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Iterated commutators of multilinear Calderón–Zygmund maximal operators on some function spaces

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Abstract

Let T^* be a multilinear Calderón–Zygmund maximal operator. In this paper, we study iterated commutators of T^* and pointwise multiplication with functions in Lipschitz spaces. More precisely, we give some new estimates for this kind of commutators under some Dini-type conditions on Lebesgue spaces, homogenous Lipschitz spaces, and homogenous Triebel–Lizorkin spaces.

MSC: 42B25; 47G10

Keywords: Maximal multilinear operators; Calderón–Zygmund operator; Commutators

1 Introduction and main results

For any $a > 0$, we say that $\omega \in \mathrm{Dini}(a)$ if

$$|\omega|_{\mathrm{Dini}(a)} = \int_0^1 \frac{\omega^a(t)}{t}\,dt < \infty,$$

where $\omega(t) : [0,\infty) \mapsto [0,\infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$.

We say that T is a multilinear Calderón–Zygmund operator with kernel of type $\omega(t)$, denoted by m-linear ω-CZO, if T can be extended to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $1 < q, q_1, \dots, q_m < \infty$ with $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$, or from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for some $1 < q_1, \dots, q_m < \infty$ with $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = 1$, and if there exists a function K defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T\vec{f}(x) = T(f_1,\dots,f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x,y_1,\dots,y_m) f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m \tag{1.1}$$

for all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and $f_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1,\dots,m$, and if there exists a constant $A > 0$ such that

$$\bigl|K(x,y_1,\dots,y_m)\bigr| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \tag{1.2}$$

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for all $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_j$ for some $j \in \{1, 2, \dots, m\}$, and

$$\begin{aligned} &\bigl|K(x, y_1, \dots, y_m) - K\bigl(x', y_1, \dots, y_m\bigr)\bigr| \\ &\quad \leq \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\biggl(\frac{|x-x'|}{|x-y_1| + \cdots + |x-y_m|}\biggr) \end{aligned} \tag{1.3}$$

whenever $|x - x'| \leq \frac{1}{m+1} \max_{1 \leq j \leq m} |x - y_j|$, and

$$\begin{aligned} &\bigl|K(x, y_1, \dots, y_j, \dots, y_m) - K\bigl(x, y_1, \dots, y_j', \dots, y_m\bigr)\bigr| \\ &\quad \leq \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\biggl(\frac{|y_j-y_j'|}{|x-y_1| + \cdots + |x-y_m|}\biggr) \end{aligned} \tag{1.4}$$

whenever $|y_j - y_j'| \leq \frac{1}{m+1} \max_{1 \leq j \leq m} |x - y_j|$.

When $\omega(x) = x^\gamma$ for some $\gamma > 0$, the m-linear ω-CZO is exactly the multilinear Calderón–Zygmund operator studied by Grafakos and Torres in [11]. The multilinear Calderón–Zygmund operators were introduced and first studied by Coifman and Meyer [5–7] and later by Grafakos and Torres [11, 12]. The study of such operators has attracted the interest of many experts; see, for example, [4, 14, 24] and the reference therein. Recently, many mathematicians are concerned to remove or replace the smoothness condition on the kernels; see, for example [1, 8–10, 13, 15, 21]. In this paper, we mainly investigate the maximal operator and give some new estimates for its iterated commutators on some function spaces.

The maximal truncated operator T^* is defined by

$$T^*(\vec{f})(x) = \sup_{\delta > 0} \bigl|T_\delta(f_1, \dots, f_m)(x)\bigr|,$$

where T_δ are the smooth truncations of T, that is,

$$T_\delta(f_1, \dots, f_m)(x) = \int_{|x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m.$$

For the maximal truncated operator T^* and a collection of locally integrable functions $\vec{b} = (b_1, \dots, b_m)$, we define the iterated commutator $T^*_{\Pi\vec{b}}$ by

$$\begin{aligned} T^*_{\Pi\vec{b}}(\vec{f})(x) &= \sup_{\delta > 0} \bigl|\bigl[b_1, \bigl[b_2, \dots \bigl[b_{m-1}, [b_m, T_\delta]_m\bigr]_{m-1} \cdots\bigr]_2\bigr]_1 (\vec{f})(x)\bigr| \\ &= \sup_{\delta > 0} \biggl|\int_{|x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2} \prod_{j=1}^m \bigl(b_j(x) - b_j(y_j)\bigr) K(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i)\, d\vec{y}\biggr|. \end{aligned}$$

The iterated commutators of multilinear singular integral operators with BMO functions have been studied by a large number of people; see, for example, [2, 18, 19]. On the other hand, commutators of multilinear singular integral operators with Lipschitz functions have been the subject of many recent papers. In 1995, Paluszyński [17] proved that the commutator generated by Calderón–Zygmund operators with classical kernel and Lipschitz functions is bounded from the Lebesgue space to the Lebesgue space and to the homogenous Triebel–Lizorkin space. The multilinear analogues of the results in [17] were given by Wang and Xu [23] and by Mo and Lu [16]. Finally, Sun and Zhang [22] relaxed the smooth condition assumed on the kernel to Dini-type condition. It is natural to ask whether, under the Dini-type condition, the iterated commutators of multilinear Calderón–Zygmund maximal operators and pointwise multiplication with functions in Lipschitz space share similar boundedness properties? In this paper, we give a positive answer. The main result reads as follows.

Theorem 1.1 *Suppose* $\omega \in \mathrm{Dini}(1)$ *and* $b_j \in \mathrm{Lip}_{\beta_j}$ *with* $0 < \beta_j < 1$ *for* $j = 1, \ldots, m$ *and* $\beta = \beta_1 + \cdots + \beta_m$. *If* $1 < p_1, \ldots, p_m < \infty$, $0 < q < \infty$, *and* $1/p_j > \beta_j/n$ *with* $1/q = 1/p_1 + \cdots + 1/p_m - \beta/n$, *then*

$$\|T^*_{\Pi\vec{b}}\vec{f}\|_{L^q} \lesssim \prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_{\beta_i}} \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

Theorem 1.2 *Suppose* $b_j \in \mathrm{Lip}_{\beta_j}$ *with* $0 < \beta_j < 1$ *for* $j = 1, \ldots, m$ *and* $\beta = \beta_1 + \cdots + \beta_m$. *If* $1 < p_1, \ldots, p_m < \infty$, $0 < 1/p_j < \beta_j/n$, $0 < \beta - n/p < 1$ *with* $1/p = 1/p_1 + \cdots + 1/p_m$, *and* ω *satisfies*

$$\int_0^1 \frac{\omega(t)}{t^{1+\beta-n/p}}\,dt < \infty, \tag{1.5}$$

then

$$\|T^*_{\Pi\vec{b}}\vec{f}\|_{\mathrm{Lip}_{\beta-n/p}} \lesssim \prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_{\beta_i}} \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

Theorem 1.3 *Suppose* $b_j \in \mathrm{Lip}_{\beta_j}$ *with* $0 < \beta_j < 1$ *for* $j = 1, \ldots, m$ *and* $\beta = \beta_1 + \cdots + \beta_m$. *If* $1 < p_1, \ldots, p_m < \infty$ *with* $1/p = 1/p_1 + \cdots + 1/p_m$ *and* ω *satisfies*

$$\int_0^1 \frac{\omega(t)}{t^{1+\beta}}\,dt < \infty, \tag{1.6}$$

then

$$\|T^*_{\Pi\vec{b}}\vec{f}\|_{\dot{F}_p^{\beta,\infty}} \lesssim \prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_{\beta_i}} \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

In the next section, we give some definitions and preliminaries. We focus on the proof of Theorem 1.1 in Sect. 3. The proof of Theorems 1.2 and 1.3 is given in Sect. 4. The notation $A \lesssim B$ stands for $A \le CB$ for some positive constant C independent of A and B.

2 Preliminaries

Definition 2.1 Given a locally integrable function f, define the fractional maximal function by

$$M_{\beta,r}f(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\beta r/n}}\int_Q \left|f(y)\right|^r dy\right)^{\frac{1}{r}}, \quad r \ge 1,$$

when $0 \le \beta < n/r$. If $\beta = 0$ and $r = 1$, then $M_{0,1}f = Mf$ denotes the usual Hardy–Littlewood maximal function. For $\delta > 0$, we denote M_δ by $M_\delta f = M(|f|^\delta)^{\frac{1}{\delta}}$.

The sharp maximal function $M^\sharp$ is given by

$$M^\sharp f(x) = \sup_{Q \ni x} \inf_{C} \frac{1}{|Q|} \int_Q \left| f(y) - C \right| dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right| dy,$$

where f_Q denotes the average of f over cube Q, and we denote $M_\delta^\sharp$ by $M_\delta^\sharp f(x) = M^\sharp(|f|^\delta)^{\frac{1}{\delta}}(x)$.

