Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Lyapunov-type inequalities for sequential fractional boundary value problems using Hilfer's fractional derivative

Wei Zhang[1] and Wenbin Liu[1*]

*Correspondence:
liuwenbin-xz@163.com
[1]School of Mathematics, China University of Mining and Technology, Xuzhou, P.R. China

Abstract

This paper is devoted to studying the Lyapunov-type inequality for sequential Hilfer fractional boundary value problems. We first provide some properties of Hilfer fractional derivative, and then establish Lyapunov-type inequalities for a sequential Hilfer fractional differential equation with two types of multi-point boundary conditions. Our results generalize and compliment the existing results in the literature.

MSC: 34A08; 34B15

Keywords: Lyapunov-type inequality; Sequential fractional differential equation; Hilfer fractional derivative; Multi-point boundary condition

1 Introduction

As is well known, Lyapunov inequality was first introduced by Lyapunov [1], who established a necessary condition for the existence of nontrivial solution of the boundary value problem (BVP for short):

$$\begin{cases} x''(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.1}$$

as the form

$$\int_a^b |q(s)|\,ds > \frac{4}{(b-a)}, \tag{1.2}$$

where $q \in C([a,b],\mathbb{R})$. Since then, Lyapunov inequality and Lyapunov-type inequality have been studied with great interest, and they have been proved to be an effective tool in the study of differential and difference equations, such as oscillation theory, disconjugacy, eigenvalue problems, etc. (see [2–5]).

In recent years, by the rise of theoretical research in fractional differential equations, there has been tremendous interest in the research of Lyapunov-type inequalities for fractional BVP, see [6–30] and the references cited therein.

In [11], Ferreira discussed the Lyapunov-type inequality for the following fractional BVP:

$$\begin{cases} (D^{\alpha}_{a+}x)(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.3}$$

where D^{α}_{a+} is the left Riemann–Liouville fractional derivative of order α, $\alpha \in (1,2]$ and $q \in C([a,b],\mathbb{R})$. The Lyapunov-type inequality for problem (1.3) was established as follows:

$$\int_a^b |q(s)|\,ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.4}$$

Furthermore, in 2016, Ferreira [12] considered the Lyapunov-type inequality for a sequential fractional BVP

$$\begin{cases} ({}_aD^{\alpha}\,{}_aD^{\beta}x)(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.5}$$

where ${}_aD^{\gamma}$, $\gamma = \alpha, \beta$ stand for the left Riemann–Liouville fractional derivative or the left Caputo fractional derivative of order γ, $\gamma \in (0,1]$ and $1 < \alpha + \beta \le 2$, $q \in C([a,b],\mathbb{R})$. Two interesting results have been obtained as follows:

(i) Take ${}_aD^{\gamma}$, $\gamma = \alpha, \beta$, is the left Riemann–Liouville fractional derivative, then problem (1.5) has a nontrivial continuous solution provided that

$$\int_a^b |q(s)|\,ds > \Gamma(\alpha+\beta)\left(\frac{4}{b-a}\right)^{\alpha+\beta-1}. \tag{1.6}$$

(ii) Take ${}_aD^{\gamma}$, $\gamma = \alpha, \beta$, is the left Caputo fractional derivative, then problem (1.5) has a nontrivial continuous solution provided that

$$\int_a^b |q(s)|\,ds > \frac{\Gamma(\alpha+\beta)}{(b-a)^{\alpha+\beta-1}}\,\frac{(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}. \tag{1.7}$$

It is worth noting that the study of the Hilfer fractional differential equations has received a significant amount of attention in the last few years. Hilfer fractional derivative was proposed by Hilfer in 2000, which is a generalization of both Riemann–Liouville and Caputo fractional derivatives (see [31]). Meanwhile, the discussion of Lyapunov-type inequalities for fractional BVP with Hilfer fractional derivative can be found in papers [25, 29, 30].

In [25], Pathak investigated Lyapunov-type inequalities for the following Hilfer fractional differential equation:

$$\left(D^{\alpha,\beta}_{a+}x\right)(t) + q(t)x(t) = 0, \quad t \in (a,b), \tag{1.8}$$

with boundary conditions

$$x(a) = x(b) = 0, \tag{1.9}$$

or

$$x(a) = x'(b) = 0, \tag{1.10}$$

where $D_{a+}^{\alpha,\beta}$ is the left Hilfer fractional derivative of order α and type β, $\alpha \in (1,2]$, $\beta \in [0,1]$, $q \in C([a,b],\mathbb{R})$. The author got two Lyapunov-type inequalities: for BVP (1.8), (1.9) as

$$\int_a^b |q(s)|\,ds \geq \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha-1}(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}, \tag{1.11}$$

for BVP (1.8), (1.10) as

$$\int_a^b (b-s)^{\alpha-2}|q(s)|\,ds \geq \frac{\Gamma(\alpha)[\alpha-1+\beta(2-\alpha)]}{(b-a)\max\{\alpha-1,\beta(2-\alpha)\}}. \tag{1.12}$$

