} &= \sup_{x\in(0,\infty)}\frac{1}{1+x^2}\left|G_n^{p_n,q_n}(t;x)-x\right| \\ &= \sup_{x\in(0,\infty)}\frac{x}{1+x^2}\left|\frac{\sqrt{p_n}-\sqrt{q_n}}{\sqrt{q_n}}-\sqrt{\frac{p_n}{q_n}}\frac{p_n^{n+1}}{[n+2]_{p_n,q_n}}\right| \\ &\leq \sup_{x\in(0,\infty)}\left|\frac{\sqrt{p_n}-\sqrt{q_n}}{\sqrt{q_n}}-\sqrt{\frac{p_n}{q_n}}\frac{p_n^{n+1}}{[n+2]_{p_n,q_n}}\right|\to 0,\quad n\to\infty.\end{aligned}$$

Thus the proof is completed. □

Theorem 3.7 *Let the sequences* (p_n), (q_n) *satisfy* $0<q_n<p_n\leq 1$ *such that* $p_n\to 1$, $q_n\to 1$, $[n]_{p_n,q_n}\to\infty$ *as* $n\to\infty$. *For every* $f\in C_{x^2}[0,\infty)$ *and* $\kappa>0$, *we have*

$$\lim_{n\to\infty}\sup_{x\in(0,\infty)}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\kappa}}=0.$$

Proof Let $x_0\in(0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}\sup_{x\in(0,\infty)}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\kappa}} &\leq \sup_{x\in(0,x_0]}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\kappa}} \\ &\quad+\sup_{x\in(x_0,\infty)}\frac{|G_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+\kappa}} \\ &\leq \left\| G_n^{p_n,q_n}(f;x)-f(x)\right\|_{C(0,x_0]} \\ &\quad+C_f\sup_{x\in(x_0,\infty)}\frac{|G_n^{p_n,q_n}((1+t^2);x)|}{(1+x^2)^{1+\kappa}} \\ &\quad+\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+\kappa}}.\end{aligned} \tag{16}$$

Since $|f(x)|\leq C_f(1+x^2)$, we have $\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+\kappa}}\leq\frac{C_f}{(1+x_0^2)^\kappa}$. Let $\epsilon>0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{C_f}{(1+x_0^2)^\kappa}<\epsilon. \tag{17}$$

In view of Lemma 2.2, while $x\in(x_0,\infty)$, we obtain

$$C_f\lim_{n\to\infty}\frac{|G_n^{p_n,q_n}((1+t^2);x)|}{(1+x^2)^{1+\kappa}}=C_f\frac{(1+x^2)}{(1+x^2)^{1+\kappa}}=\frac{C_f}{(1+x^2)^\kappa}\leq\frac{C_f}{(1+x_0^2)^\kappa}<\epsilon.$$

Using Theorem 3.4, we can see that the first term of inequality (16) implies that

$$\left\| G_n^{p_n,q_n}(f;x) - f(x) \right\|_{C(0,x_0]} < \epsilon, \quad \text{as } n \to \infty. \tag{18}$$

Combining (16)–(18), we get the desired result. □

The last result is a Voronovskaja-type asymptotic formula for the operators $G_n^{p,q}(f;x)$.

Theorem 3.8 *Let* $f \in C_B^2[0,\infty)$ *and the sequences* (p_n), (q_n) *satisfy* $0 < q_n < p_n \leq 1$ *such that* $p_n \to 1$, $q_n \to 1$ *and* $p_n^n \to \alpha$, $q_n^n \to \beta$, $[n]_{p_n,q_n} \to \infty$ *as* $n \to \infty$, *where* $0 \leq \alpha, \beta < 1$. *Then, for all* $x \in (0,\infty)$,

$$\lim_{n\to\infty} [n-1]_{p_n,q_n} \left(G_n^{p_n,q_n}(f;x) - f(x) \right) = \frac{\alpha + \beta}{2} \left(-xf'(x) + x^2 f''(x) \right). \tag{19}$$

Proof Let $x \in (0,\infty)$ be fixed. By Taylor's expansion formula, we obtain

$$f(t) = f(x) + f'(x)(t-x) + \left(\frac{1}{2} f''(x) + \Theta_{p_n,q_n}(t,x) \right)(t-x)^2,$$

where $\Theta_{p_n,q_n}(x,t)$ is bounded and $\lim_{t\to x} \Theta_{p_n,q_n}(t,x) = 0$. By applying the operator $G_n^{p_n,q_n}(f;x)$ to the relation above, we obtain

$$\begin{aligned} G_n^{p_n,q_n}(f;x) - f(x) &= f'(x) G_n^{p_n,q_n}\left((t-x);x\right) + \frac{1}{2} f''(x) G_n^{p_n,q_n}\left((t-x)^2;x\right) \\ &\quad + G_n^{p_n,q_n}\left(\Theta_{p_n,q_n}(t,x)(t-x)^2;x\right). \end{aligned}$$

Since $\lim_{t\to x} \Theta_{p_n,q_n}(t,x) = 0$, then for all $\epsilon > 0$, there exists a positive constant $\delta > 0$ which implies $|\Theta_{p_n,q_n}(t,x)| < \epsilon$ for all fixed $x \in (0,\infty)$, where n is large enough, while $|t-x| \leq \delta$, then $|\Theta_{p_n,q_n}(t,x)| < \frac{C_2}{\delta^2}(t-x)^2$, where C_2 is a positive constant. Using Lemma 2.4, we obtain

$$\begin{aligned} &[n-1]_{p_n,q_n} \left| G_n^{p_n,q_n}\left(\Theta(t,x)(t-x)^2;x\right) \right| \\ &\quad \leq \epsilon [n-1]_{p_n,q_n} G_n^{p_n,q_n}\left((t-x)^2;x\right) \\ &\qquad + \frac{C_2}{\delta^2} [n-1]_{p_n,q_n} G_n^{p_n,q_n}\left((t-x)^4;x\right) \to 0 \quad (n \to \infty). \end{aligned}$$

The proof is completed. □

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The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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