Definition 2.2 ([17]) For $\beta > 0$, the homogenous Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is the space of functions f such that

$$\|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} = \sup_{x,h \in \mathbb{R}^n, h \neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty,$$

where Δ_h^k denotes the kth difference operator.

To prove Theorems 1.1, 1.2, and 1.3, we need the following lemmas.

Lemma 2.1 ([17]) *Let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0 < \beta < 1$. For any cubes Q', Q in $\mathbb{R}^n$ such that $Q' \subset Q$, we have*

$$|b_{Q'} - b_Q| \lesssim \|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} |Q|^{\beta/n}.$$

Lemma 2.2 ([17])

(1) *For $0 < \beta < 1$ and $1 \le q < \infty$, we have*

$$\|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \approx \sup_Q \frac{1}{|Q|^{1+n/\beta}} \int_Q |f - f_Q| \approx \sup_Q \frac{1}{|Q|^{n/\beta}} \left(\int_Q |f - f_Q|^q \right)^{\frac{1}{q}}.$$

(2) *For $0 < \beta < 1$ and $1 \le p < \infty$, we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_Q \frac{1}{|Q|^{1+n/\beta}} \int_Q |f - f_Q| \right\|_{L^p}.$$

Lemma 2.3 ([20]) *Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{\omega} \in A_{\vec{p}}$. Let T be an m-linear ω-CZO with* $\omega \in \mathrm{Dini}(1)$.

(1) *If $1 < p_1, \ldots, p_m < \infty$, then*

$$\left\| T^* \vec{f} \right\|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}.$$

(2) *If $1 \le p_1, \ldots, p_m < \infty$, then*

$$\left\| T^* \vec{f} \right\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}.$$

3 Proof of Theorem 1.1

We borrow some ideas from [19]. Since the proof of Theorem 1.1 follows from similar steps in [22], we omit the proof. We just give three key lemmas.

Let $u, v \in C^\infty([0,\infty))$ be such that $|u'(t)| \le Ct^{-1}$, $|v'(t)| \le Ct^{-1}$, and

$$\chi_{[2,\infty)}(t) \le u(t) \le \chi_{[1,\infty)}(t), \qquad \chi_{[1,2]}(t) \le v(t) \le \chi_{[1/2,3]}(t).$$

For simplicity, we denote

$$K_{u,\eta}(x,y_1,\ldots,y_m) = K(x,y_1,\ldots,y_m)u\biggl(\frac{|x-y_1| + \cdots + |x-y_m|}{\eta}\biggr),$$
$$K_{v,\eta}(x,y_1,\ldots,y_m) = K(x,y_1,\ldots,y_m)v\biggl(\frac{|x-y_1| + \cdots + |x-y_m|}{\eta}\biggr),$$

and

$$U_\eta(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x,y_1,\ldots,y_m)\prod_{i=1}^m f_i(y_i)\,dy_1\cdots dy_m,$$
$$V_\eta(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K_{v,\eta}(x,y_1,\ldots,y_m)\prod_{i=1}^m f_i(y_i)\,dy_1\cdots dy_m.$$

Then we define the maximal operators

$$U^*(\vec{f})(x) = \sup_{\eta>0}\bigl|U_\eta(\vec{f})(x)\bigr| \quad \text{and} \quad V^*(\vec{f})(x) = \sup_{\eta>0}\bigl|V_\eta(\vec{f})(x)\bigr|.$$

It is easy to get $T^*(\vec{f}) \le U^*(\vec{f})(x) + V^*(\vec{f})(x)$. Next, we show that the functions $K_{u,\eta}$ and $K_{v,\eta}$ satisfy some smoothness properties.

Lemma 3.1 *For any $j = 0,1,2,\ldots,m$, we have*

$$\bigl|K_{u,\eta}(y_0,\ldots,y_j,\ldots,y_m) - K_{u,\eta}\bigl(y_0,\ldots,y_j',\ldots,y_m\bigr)\bigr|$$
$$\lesssim \frac{\omega(\frac{|y_j-y_j'|}{|y_0-y_1|+\cdots+|y_0-y_m|})}{(|y_0-y_1| + \cdots + |y_0-y_m|)^{mn}} + \frac{|y_j-y_j'|}{(|y_0-y_1| + \cdots + |y_0-y_m|)^{mn+1}}$$

and

$$\bigl|K_{v,\eta}(y_0,\ldots,y_j,\ldots,y_m) - K_{v,\eta}\bigl(y_0,\ldots,y_j',\ldots,y_m\bigr)\bigr|$$
$$\lesssim \frac{\omega(\frac{|y_j-y_j'|}{|y_0-y_1|+\cdots+|y_0-y_m|})}{(|y_0-y_1| + \cdots + |y_0-y_m|)^{mn}} + \frac{|y_j-y_j'|}{(|y_0-y_1| + \cdots + |y_0-y_m|)^{mn+1}}$$

whenever $|y_j-y_j'| \le \frac{1}{m+1}\max_{0\le j\le m}|y_0-y_j|$.

Proof We just give the estimate for $K_{u,\eta}$, since $K_{v,\eta}$ can be estimated in a similar way with slight modifications. Without loss of generality, assuming that $j = 0$, we estimate

$$\begin{aligned}
&\big|K_{u,\eta}(y_0, y_1, \dots, y_m) - K_{u,\eta}\big(y_0', y_1, \dots, y_m\big)\big| \\
&\quad = \bigg|K(y_0, y_1, \dots, y_m)u\bigg(\frac{|y_0 - y_1| + \cdots + |y_0 - y_m|}{\eta}\bigg) \\
&\qquad - K\big(y_0', y_1, \dots, y_m\big)u\bigg(\frac{|y_0' - y_1| + \cdots + |y_0' - y_m|}{\eta}\bigg)\bigg| \\
&\quad = \bigg|\big[K(y_0, y_1, \dots, y_m) - K\big(y_0', y_1, \dots, y_m\big)\big]u\bigg(\frac{|y_0' - y_1| + \cdots + |y_0' - y_m|}{\eta}\bigg) \\
&\qquad - K(y_0, y_1, \dots, y_m) \\
&\qquad \times \bigg[u\bigg(\frac{|y_0' - y_1| + \cdots + |y_0' - y_m|}{\eta}\bigg) - u\bigg(\frac{|y_0 - y_1| + \cdots + |y_0 - y_m|}{\eta}\bigg)\bigg]\bigg| \\
&\quad \lesssim \big|K(y_0, y_1, \dots, y_m) - K\big(y_0', y_1, \dots, y_m\big)\big| \\
&\qquad + \bigg|K(y_0, y_1, \dots, y_m) \\
&\qquad \times \bigg[u\bigg(\frac{|y_0' - y_1| + \cdots + |y_0' - y_m|}{\eta}\bigg) - u\bigg(\frac{|y_0 - y_1| + \cdots + |y_0 - y_m|}{\eta}\bigg)\bigg]\bigg| \\
&\quad \doteq I + II.
\end{aligned}$$

Since $|y_0 - y_0'| \le \frac{1}{m+1}\max_{0\le j\le m}|y_0 - y_j|$, by (1.3) we have

$$I \lesssim \frac{1}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{mn}}\omega\bigg(\frac{|y_0 - y_0'|}{|y_0 - y_1| + \cdots + |y_0 - y_m|}\bigg).$$

It remains to estimate II. By the mean value theorem there is t_0 between $\frac{|y_0'-y_1|+\cdots+|y_0'-y_m|}{\eta}$ and $\frac{|y_0-y_1|+\cdots+|y_0-y_m|}{\eta}$ such that

$$\begin{aligned}
&\bigg|u\bigg(\frac{|y_0' - y_1| + \cdots + |y_0' - y_m|}{\eta}\bigg) - u\bigg(\frac{|y_0 - y_1| + \cdots + |y_0 - y_m|}{\eta}\bigg)\bigg| \\
&\quad = \big|u'(t_0)\big|\bigg|\frac{|y_0' - y_1| + \cdots + |y_0' - y_m|}{\eta} - \frac{|y_0 - y_1| + \cdots + |y_0 - y_m|}{\eta}\bigg| \\
&\quad \le \frac{1}{t_0}\frac{||y_0' - y_1| - |y_0 - y_1|| + \cdots + ||y_0' - y_m| - |y_0 - y_m||}{\eta} \\
&\quad \lesssim \frac{1}{t_0}\frac{m|y_0 - y_0'|}{\eta}.
\end{aligned}$$