In [30], Wang obtained Lyapunov-type inequality for the fractional multi-point BVP involving Hilfer derivative:

$$\begin{cases} (D_{a+}^{\alpha,\beta}x)(t)+q(t)x(t)=0, & t\in(a,b), \\ x(a)=0, & x(b)=\sum_{i=1}^{m-2}\beta_i x(\xi_i), \end{cases} \tag{1.13}$$

where $q \in C([a,b],\mathbb{R})$, $D_{a+}^{\alpha,\beta}$ is the left Hilfer fractional derivative of order α and type β with $\alpha \in (1,2]$, $\beta \in [0,1]$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $\beta_i \geq 0$ $(i = 1,2,\dots,m-2)$, $(b-a)^{1-(2-\alpha)(1-\beta)} > \sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}$. The Lyapunov-type inequality for problem (1.13) is given as follows:

$$\int_a^b |q(s)|\,ds \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}L}\frac{1}{1+\sum_{i=1}^{m-2}\beta_i T(b)},$$

where

$$L = \frac{(\alpha-1)^{\alpha-1}(\alpha-1+2\beta-\alpha\beta)^{\alpha-1+2\beta-\alpha\beta}}{(2\alpha-2+2\beta-\alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}},$$

$$T(t) = \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}}, \quad t\in[a,b].$$

Motivated by these results, in this paper we study Lyapunov-type inequalities for a sequential Hilfer fractional differential equation

$$\big(D_{a+}^{\alpha_1,\beta_1}D_{a+}^{\alpha_2,\beta_2}x\big)(t)+q(t)x(t)=0, \quad t\in(a,b), \tag{1.14}$$

with multi-point boundary conditions

$$x(a)=0, \qquad x(b)=\sum_{i=1}^{m-2}\sigma_i x(\xi_i), \tag{1.15}$$

or

$$x(a) = 0, \qquad x'(b) = \sum_{i=1}^{m-2} \delta_i x(\eta_i), \tag{1.16}$$

where $q \in C([a,b],\mathbb{R})$, $D_{a+}^{\alpha_i,\beta_i}$, $i = 1, 2$, are two left Hilfer fractional derivatives of order α_i and types β_i with $\alpha_i \in (0,1]$, $1 < \alpha_1 + \alpha_2 \le 2$, $\beta_i \in [0,1]$. For the definition of Hilfer fractional derivative, see Sect. 2. A remarkable characteristic of this kind of fractional derivative is that the type β_i allows $D_{a+}^{\alpha_i,\beta_i}$ to interpolate continuously from the Riemann–Liouville case $D_{a+}^{\alpha_i,0} \equiv D_{a+}^{\alpha_i}$ to the Caputo case $D_{a+}^{\alpha_i,1} \equiv {}^{C}D_{a+}^{\alpha_i}$ (see [32]). To state our main results, we assume that the following conditions hold:

(A_1) $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b, \sigma_i \ge 0$ $(i = 1,2,\ldots,m-2)$ and

$$\Delta_1 := (b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)} - \sum_{i=1}^{m-2} \sigma_i (\xi_i - a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)} > 0.$$

(A_2) $a < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < b, \delta_i \ge 0$ $(i = 1,2,\ldots,m-2)$ and

$$\begin{aligned}\Delta_2 := {}& \bigl[\alpha_2 - (1-\alpha_1)(1-\beta_1)\bigr](b-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)} \\ & - \sum_{i=1}^{m-2} \delta_i (\eta_i - a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)} > 0.\end{aligned}$$

In the present work, we are focused on establishing the Lyapunov-type inequalities for a sequential Hilfer fractional differential equation with two types of multi-point boundary conditions. As far as we know, the Lyapunov-type inequality for fractional BVP with Hilfer derivative has seldom been considered up to now. The new insights of this paper can be presented as follows. On the one hand, we provide some properties of Hilfer fractional derivative, which are not introduced in the previous paper (see Sect. 2, Lemmas 2.6, 2.7, and 2.8). On the other hand, we extend the previous results. Many previous results are a special case of our work, which is embodied in Sect. 3. The main difficulties in this article are as follows. First, we have to construct Green's function for BVPs (1.14), (1.15) and (1.14), (1.16). Second, it is difficult to estimate the maximum of Green's function because Green's functions do not satisfy the non-negativity.

The rest of this paper is organized as follows. In Sect. 2, we recall some definitions, lemmas of fractional calculus. In Sect. 3, by constructing Green's functions and finding its corresponding maximum value, we prove our main results. Finally, a conclusion is given in Sect. 4.

2 Preliminaries

In this section, we recall some definitions and lemmas which are used throughout this paper.

Definition 2.1 ([33–35]) Let $J = [a,b]$ $(-\infty < a < b < \infty)$. The left Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $x : (a,b) \to \mathbb{R}$ is defined by

$$\bigl(I_{a+}^{\alpha} x\bigr)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\,ds \quad (t > a),$$

provided that the right-hand side integral is pointwise defined on $[a,b]$.

Definition 2.2 ([33–35]) The left-sided Riemann–Liouville fractional derivative of order $\alpha > 0$ ($n-1 < \alpha \le n$, $n \in \mathbb{N}^+$) of a function $x : (a,b) \to \mathbb{R}$ is defined by

$$\bigl(D^{\alpha}_{a+}x\bigr)(t) = \frac{d^n}{dt^n}\bigl(I^{n-\alpha}_{a+}x\bigr)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}x(s)\,ds \quad (t > a),$$

provided the right-hand side integral is pointwise defined on $[a,b]$.

Definition 2.3 ([33–35]) The left-sided Caputo fractional derivative of order $\alpha > 0$ ($n-1 < \alpha < n$, $n \in \mathbb{N}^+$) of a function $x : (a,b) \to \mathbb{R}$ is defined by

$$\bigl({}^{C}D^{\alpha}_{a+}x\bigr)(t) = \bigl(I^{n-\alpha}_{a+}x^{(n)}\bigr)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}x^{(n)}(s)\,ds \quad (t > a),$$

provided the right-hand side integral is pointwise defined on $[a,b]$.

Definition 2.4 ([32]) The left-sided Hilfer fractional derivative of order $\alpha > 0$ ($n-1 < \alpha \le n$, $n \in \mathbb{N}^+$) and type $0 \le \beta \le 1$ of a function $x : (a,b) \to \mathbb{R}$ is defined by

$$\bigl(D^{\alpha,\beta}_{a+}x\bigr)(t) = \bigl(I^{\beta(n-\alpha)}_{a+}D^n\bigl(I^{(1-\beta)(n-\alpha)}_{a+}x\bigr)\bigr)(t) \quad (t > a),$$

where $D^n = d^n/dt^n$.

Lemma 2.1 ([33–35]) *Let $\alpha > 0$, $n = [\alpha] + 1$. If $x \in L^1(a,b)$, $I^{n-\alpha}_{a+}x \in AC^n[a,b]$, then*

$$I^{\alpha}_{a+}D^{\alpha}_{a+}x(t) = x(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{k-(n-\alpha)}}{\Gamma(k-(n-\alpha)+1)}\lim_{t\to a+}\frac{d^k}{dt^k}\bigl(I^{n-\alpha}_{a+}x\bigr)(t).$$

Lemma 2.2 ([33–35]) *If $\alpha > 0$, $\lambda > -1$, then*

$$I^{\alpha}_{a+}(t-a)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)}(t-a)^{\alpha+\lambda}, \qquad D^{\alpha}_{a+}(t-a)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}(t-a)^{\lambda-\alpha}.$$

Lemma 2.3 ([36]) *Let $\alpha > 0$, $n = [\alpha] + 1$, $0 \le \beta \le 1$. If $x \in L^1(a,b)$, $I^{(n-\alpha)(1-\beta)}_{a+}x \in AC^n[a,b]$, then*

$$\bigl(I^{\alpha}_{a+}D^{\alpha,\beta}_{a+}x\bigr)(t) = x(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)}\lim_{t\to a+}\frac{d^k}{dt^k}\bigl(I^{(n-\alpha)(1-\beta)}_{a+}x\bigr)(t).$$

Lemma 2.4 ([35]) *Let $\alpha > 0$, $n \in N$, and $D = d/dx$. If the fractional derivatives $(D^{\alpha}_{a+}x)(t)$ and $(D^{\alpha+n}_{a+}x)(t)$ exist, then*

$$\bigl(D^nD^{\alpha}_{a+}x\bigr)(t) = \bigl(D^{\alpha+n}_{a+}x\bigr)(t).$$

Lemma 2.5 ([35]) *Let $\alpha > 0$, $n \in N$, and $D = d/dx$. If the fractional derivatives $(D^nx)(t)$ and $({}^{C}D^{\alpha+n}_{a+}x)(t)$ exist, then*

$$\bigl({}^{C}D^{\alpha}_{a+}D^nx\bigr)(t) = \bigl({}^{C}D^{\alpha+n}_{a+}x\bigr)(t).$$