Again, since $|y_0 - y_0'| \lesssim \frac{1}{m+1}\max_{0\le j\le m}|y_0 - y_j|$, we have

$$\begin{aligned}
\big|y_0' - y_1\big| + \cdots + \big|y_0' - y_m\big| &= \big|y_0 - y_1 + y_0' - y_0\big| + \cdots + \big|y_0 - y_m + y_0' - y_0\big| \\
&\ge |y_0 - y_1| + \cdots + |y_0 - y_m| - m\big|y_0 - y_0'\big|
\end{aligned}$$

$$\geq |y_0 - y_1| + \cdots + |y_0 - y_m| - \frac{m}{m+1} \max_{0\leq j\leq m} |y_0 - y_0'|$$
$$\geq \frac{|y_0 - y_1| + \cdots + |y_0 - y_m|}{m+1}.$$

From this,

$$\frac{1}{t_0} \lesssim \max\left\{ \frac{\eta}{|y_0' - y_1| + \cdots + |y_0' - y_m|}, \frac{\eta}{|y_0 - y_1| + \cdots + |y_0 - y_m|} \right\}$$
$$\lesssim \frac{\eta}{|y_0 - y_1| + \cdots + |y_0 - y_m|},$$

and therefore

$$\left| u\left(\frac{|y_0' - y_1| + \cdots + |y_0' - y_m|}{\eta} \right) - u\left(\frac{|y_0 - y_1| + \cdots + |y_0 - y_m|}{\eta} \right) \right|$$
$$\lesssim \frac{|y_0 - y_0'|}{|y_0 - y_1| + \cdots + |y_0 - y_m|}.$$

This, together with the size condition (1.2), implies that

$$II \lesssim \frac{|y_0 - y_0'|}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{mn+1}}.$$

This ends the proof of Lemma 3.1. □

Lemma 3.2 *Let* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_2}$ *and* $\vec{\omega} \in A_{\vec{p}}$. *Then we have*:

(1) *If* $1 < p_1, \ldots, p_m < \infty$, *then*

$$\| U^* \vec{f} \|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}.$$

(2) *If* $1 \leq p_1, \ldots, p_m < \infty$, *then*

$$\| U^* \vec{f} \|_{L^{p,\infty}(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}.$$

Similar estimates hold for V^*.

Proof Lemma 3.2 is a consequence of Lemma 2.3, Lemma 3.1, and Theorem 1.3 in [3]. □

For the maximal truncated operator T^* and a collection of locally integrable functions $\vec{b} = (b_1, \ldots, b_m)$, we define the commutator $T^*_{\Sigma\vec{b}}$ by

$$T^*_{\Sigma\vec{b}}(f_1, \ldots, f_m) = \sum_{j=1}^m T^{*j}_{\vec{b}}(\vec{f}),$$

where

$$T^{*j}_{\vec{b}}(\vec{f})(x) = \left[b_j, T^*\right]_j (\vec{f})(x) = \sup_{\delta>0} \left| b_j(x) T_\delta(f_1, \ldots, f_m)(x) - T_\delta(f_1, \ldots, b_j f_j, \ldots, f_m)(x) \right|.$$

Next, we give the key lemma, which plays important role in the proof of Theorem 1.1. We just consider the case $m = 2$ for simplicity.

Lemma 3.3 *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Then we have*:

(i) *If $b_1 \in \mathrm{Lip}_{\beta_1}$ and $b_2 \in \mathrm{Lip}_{\beta_2}$ with $0 < \beta_1, \beta_2 < 1$, $0 < \delta < \epsilon < 1/2$, then*

$$\begin{aligned} &M_\delta^\sharp T^*_{\Pi\vec{b}}(f_1,f_2)(x) \\ &\quad \lesssim \Biggl\{ \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_{\beta_i}} M_{\epsilon,\beta}\bigl(T^*(f_1,f_2)\bigr)(x) + \|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{\epsilon,\beta_1}\bigl(T^{*2}_{\vec{b}}(f_1,f_2)\bigr)(x) \\ &\qquad + \|b_2\|_{\mathrm{Lip}_{\beta_1}} M_{\epsilon,\beta_2}\bigl(T^{*1}_{\vec{b}}(f_1,f_2)\bigr)(x) \\ &\qquad + \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_{\beta_i}} M_{1,\beta_1}(f_1)(x) M_{1,\beta_2}(f_2)(x) \Biggr\}. \end{aligned} \tag{3.1}$$

(ii) *Suppose that $b_j \in \mathrm{Lip}_\beta$, $j = 1,2$, $0 < \beta < 1$, and $0 < \delta < \epsilon < 1/2 < 1 < n/\beta$. Then*

$$\begin{aligned} &M_\delta^\sharp T^*_{\Sigma\vec{b}}(f_1,f_2)(x) \\ &\quad \lesssim \|b\|_{\mathrm{Lip}_\beta} \bigl\{ M_{\epsilon,\beta}\bigl(T^*(f_1,f_2)\bigr)(x) + M_{1,\beta}(f_1)(x) M(f_2)(x) \\ &\qquad + M_{1,\beta}(f_2)(x) M(f_1)(x) \bigr\}. \end{aligned} \tag{3.2}$$

Proof (i) We need two auxiliary maximal operators. The key role in the proof is played by the maximal operators $U^*_{\Pi b}$ and $V^*_{\Pi b}$ defined by

$$\begin{aligned} U^*_{\Pi b}(\vec{f})(x) &= \sup_{\eta>0} \bigl| \bigl[b_1, [b_2, U_\eta]_2\bigr]_1 (\vec{f})(x) \bigr| \\ &= \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x,y_1,y_2) \prod_{j=1}^{2} \bigl(b_j(x) - b_j(y_j)\bigr) \prod_{i=1}^{2} f_i(y_i)\, dy_1\, dy_2 \right|, \\ V^*_{\Pi b}(\vec{f})(x)) &= \sup_{\eta>0} \bigl| \bigl[b_1, [b_2, V_\eta]_2\bigr]_1 (\vec{f})(x) \bigr| \\ &= \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^2} K_{v,\eta}(x,y_1,y_2) \prod_{j=1}^{2} b_j(x) - b_j(y_j)) \prod_{i=1}^{2} f_i(y_i)\, dy_1\, dy_2 \right|. \end{aligned}$$

It is easy to get that $T^*_{\Pi b}(\vec{f}) \le U^*_{\Pi b}(\vec{f})(x) + V^*_{\Pi b}(\vec{f})(x)$. We need to prove (3.1) for $U^*_{\Pi\vec{b}}$ and $V^*_{\Pi\vec{b}}$. We just give the proof for $U^*_{\Pi\vec{b}}$, since the proof for $V^*_{\Pi\vec{b}}$ is almost the same. Fix $x \in \mathbb{R}^n$ and denote by $Q = Q(x_Q, l)$ the cube centered at x_Q and containing x with side length l. Denote $c = \sup_{\eta>0} |c_\eta|$ and $\lambda_i = (b_i)_{Q^*} = \frac{1}{|Q^*|} \int_{Q^*} b_i(y)\, dy$, where $Q^* = 8\sqrt{n}Q$. For any $z \in \mathbb{R}^n$, we have

$$\begin{aligned} \bigl| U^*_{\Pi\vec{b}}(f_1,f_2)(z) - c \bigr| &\le \bigl| \bigl(b_1(z) - \lambda_1\bigr)\bigl(b_2(z) - \lambda_2\bigr) U^*(f_1,f_2)(z) \bigr| \\ &\quad + \sup_\eta \bigl| \bigl(b_1(z) - \lambda_1\bigr) [b_2, U_\eta]_2 (f_1,f_2)(z) \bigr| \end{aligned}$$

$$
\begin{aligned}
&+ \sup_{\eta} \bigl| \bigl(b_2(z)-\lambda_2\bigr)[b_1, U_\eta]_1(f_1,f_2)(z) \bigr| \\
&+ \Bigl| U^*\bigl((b_1-\lambda_1)f_1, (b_2-\lambda_2)f_2\bigr)(z) - \sup_{\eta>0} |c_\eta| \Bigr|.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left(\frac{1}{|Q|} \int_Q \bigl| \bigl| U^*_{\Pi \vec{b}}(f_1,f_2)(z) \bigr|^\delta - |c|^\delta \bigr| \, dz \right)^{\frac{1}{\delta}} \\
&\quad \leq \left(\frac{1}{|Q|} \int_Q \Bigl| U^*_{\Pi \vec{b}}(f_1,f_2)(z) - \sup_{\eta>0} |c_\eta| \Bigr|^\delta \, dz \right)^{\frac{1}{\delta}} \\
&\quad \leq \left(\frac{1}{|Q|} \int_Q \bigl| \bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr) U^*(f_1,f_2)(z) \bigr|^\delta \, dz \right)^{\frac{1}{\delta}} \\
&\qquad + \left(\frac{1}{|Q|} \int_Q \bigl| \bigl(b_1(z)-\lambda_1\bigr)\bigl[b_2, U^*\bigr]_2(f_1,f_2)(z) \bigr|^\delta \, dz \right)^{\frac{1}{\delta}} \\
&\qquad + \left(\frac{1}{|Q|} \int_Q \bigl| \bigl(b_2(z)-\lambda_2\bigr)\bigl[b_1, U^*\bigr]_1(f_1,f_2)(z) \bigr|^\delta \, dz \right)^{\frac{1}{\delta}} \\
&\qquad + \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} \bigl| U_\eta\bigl((b_1-\lambda_1)f_1, (b_2-\lambda_2)f_2\bigr)(z) - c_\eta \bigr|^\delta \, dz \right)^{\frac{1}{\delta}} \\
&\quad \doteq T_1 + T_2 + T_3 + T_4.
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
T_1 &\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_{\beta_i}} \left(\frac{1}{|Q|^{1-\frac{\delta\beta}{n}}} \int_Q \bigl| U^*(f_1,f_2)(z) \bigr|^\delta \, dz \right)^{\frac{1}{\delta}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_{\beta_i}} M_{\epsilon,\beta}\bigl(U^*(f_1,f_2)\bigr)(x).
\end{aligned}
$$