Lemma 2.6 *Let $\alpha > 0$, $n = [\alpha] + 1$, $0 \leq \beta \leq 1$, $m \in \mathbb{N}$, and $D = d/dx$. If the fractional derivatives $(D^m x)(t)$ and $(D_{a+}^{\alpha+m,\beta} x)(t)$ exist, then*

$$\bigl(D_{a+}^{\alpha,\beta} D^m x\bigr)(t) = \bigl(D_{a+}^{\alpha+m,\beta} x\bigr)(t),$$

provided that

$$x^{(j)}(a) = 0, \quad j = 0, 1, 2, \ldots, m-1.$$

Proof Since $x^{(j)}(a) = 0$, $j = 0, 1, 2, \ldots, m-1$, then we get

$$\bigl(I_{a+}^m D^m x\bigr)(t) = x(t),$$

which yields the following equalities hold:

$$\begin{aligned}
\bigl(D_{a+}^{\alpha,\beta} D^m x\bigr)(t) &= \bigl(I_{a+}^{\beta(n-\alpha)} D^n I_{a+}^{(1-\beta)(n-\alpha)} D^m x\bigr)(t) \\
&= \bigl(I_{a+}^{\beta(n-\alpha)} D^{n+m} I_{a+}^{(1-\beta)(n-\alpha)} \bigl(I_{a+}^m D^m x\bigr)\bigr)(t) \\
&= \bigl(I_{a+}^{\beta(n-\alpha)} D^{n+m} I_{a+}^{(1-\beta)(n-\alpha)} x\bigr)(t) = \bigl(D_{a+}^{\alpha+m,\beta} x\bigr)(t).
\end{aligned}$$

The proof is completed. □

Lemma 2.7 *Let $\alpha > 0$, $n = [\alpha] + 1$, $0 \leq \beta \leq 1$. If $x \in C[a,b]$, $I_{a+}^{1-\beta(n-\alpha)} x \in AC[a,b]$, then*

$$D_{a+}^{\alpha,\beta} I_{a+}^{\alpha} x(t) = x(t). \tag{2.1}$$

Proof On the one hand, if $\beta(n-\alpha) = 0$, i.e., $\beta = 0$ or $\alpha = n$, then

$$D_{a+}^{\alpha,\beta} I_{a+}^{\alpha} x(t) = D_{a+}^{\alpha} I_{a+}^{\alpha} x(t) = x(t)$$

or

$$D_{a+}^{\alpha,\beta} I_{a+}^{\alpha} x(t) = D^n I_{a+}^{n} x(t) = x(t).$$

On the other hand, if $\beta(n-\alpha) \neq 0$, by Definitions 2.2, 2.4 and Lemma 2.1, we have

$$\begin{aligned}
D_{a+}^{\alpha,\beta} I_{a+}^{\alpha} x(t) &= \bigl(I_{a+}^{\beta(n-\alpha)} D^n \bigl(I_{a+}^{n-\beta(n-\alpha)} x\bigr)\bigr)(t) \\
&= \bigl(I_{a+}^{\beta(n-\alpha)} D_{a+}^{\beta(n-\alpha)} x\bigr)(t) = x(t) - \frac{(t-a)^{\beta(n-\alpha)-1}}{\Gamma(\beta(n-\alpha))} \lim_{t\to a+} \bigl(I_{a+}^{1-\beta(n-\alpha)} x\bigr)(t).
\end{aligned}$$

Since $x \in C[a,b]$, one has

$$\lim_{t\to a+} \bigl(I_{a+}^{1-\beta(n-\alpha)} x\bigr)(t) = 0.$$

Thus, (2.1) holds for $\beta(n-\alpha) \neq 0$. The proof is completed. □

Lemma 2.8 *Let $\alpha > 0$, $n = [\alpha] + 1$, $0 \le \beta \le 1$, $\lambda > -1$, then*

$$D_{a+}^{\alpha,\beta}(t-a)^{\lambda+\beta(n-\alpha)} = \frac{\Gamma(\lambda+1+\beta(n-\alpha))}{\Gamma(\lambda+1-\alpha+\beta(n-\alpha))}(t-a)^{\beta(n-\alpha)+\lambda-\alpha}.$$

In particular,

$$D_{a+}^{\alpha,\beta}(t-a)^{\alpha-j+\beta(n-\alpha)} = 0, \quad j = 1,2,\ldots,n.$$

Proof For $\lambda > -1$, by Definition 2.4 and Lemma 2.2, we have

$$\begin{aligned} D_{a+}^{\alpha,\beta}(t-a)^{\lambda+\beta(n-\alpha)} &= I_{a+}^{\beta(n-\alpha)}D^n I_{a+}^{(1-\beta)(n-\alpha)}(t-a)^{\lambda+\beta(n-\alpha)} \\ &= \frac{\Gamma(\lambda+1+\beta(n-\alpha))}{\Gamma(\lambda+1+n-\alpha)} I_{a+}^{\beta(n-\alpha)}D^n(t-a)^{\lambda+n-\alpha} \\ &= \frac{\Gamma(\lambda+1+\beta(n-\alpha))}{\Gamma(\lambda-\alpha+1)} I_{a+}^{\beta(n-\alpha)}(t-a)^{\lambda-\alpha} \\ &= \frac{\Gamma(\lambda+1+\beta(n-\alpha))}{\Gamma(\lambda-\alpha+\beta(n-\alpha)+1)}(t-a)^{\lambda-\alpha+\beta(n-\alpha)}. \end{aligned}$$

In particular,

$$\begin{aligned} D_{a+}^{\alpha,\beta}(t-\alpha)^{\alpha-j+\beta(n-\alpha)} &= \frac{\Gamma(\alpha-j+1+\beta(n-\alpha))}{\Gamma(n-j+1)} I_{a+}^{\beta(n-\alpha)}D^n(t-\alpha)^{n-j} \\ &= 0, \quad \text{as } j = 1,2,\ldots,n. \end{aligned}$$

Thus the proof of Lemma 2.8 is completed. □

3 Main result

Take the Banach space $(X, \|\cdot\|_\infty)$,

$$X = C[a,b] \quad \text{with the norm } \|x\|_\infty = \max_{t\in[a,b]}\bigl|x(t)\bigr|.$$

Lemma 3.1 *Let $0 < \alpha_1, \alpha_2 \le 1$, $1 < \alpha_1 + \alpha_2 \le 2$, $0 \le \beta_1, \beta_2 \le 1$. Assume that (A_1) holds. Then, for $y \in X$, the function $x \in X$ is a solution of the following BVP*:

$$D_{a+}^{\alpha_1,\beta_1}D_{a+}^{\alpha_2,\beta_2}x(t) + y(t) = 0, \quad t \in (a,b), \tag{3.1}$$

$$x(a) = 0, \qquad x(b) = \sum_{i=1}^{m-2}\sigma_i x(\xi_i), \tag{3.2}$$

if and only if x satisfies the integral equation

$$x(t) = \int_a^b K(t,s)y(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_1}\int_a^b \sum_{i=1}^{m-2}\sigma_i K(\xi_i,s)y(s)\,ds, \quad t \in [a,b], \tag{3.3}$$

where

$$K(t,s) = \frac{(b-a)^{(1-\alpha_1)(1-\beta_1)-\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}\begin{cases} k_1(t,s), & a \le s \le t \le b, \\ k_2(t,s), & a \le t \le s \le b, \end{cases}$$

and

$$k_1(t,s) = (t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-s)^{\alpha_1+\alpha_2-1} - (b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(t-s)^{\alpha_1+\alpha_2-1},$$
$$k_2(t,s) = (t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-s)^{\alpha_1+\alpha_2-1}.$$

Proof By using Lemma 2.3 twice and combining Lemma 2.2, we get that x is a solution of (3.1) if and only if

$$x(t) = -I_{a+}^{\alpha_1+\alpha_2}y(t) + c_1\frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Gamma(\alpha_2+1-(1-\alpha_1)(1-\beta_1))} + c_2\frac{(t-a)^{-(1-\alpha_2)(1-\beta_2)}}{\Gamma(1-(1-\alpha_2)(1-\beta_2))},$$

where $c_1, c_2 \in \mathbb{R}$. Considering the boundary conditions $x(a) = 0$, $x(b) = \sum_{i=1}^{m-2}\sigma_i x(\xi_i)$, we get

$$c_2 = 0, \qquad c_1 = \frac{\Gamma[\alpha_2+1-(1-\alpha_1)(1-\beta_1)]}{(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}\left[I_{a+}^{\alpha_1+\alpha_2}y(t)|_{t=b} + \sum_{i=1}^{m-2}\sigma_i x(\xi_i)\right].$$