In a similar way, we can prove that

$$
T_2 + T_3 \lesssim \|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{\epsilon,\beta_1}\bigl(\bigl[b_2, U^*\bigr]_2(f_1,f_2)\bigr)(x) + \|b_2\|_{\mathrm{Lip}_{\beta_2}} M_{\epsilon,\beta_2}\bigl(\bigl[b_1, U^*\bigr]_1(f_1,f_2)\bigr)(x).
$$

It remains to estimate the last term T_4. Take now $c_\eta = U_\eta((b_1-\lambda_1)f_1^\infty, (b_2-\lambda_2)f_2^\infty)(x)$. Then $T_4 \leq T_{41} + T_{42} + T_{43} + T_{44}$, where

$$
\begin{aligned}
T_{41} &= \left(\frac{1}{|Q|} \int_Q \bigl| U^*\bigl((b_1-\lambda_1)f_1^0, (b_2-\lambda_2)f_2^0\bigr)(z) \bigr|^\delta \, dx \right)^{\frac{1}{\delta}}; \\
T_{42} &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta} \bigl| U_\eta\bigl((b_1-\lambda_1)f_1^0, (b_2-\lambda_2)f_2^\infty\bigr)(z) \bigr|^\delta \, dz \right)^{\frac{1}{\delta}}; \\
T_{43} &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta} \bigl| U_\eta\bigl((b_1-\lambda_1)f_1^\infty, (b_2-\lambda_2)f_2^0\bigr)(z) \bigr|^\delta \, dz \right)^{\frac{1}{\delta}};
\end{aligned}
$$

$$T_{44} = \left(\frac{1}{|Q|} \int_Q \sup_\eta \bigl| U_\eta\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty\bigr)(z) \right.$$
$$\left. - U_\eta\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty\bigr)(x) \bigr|^\delta \, dz \right)^{\frac{1}{\delta}}.$$

By the Kolmogorov inequality and by Lemma 3.2,

$$\begin{aligned} T_{41} &\lesssim \bigl\| U^*\bigl((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^0\bigr) \bigr\|_{L^{1/2,\infty}(Q, \frac{dx}{|Q|})} \\ &\lesssim \frac{1}{|Q|} \int_Q \bigl| (b_1 - \lambda_1)f_1^0(z) \bigr| \, dz \frac{1}{|Q|} \int_Q \bigl| (b_2 - \lambda_2)f_2^0(z) \bigr| \, dz \\ &\lesssim \prod_{i=1}^2 \|b_i\|_{\mathrm{Lip}_{\beta_i}} M_{1,\beta_i}(f_i)(x). \end{aligned}$$

Next, by Hölder's inequality and by the size condition (1.2),

$$\begin{aligned} T_{42} &\le \frac{1}{|Q|} \int_Q \sup_\eta \bigl| U_\eta\bigl((b_1 - \lambda_1)f_1^0, (b_2 - \lambda_2)f_2^\infty\bigr)(z) \bigr| \, dz \\ &\lesssim \frac{1}{|Q|} \int_Q \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \frac{|(b_1(y_1) - \lambda_1)f_1^0(y_1)||(b_2(y_2) - \lambda_2)f_2^\infty(y_2)| \, dy_1 \, dy_2}{(|z - y_1| + |z - y_2|)^{2n}} \, dz \\ &\lesssim \|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{1,\beta_1}(f_1)(x)|Q| \sum_{k=1}^\infty \int_{2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q} \frac{|f_2(y_2)(b_2(y_2) - \lambda_2)| \, dy_2}{|y_2 - x_Q|^{2n}} \\ &\lesssim \|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{1,\beta_1}(f_1)(x) \|b_2\|_{\mathrm{Lip}_{\beta_2}} M_{1,\beta_2}(f_2)(x). \end{aligned}$$

The operator T_{43} can be estimated in the same way. Finally, we estimate T_{44}. By Lemma 3.1 we have

$$\begin{aligned} T_{44} &\lesssim \frac{1}{|Q|} \int_Q \sup_\eta \bigl| U_\eta\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty\bigr)(z) \\ &\quad - U_\eta\bigl((b_1 - \lambda_1)f_1^\infty, (b_2 - \lambda_2)f_2^\infty\bigr)(x) \bigr| \, dz \\ &\lesssim \frac{1}{|Q|} \int_Q \int_{(\mathbb{R}^n \setminus Q^*)^2} \sup_\eta \bigl| K_{\mu,\eta}(z, \vec{y}) - K_{\mu,\eta}(x_Q, \vec{y}) \bigr| \prod_{i=1}^2 \bigl| (b_i(y_i) - \lambda_i)f_i^\infty(y_i) \bigr| \, dy_1 \, dy_2 \, dz \\ &\lesssim \frac{1}{|Q|} \int_Q \int_{(\mathbb{R}^n \setminus Q^*)^2} \frac{1}{(|x_Q - y_1| + |x_Q - y_2|)^{2n}} \omega\left(\frac{|z - x_Q|}{|x_Q - y_1| + |x_Q - y_2|} \right) \\ &\quad \times \prod_{i=1}^2 \bigl| (b_i(y_i) - \lambda_i)f_i^\infty(y_i) \bigr| \, dy_1 \, dy_2 \, dz \\ &\quad + \frac{1}{|Q|} \int_Q \int_{(\mathbb{R}^n \setminus Q^*)^2} \frac{|z - x_Q|}{(|x_Q - y_1| + |x_Q - y_2|)^{2n+1}} \prod_{i=1}^2 \bigl| (b_i(y_i) - \lambda_i)f_i^\infty(y_i) \bigr| \, dy_1 \, dy_2 \, dz \\ &\lesssim \frac{1}{|Q|} \int_Q \sum_{k=1}^\infty \int_{(2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q)^2} \frac{1}{(|2^{k+3}\sqrt{n}Q|)^2} \omega\bigl(2^{-k}\bigr) \\ &\quad \times \prod_{i=1}^2 \bigl| (b_i(y_i) - \lambda_i)f_i^\infty(y_i) \bigr| \, dy_1 \, dy_2 \, dz \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus Q^*}\frac{|z-x_Q|^{1/2}}{|x_Q-y_1|^{n+1/2}}\left|\big(b_1(y_1)-\lambda_1\big)f_1^\infty(y_1)\right|dy_1\\
&\times\int_{\mathbb{R}^n\setminus Q^*}\frac{|z-x_Q|^{1/2}}{|x_Q-y_2|^{n+1/2}}\left|\big(b_2(y_2)-\lambda_2\big)f_2^\infty(y_2)\right|dy_2\,dz\\
\lesssim{}&\sum_{k=1}^\infty\frac{1}{(|2^{k+3}\sqrt{n}Q|)^2}\int_{(2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q)^2}\omega\big(2^{-k}\big)\prod_{i=1}^2\left|\big(b_i(y_i)-\lambda_i\big)f_i^\infty(y_i)\right|d\vec{y}\\
&+\sum_{k=1}^\infty 2^{-\frac{k}{2}}\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\left|\big(b_1(y_1)-\lambda_1\big)f_1^\infty(y_1)\right|dy_1\\
&\times\sum_{k=1}^\infty 2^{-\frac{k}{2}}\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\left|\big(b_2(y_2)-\lambda_2\big)f_2^\infty(y_2)\right|dy_2\\
\lesssim{}&\|b_1\|_{\mathrm{Lip}_{\beta_1}}M_{1,\beta_1}(f_1)(x)\|b_2\|_{\mathrm{Lip}_{\beta_2}}M_{1,\beta_2}(f_2)(x).
\end{aligned}
$$

Combining the obtained estimates proves (3.1).

(ii) It is sufficient to prove (3.2) for the operator with only one symbol. Set

$$
\begin{aligned}
U_{\vec{b}}^{*1}(\vec{f})(x)&=\sup_{\eta>0}\left|b(x)U_\eta(f_1,f_2)(x)-U_\eta(bf_1,f_2)(x)\right|\\
&=\sup_{\eta>0}\left|\big(b(x)-\lambda\big)U_\eta(f_1,f_2)(x)-U_\eta\big((b-\lambda)f_1,f_2\big)(x)\right|,
\end{aligned}
$$

where $\lambda=b_{Q^*}=\frac{1}{|Q^*|}\int_{Q^*}b(y)\,dy$. Let $c=\sup_{\eta>0}|c_\eta|$. Then

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\left|\left|U_{\vec{b}}^{*1}(f_1,f_2)(z)\right|^\delta-|c|^\delta\right|dz\right)^{\frac{1}{\delta}}\\
&\quad\lesssim\left(\frac{1}{|Q|}\int_Q\left|U_{\vec{b}}^{*1}(f_1,f_2)(z)-\sup_{\eta>0}|c_\eta|\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad\lesssim\left(\frac{1}{|Q|}\int_Q\left|\big(b(z)-\lambda\big)U^*(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\qquad+\left(\frac{1}{|Q|}\int_Q\sup_{\eta>0}\left|U_\eta\big((b-\lambda)f_1,f_2\big)(z)-c_\eta\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\quad=:(P_1+P_2).
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
P_1&\lesssim\|b\|_{\mathrm{Lip}_\beta}\left(\frac{1}{|Q|^{1-\frac{\delta\beta}{n}}}\int_Q\left|U^*(f_1,f_2)(z)\right|^\delta dz\right)^{\frac{1}{\delta}}\\
&\lesssim\|b\|_{\mathrm{Lip}_\beta}M_{\epsilon,\beta}\big(U^*(f_1,f_2)\big)(x).
\end{aligned}
$$

Set $c_\eta=U_\eta((b-\lambda)f_1^\infty,f_2^\infty)(x)$. Then $P_2\le P_{21}+P_{22}+P_{23}+P_{24}$, where

$$
P_{21}=\left(\frac{1}{|Q|}\int_Q\left|U^*\big((b-\lambda)f_1^0,f_2^0\big)(z)\right|^\delta dx\right)^{\frac{1}{\delta}};
$$

$$P_{22} = \left(\frac{1}{|Q|}\int_Q \sup_\eta \bigl|U_\eta\bigl((b-\lambda)f_1^0,f_2^\infty\bigr)(z)\bigr|^\delta\,dz\right)^{\frac{1}{\delta}};$$

$$P_{23} = \left(\frac{1}{|Q|}\int_Q \sup_\eta \bigl|U_\eta\bigl((b-\lambda)f_1^\infty,f_2^0\bigr)(z)\bigr|^\delta\,dz\right)^{\frac{1}{\delta}};$$

$$P_{24} = \left(\frac{1}{|Q|}\int_Q \sup_\eta \bigl|U_\eta\bigl((b-\lambda)f_1^\infty,f_2^\infty\bigr)(z) - U_\eta\bigl((b-\lambda)f_1^\infty,f_2^\infty\bigr)(x)\bigr|^\delta\,dz\right)^{\frac{1}{\delta}}.$$

By the Kolmogorov inequality and by Lemma 3.2,

$$\begin{aligned}
P_{21} &\lesssim \bigl\|U^*\bigl((b-\lambda)f_1^0,f_2^0\bigr)\bigr\|_{L^{1/2,\infty}(Q,\frac{dx}{|Q|})}\\
&\lesssim \frac{1}{|Q|}\int_Q \bigl|(b-\lambda)f_1^0(z)\bigr|\,dz \frac{1}{|Q|}\int_Q \bigl|f_2^0(z)\bigr|\,dz\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}\bigl|Q^*\bigr|^{\beta/n}\frac{1}{|Q|}\int_Q \bigl|f_1^0(z)\bigr|\,dz \frac{1}{|Q|}\int_Q \bigl|f_2^0(z)\bigr|\,dz\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} M_{1,\beta}(f_1)(x)M(f_2)(x).
\end{aligned}$$

Next, by the size condition (1.2),

$$\begin{aligned}
P_{22} &\lesssim \frac{1}{|Q|}\int_Q \sup_\eta \bigl|U_\eta\bigl((b-\lambda)f_1^0,f_2^\infty\bigr)(z)\bigr|\,dz\\
&\lesssim \frac{1}{|Q|}\int_Q\int_{Q^*}\int_{(Q^*)^c} \frac{1}{(|z-y_1|+|z-y_2|)^{2n}}\bigl|\bigl(b(y_1)-\lambda\bigr)f_1(y_1)\bigr|\bigl|f_2(y_2)\bigr|\,dy_2\,dy_1\,dz\\
&\lesssim \int_{Q^*}\bigl|\bigl(b(y_1)-\lambda\bigr)f_1(y_1)\bigr|\,dy_1 \int_{\mathbb{R}^n\setminus Q^*}\frac{|f_2(y_2)|\,dy_2}{|x_Q-y_2|^{2n}}\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}|Q|^{\frac{\beta}{n}}\int_{Q^*} f_1(y_1)\,dy_1 \sum_{k=1}^\infty \frac{1}{|2^{k+1}Q|^2}\int_{2^{k+1}Q^*\setminus 2^kQ^*}\bigl|f_2(y_2)\bigr|\,dy_2\\
&\lesssim \|b\|_{\mathrm{Lip}_\beta} M_{1,\beta}(f_1)(x)M(f_2)(x).
\end{aligned}$$

Similarly,

$$P_{23} \lesssim \|b\|_{\mathrm{Lip}_\beta} M_{1,\beta}(f_1)(x)M(f_2)(x).$$

By Lemma 3.1 we obtain

$$\begin{aligned}
P_{24} &\lesssim \frac{1}{|Q|}\int_Q \sup_\eta \bigl|U_\eta\bigl((b-\lambda)f_1^\infty,f_2^\infty\bigr)(z) - U_\eta\bigl((b-\lambda)f_1^\infty,f_2^\infty\bigr)(x)\bigr|\,dz\\
&\lesssim \frac{1}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2} \sup_\eta \bigl|K_{\mu,\eta}(z,\vec{y}) - K_{\mu,\eta}(x_Q,\vec{y})\bigr|\bigl|\bigl(b(y_1)-\lambda\bigr)\bigr|\Bigl|\prod_{i=1}^2 \bigl|f_i^\infty(y_i)\bigr|\,dy_1\,dy_2\,dz\\
&\lesssim \frac{1}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2} \frac{\omega(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|})}{(|z-y_1|+|z-y_2|)^{2n}}\bigl|\bigl(b(y_1)-\lambda\bigr)\bigr|\Bigl|\prod_{i=1}^2 \bigl|f_i^\infty(y_i)\bigr|\,dy_1\,dy_2\,dz\\
&\quad + \frac{1}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2} \frac{|z-x_Q|}{(|z-y_1|+|z-y_2|)^{2n+1}}\bigl|\bigl(b(y_1)-\lambda\bigr)\bigr|\Bigl|\prod_{i=1}^2 \bigl|f_i^\infty(y_i)\bigr|\,dy_1\,dy_2\,dz
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{(2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q)^2}\frac{1}{(|2^{k+3}\sqrt{n}Q|)^2}\omega(2^{-k})\big|\big(b(y_1)-\lambda\big)\big| \\
&\quad \times \prod_{i=1}^{2}\big|f_i^{\infty}(y_i)\big|\,dy_1\,dy_2\,dz + \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus Q^*}\frac{|z-x_Q|^{1/2}}{|x_Q-y_1|^{n+1/2}}\big|\big(b(y_1)-\lambda\big)f_1^{\infty}(y_1)\big|\,dy_1 \\
&\quad \times \int_{\mathbb{R}^n\setminus Q^*}\frac{|z-x_Q|^{1/2}}{|x_Q-y_2|^{n+1/2}}\big|f_2^{\infty}(y_2)\big|\,dy_2\,dz \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{\infty}\frac{\omega(2^{-k})}{(|2^{k+3}\sqrt{n}Q|)^{1-\beta/n}}\int_{2^{k+3}\sqrt{n}Q}\big|f_1^{\infty}(y_1)\big|\,dy_1\frac{1}{|2^kQ^*|}\int_{2^{k+3}\sqrt{n}Q}\big|f_2^{\infty}(y_2)\big|\,dy_2 \\
&\quad + \sum_{k=1}^{\infty}2^{-\frac{k}{2}}\frac{1}{|2^{k+3}\sqrt{n}Q|^{1-\beta/n}}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\big|f_1^{\infty}(y_1)\big|\,dy_1 \\
&\quad \times \sum_{k=1}^{\infty}2^{-\frac{k}{2}}\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\big|f_2^{\infty}(y_2)\big|\,dy_2 \\
&\lesssim \|b\|_{\mathrm{Lip}_\beta}M_{1,\beta}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

Thus we finish the proof of (3.2). Then Lemma 3.3 is proved. □

4 Proofs of Theorems 1.2 and 1.3

The main ideas in this section are from [17] and [19]. We should also mention that the proof of this part is similar to that of Theorem 1.2 and Theorem 1.3 in [22]; we just give the different part of the proof.