Thus,

$$\begin{aligned} x(t) &= -I_{a+}^{\alpha_1+\alpha_2}y(t) + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}[I_{a+}^{\alpha_1+\alpha_2}y(t)|_{t=b}]}{(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}} \\ &\quad + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}\sum_{i=1}^{m-2}\sigma_i x(\xi_i) \\ &= \int_a^b K(t,s)y(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}\sum_{i=1}^{m-2}\sigma_i x(\xi_i). \end{aligned} \tag{3.4}$$

Therefore,

$$\sum_{i=1}^{m-2}\sigma_i x(\xi_i) = \int_a^b \sum_{i=1}^{m-2}\sigma_i K(\xi_i,s)y(s)\,ds + \frac{\sum_{i=1}^{m-2}\sigma_i(\xi_i-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}\sum_{i=1}^{m-2}\sigma_i x(\xi_i).$$

That is,

$$\sum_{i=1}^{m-2}\sigma_i x(\xi_i) = \frac{(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_1}\int_a^b \sum_{i=1}^{m-2}\sigma_i K(\xi_i,s)y(s)\,ds. \tag{3.5}$$

By substituting (3.5) into (3.4), we obtain

$$x(t) = \int_a^b K(t,s)y(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_1}\int_a^b \sum_{i=1}^{m-2}\sigma_i K(\xi_i,s)y(s)\,ds.$$

The proof is completed. □

Lemma 3.2 *Let $0 < \alpha_1, \alpha_2 \leq 1$, $1 < \alpha_1 + \alpha_2 \leq 2$, $0 \leq \beta_1, \beta_2 \leq 1$. Assume that (A_2) holds. Then, for $y \in X$, the function $x \in X$ is a solution of the following BVP*:

$$D_{a+}^{\alpha_1,\beta_1} D_{a+}^{\alpha_2,\beta_2} x(t) + y(t) = 0, \quad t \in (a,b), \tag{3.6}$$

$$x(a) = 0, \qquad x'(b) = \sum_{i=1}^{m-2} \delta_i x(\eta_i), \tag{3.7}$$

if and only if x satisfies the integral equation

$$x(t) = \int_a^b H(t,s)y(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_2} \int_a^b \sum_{i=1}^{m-2} \delta_i H(\eta_i,s)y(s)\,ds, \quad t \in [a,b], \tag{3.8}$$

where

$$H(t,s) = \frac{(b-s)^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2)[\alpha_2-(1-\alpha_1)(1-\beta_1)]} H_1(t,s),$$

and

$$H_1(t,s) = \begin{cases} h_1(t,s), & a \leq s \leq t \leq b, \\ h_2(t,s), & a \leq t \leq s \leq b, \end{cases}$$

$$\begin{aligned} h_1(t,s) &= (\alpha_1+\alpha_2-1)(b-a)^{2-(\alpha_1+\alpha_2)-\beta_1(1-\alpha_1)}(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)} \\ &\quad - \big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big]\frac{(t-s)^{\alpha_1+\alpha_2-1}}{(b-s)^{\alpha_1+\alpha_2-2}}, \\ h_2(t,s) &= (\alpha_1+\alpha_2-1)(b-a)^{2-(\alpha_1+\alpha_2)-\beta_1(1-\alpha_1)}(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}. \end{aligned}$$

Proof By a similar method used in Lemma 3.1, we obtain

$$x(t) = -I_{a+}^{\alpha_1+\alpha_2} y(t) + c_1 \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Gamma[\alpha_1+\alpha_2+\beta_1(1-\alpha_1)]},$$

where $c_1 \in \mathbb{R}$. Then, taking derivative to the both sides of the above equality, we have

$$\begin{aligned} x'(t) &= -\frac{\alpha_1+\alpha_2-1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t (t-s)^{\alpha_1+\alpha_2-2} y(s)\,ds \\ &\quad + c_1 \frac{[\alpha_2-(1-\alpha_1)(1-\beta_1)](t-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}}{\Gamma[\alpha_1+\alpha_2+\beta_1(1-\alpha_1)]}. \end{aligned}$$

Using the boundary condition $x'(b) = \sum_{i=1}^{m-2} \delta_i x(\eta_i)$, we get

$$\begin{aligned} c_1 = {}& \frac{\Gamma[\alpha_1+\alpha_2+\beta_1(1-\alpha_1)]}{[\alpha_2-(1-\alpha_1)(1-\beta_1)](b-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}} \\ & \times \left[\frac{\alpha_1+\alpha_2-1}{\Gamma(\alpha_1+\alpha_2)} \int_a^b (b-s)^{\alpha_1+\alpha_2-2} y(s)\,ds + \sum_{i=1}^{m-2} \delta_i x(\eta_i) \right]. \end{aligned}$$