We begin with the proof of Theorem 1.2.

Proof For any cube Q centered at x_Q, the theorem will be proved if we can show that

$$
\sup_Q \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\big|U^*_{\Pi\vec{b}}(\vec{f})(z)-\big(U^*_{\Pi\vec{b}}(\vec{f})\big)_Q\big|\,dz \lesssim \|b_1\|_{\dot{\Lambda}_{\beta_1}}\|b_2\|_{\dot{\Lambda}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}. \quad (4.1)
$$

Set $c = c_1 + c_2 + c_3$, which will be determined later. We estimate

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\big|U^*_{\Pi\vec{b}}(\vec{f})(z)-\big(U^*_{\Pi\vec{b}}(\vec{f})\big)_Q\big|\,dz \\
&\quad\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\big|U^*_{\Pi\vec{b}}(f_1,f_2)(z)-c\big|\,dz \\
&\quad\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\big|U^*_{\Pi\vec{b}}(f_1^0,f_2^0)(z)\big|\,dz \\
&\qquad + \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\big|U^*_{\Pi\vec{b}}(f_1^0,f_2^{\infty})(z)-c_1\big|\,dz \\
&\qquad + \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\big|U^*_{\Pi\vec{b}}(f_1^{\infty},f_2^0)(z)-c_2\big|\,dz \\
&\qquad + \frac{1}{|Q|^{1+\beta/n-1/p}}\int_Q\big|U^*_{\Pi\vec{b}}(f_1^{\infty},f_2^{\infty})(z)-c_3\big|\,dz \\
&\quad\doteq M_1+M_2+M_3+M_4.
\end{aligned}
$$

We estimate these terms separately. For the first term, we can choose $1 < q, q_j < \infty$, $q_j < n/\beta_j < p_j$, $j = 1, 2$, with $1/q = 1/q_1 + 1/q_2 - (\beta_1 + \beta_2)/n$. By Hölder's inequality and by Theorem 1.1 we have

$$\begin{aligned} M_1 &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \biggl(\int_Q \bigl| U^*_{\Pi \vec{b}}(f_1^0, f_2^0)(z) \bigr|^q \,dz \biggr)^{1/q} |Q|^{1-1/q} \\ &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} |Q|^{1-1/q} \bigl\| f_1^0 \bigr\|_{L^{q_1}} \bigl\| f_2^0 \bigr\|_{L^{q_2}} \\ &\lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \end{aligned}$$

To get M_2, we take $c_1 = T((b_1 - \lambda_1)f_1^0, f_2^\infty)(x_Q)$. Then

$$\begin{aligned} M_2 &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \biggl| \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \bigl(b_1(z) - \lambda_1\bigr)\bigl(b_2(z) - \lambda_2\bigr) \\ &\quad \times K_{\mu,\eta}(z, y_1, y_2) f_1(y_1) f_2(y_2) \,dy_1 \,dy_2 \biggr| \,dz \\ &\quad + \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \biggl| \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \bigl(b_1(z) - \lambda_1\bigr)\bigl(b_2(y_2) - \lambda_2\bigr) \\ &\quad \times K_{\mu,\eta}(z, y_1, y_2) f_1(y_1) f_2(y_2) \,dy_1 \,dy_2 \biggr| \,dz \\ &\quad + \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \biggl| \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \bigl(b_1(y_1) - \lambda_1\bigr)\bigl(b_2(z) - \lambda_2\bigr) \\ &\quad \times K_{\mu,\eta}(z, y_1, y_2) f_1(y_1) f_2(y_2) \,dy_1 \,dy_2 \biggr| \,dz \\ &\quad + \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \biggl| \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \bigl(b_1(y_1) - \lambda_1\bigr)\bigl(b_2(y_2) - \lambda_2\bigr) \\ &\quad \times \bigl[K_{\mu,\eta}(z, y_1, y_2) - K_{\mu,\eta}(x_Q, y_1, y_2)\bigr] f_1(y_1) f_2(y_2) \,dy_1 \,dy_2 \biggr| \,dz \\ &\doteq M_{21} + M_{22} + M_{23} + M_{24}. \end{aligned}$$

Using the size condition (1.2) and the estimate in [22, p. 5013], we have

$$\begin{aligned} M_{21} &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \bigl| \bigl(b_1(z) - \lambda_1\bigr)\bigl(b_2(z) - \lambda_2\bigr) \bigr| \\ &\quad \times \frac{1}{(|z - y_1| + |z - y_2|)^{2n}} \bigl| f_1(y_1) f_2(y_2) \bigr| \,dy_1 \,dy_2 \,dz \\ &\lesssim \|b_1\|_{\dot{\Lambda}_{\beta_1}} \|b_2\|_{\dot{\Lambda}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \end{aligned}$$

In a similar way, we get $M_{23} + M_{22} \lesssim \|b_1\|_{\dot{\Lambda}_{\beta_1}} \|b_2\|_{\dot{\Lambda}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$.
By Minkowski's inequality and by Lemma 3.1,

$$\begin{aligned} M_{24} &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \bigl| \bigl(b_1(y_1) - \lambda_1\bigr)\bigl(b_2(y_2) - \lambda_2\bigr) \bigr| \\ &\quad \times \bigl| K_{\mu,\eta}(z, y_1, y_2) - K_{\mu,\eta}(x_Q, y_1, y_2) \bigr| \bigl| f_1(y_1) f_2(y_2) \bigr| \,dy_1 \,dy_2 \,dz \end{aligned}$$

$$
\begin{aligned}
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \bigl|\bigl(b_1(y_1)-\lambda_1\bigr)\bigl(b_2(y_2)-\lambda_2\bigr)\bigr| \\
&\quad \times \left(\frac{\omega(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|})}{(|z-y_1|+|z-y_2|)^{2n}} + \frac{|z-x_Q|}{(|x-y_1|+|x-y_2|)^{2n+1}} \right) \bigl|f_1(y_1)f_2(y_2)\bigr| \, dy_1 \, dy_2 \, dz \\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \frac{1}{|Q|^{1+\beta_2/n-1/p}} \\
&\quad \times \int_Q \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \left(\frac{\omega(\frac{|z-x_Q|}{|x_Q-y_2|})}{(|z-y_1|+|z-y_2|)^{2n-\beta_2}} + \frac{2^{-k}}{(|z-y_1|+|z-y_2|)^{2n-\beta_2}} \right) \\
&\quad \times \bigl|f_1(y_1)f_2(y_2)\bigr| \, dy_1 \, dy_2 \, dz \\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \frac{1}{|Q|^{1+\beta_2/n-1/p}} \\
&\quad \times \int_Q \int_{Q^*} \int_{\mathbb{R}^n \setminus Q^*} \frac{\omega(2^{-k})+2^{-k}}{(|z-y_1|+|z-y_2|)^{2n-\beta_2}} \bigl|f_1(y_1)f_2(y_2)\bigr| \, dy_1 \, dy_2 \, dz \\
&\lesssim \frac{\|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{\beta_2/n-1/p}} \int_{Q^*} \bigl|f_1(y_1)\bigr| \, dy_1 \sum_{k=1}^{\infty} \frac{\omega(2^{-k})+2^{-k}}{|2^{k+3}\sqrt{n}Q|^{2-\beta_2/n}} \\
&\quad \times \int_{2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q} \bigl|f_2(y_2)\bigr| \, dy_2 \\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{k=1}^{\infty} \bigl(\omega\bigl(2^{-k}\bigr)+2^{-k}\bigr) 2^{-kn(1-\beta_2/n+1/p_2)} \\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}},
\end{aligned}
$$

where we have used assumption (1.5) and the inequality $1-\beta_2/n+1/p_2>0$.

Thus $M_2 \lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$. Similarly, $M_3 \lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$.

We deal with M_4 as follows:

$$
\begin{aligned}
M_4 &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_{\eta} \left| \int_{(\mathbb{R}^n \setminus Q^*)^2} \bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr) \right. \\
&\quad \left. \times K_{\mu,\eta}(z,y_1,y_2) f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \right| dz \\
&\quad + \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_{\eta} \left| \int_{(\mathbb{R}^n \setminus Q^*)^2} \bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(y_2)-\lambda_2\bigr) \right. \\
&\quad \left. \times \bigl[K_{\mu,\eta}(z,y_1,y_2) - K_{\mu,\eta}(x_Q,y_1,y_2)\bigr] f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \right| dz \\
&\quad + \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_{\eta} \left| \int_{(\mathbb{R}^n \setminus Q^*)^2} \bigl(b_1(y_1)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr) \right. \\
&\quad \left. \times \bigl[K_{\mu,\eta}(z,y_1,y_2) - K_{\mu,\eta}(x_Q,y_1,y_2)\bigr] f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \right| dz \\
&\quad + \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_{\eta} \left| \int_{(\mathbb{R}^n \setminus Q^*)^2} \bigl(b_1(y_1)-\lambda_1\bigr)\bigl(b_2(y_2)-\lambda_2\bigr) \right. \\
&\quad \left. \times \bigl[K_{\mu,\eta}(z,y_1,y_2) - K_{\mu,\eta}(x_Q,y_1,y_2)\bigr] f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \right| dz \\
&\doteq M_{41} + M_{42} + M_{43} + M_{44}.
\end{aligned}
$$

By Minkowski's inequality and by the size condition (1.2),

$$\begin{aligned}
M_{41} &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \int_{(\mathbb{R}^n\setminus Q^*)^2} \bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr)\bigr| \\
&\quad \times \bigl|K_{\mu,\eta}(z,y_1,y_2)\bigr|\bigl|f_1(y_1)f_2(y_2)\bigr|\,dy_1\,dy_2\,dz \\
&\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \int_{(\mathbb{R}^n\setminus Q^*)^2} \bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr)\bigr| \\
&\quad \times \frac{1}{(|z-y_1|+|z-y_2|)^{2n}} \bigl|f_1(y_1)f_2(y_2)\bigr|\,dy_1\,dy_2\,dz \\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
\end{aligned}$$

Using Minkowski's inequality along with Lemma 3.1, we obtain

$$\begin{aligned}
M_{42} &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \int_{(\mathbb{R}^n\setminus Q^*)^2} \bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(y_2)-\lambda_2\bigr)\bigr| \\
&\quad \times \bigl|K_{\mu,\eta}(z,y_1,y_2)-K_{\mu,\eta}(x_Q,y_1,y_2)\bigr|\bigl|f_1(y_1)f_2(y_2)\bigr|\,dy_1\,dy_2\,dz \\
&\lesssim \frac{\|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{\beta_2/n-1/p}} \sum_{k=1}^\infty \int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q} \frac{|f_1(y_1)|}{|y_1-x_Q|^n}\,dy_1 \\
&\quad \times \sum_{k=1}^\infty \int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q} \frac{|f_2(y_2)|(\omega(2^{-k})+\frac{|z-x_Q|}{|x_Q-y_1|+|x_Q-y_2|})}{|y_2-x_Q|^{n-\beta_2}}\,dy_2 \\
&\lesssim \frac{\|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{\beta_2/n-1/p}} \sum_{k=1}^\infty \frac{1}{|2^{k+3}\sqrt{n}Q|} \int_{2^{k+3}\sqrt{n}Q} f_1(y_1)\,dy_1 \\
&\quad \times \sum_{k=1}^\infty \bigl(\omega\bigl(2^{-k}\bigr)+2^{-k}\bigr) \frac{1}{|2^{k+3}\sqrt{n}Q|^{1-\beta_2/n}} \int_{2^{k+3}\sqrt{n}Q} f_2(y_2)\,dy_2 \\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{k=1}^\infty \bigl(\omega\bigl(2^{-k}\bigr)+2^{-k}\bigr)2^{kn(\beta_2/n-1/p)} \\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}},
\end{aligned}$$

where we have used assumption (1.5) and the inequality $0 < \beta - n/p < 1$.

Similarly, $M_{43} \lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$.

Now we estimate M_{44}:

$$\begin{aligned}
M_{44} &\lesssim \frac{1}{|Q|^{1+\beta/n-1/p}} \int_Q \sup_\eta \int_{(\mathbb{R}^n\setminus Q^*)^2} \bigl|\bigl(b_1(y_1)-\lambda_1\bigr)\bigl(b_2(y_2)-\lambda_2\bigr)\bigr| \\
&\quad \times \bigl|K_{\mu,\eta}(z,y_1,y_2)-K_{\mu,\eta}(x_Q,y_1,y_2)\bigr|\bigl|f_1(y_1)f_2(y_2)\bigr|\,dy_1\,dy_2\,dz \\
&\lesssim \frac{\|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}}}{|Q|^{1+\beta/n-1/p}} \int_Q \sum_{k=1}^\infty \int_{(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2} \frac{|f_1(y_1)|}{|y_2-x_Q|^{2n-\beta_1-\beta_2}} \\
&\quad \times \biggl(\omega\biggl(\frac{|z-x_Q|}{|y_2-x_Q|}\biggr)+\frac{|z-x_Q|}{|x_Q-y_1|+|x_Q-y_2|}\biggr)\,dy_1\,dy_2\,dz
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\sum_{k=1}^{\infty}\bigl(\omega\bigl(2^{-k}\bigr)+2^{-k}\bigr)2^{kn(\beta/n-1/p)}\\
&\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}$$

Putting the estimates for M_1, M_2, M_3, M_4 together, we get (4.1). Thus the proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3.

Proof We use the same notations as in previous sections. Then we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q\bigl|U^*_{\Pi\vec{b}}(\vec{f})(z)-\bigl(U^*_{\Pi\vec{b}}(\vec{f})\bigr)_Q\bigr|\,dz\\
&\quad\lesssim \frac{1}{|Q|^{1+\beta/n}}\int_Q\bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigl(b_2(z)-\lambda_2\bigr)U^*(f_1,f_2)(z)\bigr|\,dz\\
&\qquad+\frac{1}{|Q|^{1+\beta/n}}\int_Q\bigl|\bigl(b_2(z)-\lambda_2\bigr)U^{*,1}_{\vec{b}}(f_1,f_2)(z)-c_1\bigr|\,dz\\
&\qquad+\frac{1}{|Q|^{1+\beta/n}}\int_Q\bigl|\bigl(b_1(z)-\lambda_1\bigr)U^{*,2}_{\vec{b}}(f_1,f_2)(z)-c_2\bigr|\,dz\\
&\qquad+\frac{1}{|Q|^{1+\beta/n}}\int_Q\bigl|U^*\bigl((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2\bigr)(z)-c_3\bigr|\,dz\\
&\quad\doteq N_1+N_2+N_3+N_4.
\end{aligned}$$

For $1<r<p$, by the Hölder inequality, we have

$$N_1\lesssim \|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}}M_r\bigl(U^*(f_1,f_2)\bigr)(x).$$

In what follows, we just give an estimate for N_2, since N_3 and N_4 can be estimated in a similar way. Let

$$\begin{aligned}
c_1' &= \|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{\beta_2/n}\sup_\eta\biggl|\int_{(\mathbb{R}^n)^2}\bigl(b_1(y_1)-\lambda_1\bigr)K_{\mu,\eta}(x_Q,y_1,y_2)f_1^\infty(y_1)f_2^0(y_2)\,dy_1\,dy_2\biggr|\\
&\quad+\|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{\beta_2/n}\sup_\eta\biggl|\int_{(\mathbb{R}^n)^2}\bigl(b_1(y_1)-\lambda_1\bigr)K_{\mu,\eta}(x_Q,y_1,y_2)f_1^0(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\biggr|\\
&\quad+\|b_2\|_{\dot{\wedge}_{\beta_2}}|Q|^{\beta_2/n}\sup_\eta\biggl|\int_{(\mathbb{R}^n)^2}\bigl(b_1(y_1)-\lambda_1\bigr)K_{\mu,\eta}(x_Q,y_1,y_2)f_1^\infty(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\biggr|.
\end{aligned}$$

Observe that

$$\begin{aligned}
U^{*,1}_{\vec{b}}(f_1,f_2)(z) &< \bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigr|U^*(f_1,f_2)(z)+U^*\bigl((b_1-\lambda_1)f_1^0,f_2^0\bigr)(z)\\
&\quad+\sup_\eta\biggl|\int_{(\mathbb{R}^n)^2}\bigl(b_1(y_1)-\lambda_1\bigr)K_{\mu,\eta}(x,y_1,y_2)f_1^\infty(y_1)f_2^0(y_2)\,dy_1\,dy_2\biggr|\\
&\quad+\sup_\eta\biggl|\int_{(\mathbb{R}^n)^2}\bigl(b_1(y_1)-\lambda_1\bigr)K_{\mu,\eta}(x,y_1,y_2)f_1^0(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\biggr|\\
&\quad+\sup_\eta\biggl|\int_{(\mathbb{R}^n)^2}\bigl(b_1(y_1)-\lambda_1\bigr)K_{\mu,\eta}(x,y_1,y_2)f_1^\infty(y_1)f_2^\infty(y_2)\,dy_1\,dy_2\biggr|.
\end{aligned}$$