Hence,

$$\begin{aligned} x(t) &= -I_{a+}^{\alpha_1+\alpha_2}y(t) + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(\alpha_1+\alpha_2-1)\int_a^b (b-s)^{\alpha_1+\alpha_2-2}y(s)\,ds}{[\alpha_2-(1-\alpha_1)(1-\beta_1)](b-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}\Gamma(\alpha_1+\alpha_2)} \\ &\quad + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{[\alpha_2-(1-\alpha_1)(1-\beta_1)](b-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}}\sum_{i=1}^{m-2}\delta_i x(\eta_i) \\ &= \int_a^b H(t,s)y(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{[\alpha_2-(1-\alpha_1)(1-\beta_1)](b-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}} \\ &\quad \times \sum_{i=1}^{m-2}\delta_i x(\eta_i). \end{aligned} \tag{3.9}$$

Therefore,

$$\sum_{i=1}^{m-2}\delta_i x(\eta_i) = \int_a^b \sum_{i=1}^{m-2}\delta_i H(\eta_i,s)y(s)\,ds + \frac{\sum_{i=1}^{m-2}\delta_i(\eta_i-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}\sum_{i=1}^{m-2}\delta_i x(\eta_i)}{[\alpha_2-(1-\alpha_1)(1-\beta_1)](b-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}}.$$

It follows that

$$\sum_{i=1}^{m-2}\delta_i x(\eta_i) = \frac{\alpha_2-(1-\alpha_1)(1-\beta_1)}{\Delta_2(b-a)^{1-\alpha_2+(1-\alpha_1)(1-\beta_1)}}\int_a^b \sum_{i=1}^{m-2}\delta_i H(\eta_i,s)y(s)\,ds. \tag{3.10}$$

If we plug (3.10) back into (3.9), we obtain

$$x(t) = \int_a^b H(t,s)y(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_2}\int_a^b \sum_{i=1}^{m-2}\delta_i H(\eta_i,s)y(s)\,ds,$$

which completes the proof. □

Lemma 3.3 (See [14]) *If* $1<\omega<2$, *then*

$$\frac{2-\omega}{(\omega-1)^{(\omega-1)/(\omega-2)}} \le \frac{(\omega-1)^{\omega-1}}{\omega^{\omega}}.$$

Lemma 3.4 *The functions* $K(t,s)$ *and* $H_1(t,s)$ *defined in* (3.3) *and* (3.8) *satisfy the following properties*:

(i) $K(t,s)$ *and* $H_1(t,s)$ *are two continuous functions for any* $(t,s)\in[a,b]\times[a,b]$;

(ii) $|K(t,s)| \le \frac{[(b-a)(\alpha_1+\alpha_2-1)]^{\alpha_1+\alpha_2-1}[\alpha_2-(1-\alpha_1)(1-\beta_1)]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Gamma(\alpha_1+\alpha_2)[2\alpha_2-(1-\alpha_1)(2-\beta_1)]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}}$ *for all* $(t,s)\in[a,b]\times[a,b]$;

(iii) $|H_1(t,s)| \le (b-a)\max\{\alpha_1+\alpha_2-1,\beta_1(1-\alpha_1)\}$ *for every* $(t,s)\in[a,b]\times[a,b]$.

Proof Obviously, (i) is true. To prove (ii), for $(t,s)\in[a,b]\times[a,b]$, it is straightforward to show that

$$0 \le k_2(t,s) \le k_2(s,s).$$

Differentiating $k_1(t,s)$ with respect to s, we have

$$\begin{aligned}\frac{\partial k_1(t,s)}{\partial s} &= -(\alpha_1+\alpha_2-1)(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-s)^{\alpha_1+\alpha_2-2}\\ &\quad+(\alpha_1+\alpha_2-1)(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(t-s)^{\alpha_1+\alpha_2-2}\\ &= (\alpha_1+\alpha_2-1)(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(t-s)^{\alpha_1+\alpha_2-2}\\ &\quad\times\left[1-\left(\frac{t-s}{b-s}\right)^{2-(\alpha_1+\alpha_2)}\left(\frac{t-a}{b-a}\right)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}\right]\\ &\geq 0,\end{aligned}$$

which shows $k_1(t,s)$ is increasing with respect to $s\in[a,t]$. Thus,

$$k_1(t,a)\leq k_1(t,s)\leq k_1(t,t).$$

Since

$$\begin{aligned}k_1(t,a) &= (t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-a)^{\alpha_1+\alpha_2-1}-(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(t-a)^{\alpha_1+\alpha_2-1}\\ &= (t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-a)^{\alpha_1+\alpha_2-1}\left[1-\left(\frac{b-a}{t-a}\right)^{\beta_1(1-\alpha_1)}\right]\leq 0.\end{aligned}$$