From this we have

$$\begin{aligned}
N_2 &\lesssim \frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl| \|b_2\|_{\dot{\Lambda}_{\beta_2}} |Q|^{\beta_2/n} U_{\vec{b}}^{*,1}(f_1,f_2)(z) - c_1' \bigr| \, dz \\
&\lesssim \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \bigl|\bigl(b_1(z)-\lambda_1\bigr)\bigr| U^*(f_1,f_2)(z)\, dz + \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q U^*\bigl((b_1-\lambda_1)f_1^0,f_2^0\bigr)(z)\, dz \\
&\quad + \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \sup_\eta \biggl| \int_{(\mathbb{R}^n)^2} \bigl(b_1(y_1)-\lambda_1\bigr) \\
&\quad \times \bigl[K_{\mu,\eta}(z,y_1,y_2) - K_{\mu,\eta}(x_Q,y_1,y_2)\bigr] f_1^0(y_1) f_2^\infty(y_2)\, dy_1\, dy_2 \biggr| \, dz \\
&\quad + \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \sup_\eta \biggl| \int_{(\mathbb{R}^n)^2} \bigl(b_1(y_1)-\lambda_1\bigr) \\
&\quad \times \bigl[K_{\mu,\eta}(z,y_1,y_2) - K_{\mu,\eta}(x_Q,y_1,y_2)\bigr] f_1^\infty(y_1) f_2^0(y_2)\, dy_1\, dy_2 \biggr| \, dz \\
&\quad + \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \sup_\eta \biggl| \int_{(\mathbb{R}^n)^2} \bigl(b_1(y_1)-\lambda_1\bigr) \\
&\quad \times \bigl[K_{\mu,\eta}(z,y_1,y_2) - K_{\mu,\eta}(x_Q,y_1,y_2)\bigr] f_1^\infty(y_1) f_2^\infty(y_2)\, dy_1\, dy_2 \biggr| \, dz \\
&\doteq N_{21} + N_{22} + N_{23} + N_{24} + N_{25}.
\end{aligned}$$

By the Hölder inequality,

$$\begin{aligned}
N_{21} &\lesssim \|b_2\|_{\dot{\Lambda}_{\beta_2}} \biggl(\frac{1}{|Q|^{r'\beta_1/n+1}} \int_Q \bigl|b_1(z)-\lambda_1\bigr|^{r'}\, dz \biggr)^{1/r'} \biggl(\frac{1}{|Q|} \int_Q \bigl|U^*(f_1,f_2)(z)\bigr|^r\, dz \biggr)^{1/r} \\
&\lesssim \|b_1\|_{\dot{\Lambda}_{\beta_1}} \|b_2\|_{\dot{\Lambda}_{\beta_2}} M_r\bigl(U^*(f_1,f_2)\bigr)(x).
\end{aligned}$$

Take $1 < q_1 < p_1$, $1 < q_2 < p_2$, and $1 < q < \infty$ such that $1/q = 1/q_1 + 1/q_2$. Then by the Hölder inequality and by Lemma 3.2,

$$\begin{aligned}
N_{22} &\lesssim \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{\beta_1/n+1/q}} \biggl(\int_Q \bigl|U^*\bigl((b_1-\lambda_1)f_1^0,f_2^0\bigr)(z)\bigr|^q\, dz \biggr)^{1/q} \\
&\lesssim \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{\beta_1/n+1/q}} \bigl\|(b_1-\lambda_1)f_1^0\bigr\|_{L^{q_1}} \bigl\|f_2^0\bigr\|_{L^{q_2}} \\
&\lesssim \|b_1\|_{\dot{\Lambda}_{\beta_1}} \|b_2\|_{\dot{\Lambda}_{\beta_2}} M_{q_1}(f_1)(x) M_{q_2}(f_2)(x).
\end{aligned}$$

For any $y_2 \in (Q^*)^c$, we have $|y_2 - x_Q| \sim |y_2 - z|$ and $|z - x_Q| \le \frac{|y_2-z|}{2} \le \frac{1}{2}\max\{|z-y_1|, |z-y_2|\}$. Then by Minkowski's inequality and by Lemma 3.1,

$$\begin{aligned}
N_{23} &\lesssim \frac{\|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \sup_\eta \int_{(\mathbb{R}^n)^2} \bigl|\bigl(b_1(y_1)-\lambda_1\bigr)\bigr| \\
&\quad \times \bigl|K_{\mu,\eta}(z,y_1,y_2) - K_{\mu,\eta}(x_Q,y_1,y_2)\bigr| \bigl|f_1^0(y_1) f_2^\infty(y_2)\bigr|\, dy_1\, dy_2\, dz \\
&\lesssim \frac{\|b_1\|_{\dot{\Lambda}_{\beta_1}} \|b_2\|_{\dot{\Lambda}_{\beta_2}}}{|Q|} \int_Q \int_{(\mathbb{R}^n)^2} \frac{|f_1^0(y_1) f_2^\infty(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\omega\left(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right)+\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right) dy_1\,dy_2\,dz\\
&\lesssim \frac{\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|}\int_Q\int_{Q^*}\big|f_1(y_1)\big|\int_{(Q^*)^c}\frac{|f_2(y_2)|}{|z-y_2|^{2n}}\\
&\times \left(\omega\left(\frac{|z-x_Q|}{|z-y_2|}\right)+\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right) dy_2\,dy_1\,dz\\
&\lesssim \frac{\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|}\int_Q\int_{Q^*}\big|f_1(y_1)\big|\\
&\times \sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{(\omega(2^{-k})+2^{-k})|f_2(y_2)|}{|2^k\sqrt{n}Q|^2}\,dy_2\,dy_1\,dz\\
&\lesssim \|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\frac{1}{|Q|}\int_{Q^*}\big|f_1(y_1)\big|\,dy_1\\
&\times \sum_{k=1}^{\infty}\frac{|Q|}{|2^{k+3}\sqrt{n}Q|}\big(\omega\big(2^{-k}\big)+2^{-k}\big)\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\big|f_2(y_2)\big|\,dy_2\\
&\lesssim C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M(f_1)(x)M(f_2)(x).
\end{aligned}
$$

Similarly, $N_{24}\lesssim \|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M(f_1)(x)M(f_2)(x)$.

For any $y_1,y_2\in(Q^*)^c$, we have $|y_1-x_Q|\sim|y_1-z|$ and $|y_2-x_Q|\sim|y_2-z|$. Then by Minkowski's inequality and by Lemma 3.1,

$$
\begin{aligned}
N_{25}&\lesssim \frac{\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\sup_{\eta}\int_{(\mathbb{R}^n)^2}\big|\big(b_1(y_1)-\lambda_1\big)\big|\\
&\times \big|K_{\mu,\eta}(z,y_1,y_2)-K_{\mu,\eta}(x_Q,y_1,y_2)\big|\big|f_1^\infty(y_1)f_2^\infty(y_2)\big|\,dy_1\,dy_2\,dz\\
&\lesssim \frac{\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\int_{(\mathbb{R}^n)^2}\frac{|y_1-x_Q|^{\beta_1}|f_1^0(y_1)f_2^\infty(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}}\\
&\times \left(\omega\left(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right)+\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right) dy_1\,dy_2\,dz\\
&\lesssim \frac{\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\int_{((Q^*)^c)^2}\frac{|f_1(y_1)||f_2(y_2)|}{|y_1-x_Q|^{2n-\beta_1}}\\
&\times \left(\omega\left(\frac{|z-x_Q|}{|z-y_1|}\right)+\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right) dy_1\,dy_2\,dz\\
&\lesssim \frac{\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{|f_1(y_1)||f_2(y_2)|}{|y_1-x_Q|^{2n-\beta_1}}\\
&\times \left(\omega\left(\frac{|z-x_Q|}{|z-y_1|}\right)+\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\right) dy_1\,dy_2\,dz\\
&\lesssim \|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}\sum_{k=1}^{\infty}\frac{2^{k\beta_1}(\omega(2^{-k})+2^{-k})}{|2^{k+3}\sqrt{n}Q|^2}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\big|f_1(y_1)\big|\,dy_1\\
&\times \int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\big|f_2(y_2)\big|\,dy_2\\
&\lesssim \|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}}M(f_1)(x)M(f_2)(x),
\end{aligned}
$$

where assumption (1.6) was used.

Combining the estimates for $N_{21}, N_{22}, N_{23}, N_{24}, N_{25}$, we get

$$N_2 \lesssim \|b_1\|_{\dot{\Lambda}_{\beta_1}} \|b_2\|_{\dot{\Lambda}_{\beta_2}} \left\{M_r\left(U^*(f_1,f_2)\right)(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x)\right\}.$$

The rest of the proof is the same as in [22], and hence we proved Theorem 1.3. □

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