Thus,

$$\left|k_1(t,s)\right|\leq \max\Big\{\max_{t\in[a,b]}k_1(t,t),\max_{t\in[a,b]}\big(-k_1(t,a)\big)\Big\}.$$

We consider the functions

$$\begin{aligned}f(t) &= k_1(t,t)=(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-t)^{\alpha_1+\alpha_2-1},\quad t\in[a,b],\\ \tilde{f}(t) &= -k_1(t,a)\\ &= (b-a)^{\alpha_1+\alpha_2-1}(t-a)^{\alpha_1+\alpha_2-1}\big[(b-a)^{\beta_1(1-\alpha_1)}-(t-a)^{\beta_1(1-\alpha_1)}\big],\quad t\in[a,b].\end{aligned}$$

Differentiating $f(t)$ on (a,b), we have

$$\begin{aligned}f'(t) &= \big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big](t-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}(b-t)^{\alpha_1+\alpha_2-1}\\ &\quad-(\alpha_1+\alpha_2-1)(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-t)^{\alpha_1+\alpha_2-2}\\ &= (t-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}(b-t)^{\alpha_1+\alpha_2-2}\\ &\quad\times\big\{\big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big](b-t)-(\alpha_1+\alpha_2-1)(t-a)\big\},\end{aligned}$$

and

$$\begin{aligned}f''(t) &= \big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big]\big[\alpha_2-1-(1-\alpha_1)(1-\beta_1)\big]\\ &\quad\times(t-a)^{\alpha_2-2-(1-\alpha_1)(1-\beta_1)}(b-t)^{\alpha_1+\alpha_2-1}\\ &\quad-2(\alpha_1+\alpha_2-1)\big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big]\end{aligned}$$

$$
\begin{aligned}
&\times (t-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)}(b-t)^{\alpha_1+\alpha_2-2} \\
&+ (\alpha_1+\alpha_2-1)(\alpha_1+\alpha_2-2)(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(b-t)^{\alpha_1+\alpha_2-3}.
\end{aligned}
$$

By calculating, we get $f'(t)=0$ has a unique zero in (a,b) as follows:

$$
\begin{aligned}
t=t^* &= a+\frac{\alpha_2-(1-\alpha_1)(1-\beta_1)}{2\alpha_2-(1-\alpha_1)(2-\beta_1)}(b-a) \\
&= b-\frac{\alpha_1+\alpha_2-1}{2\alpha_2-(1-\alpha_1)(2-\beta_1)}(b-a)\in(a,b).
\end{aligned}
\tag{3.11}
$$

Because

$$
\begin{aligned}
&\alpha_2-(1-\alpha_1)(1-\beta_1)\geq 0, \qquad \alpha_2-1-(1-\alpha_1)(1-\beta_1)\leq 0, \\
&\alpha_1+\alpha_2-1>0, \qquad \alpha_1+\alpha_2-2\leq 0,
\end{aligned}
$$

it is easy to verify that

$$
f''\bigl(t^*\bigr)\leq 0.
$$

Hence, we obtain

$$
\begin{aligned}
\max_{t\in[a,b]} f(t) &= f\bigl(t^*\bigr) \\
&= \biggl[\frac{\alpha_2-(1-\alpha_1)(1-\beta_1)}{2\alpha_2-(1-\alpha_1)(2-\beta_1)}(b-a)\biggr]^{\alpha_2-(1-\alpha_1)(1-\beta_1)} \\
&\quad\times\biggl[\frac{\alpha_1+\alpha_2-1}{2\alpha_2-(1-\alpha_1)(2-\beta_1)}(b-a)\biggr]^{\alpha_1+\alpha_2-1} \\
&= \bigl[\alpha_2-(1-\alpha_1)(1-\beta_1)\bigr]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(\alpha_1+\alpha_2-1)^{\alpha_1+\alpha_2-1} \\
&\quad\times\biggl[\frac{b-a}{2\alpha_2-(1-\alpha_1)(2-\beta_1)}\biggr]^{\alpha_1+2\alpha_2-1-(1-\alpha_1)(1-\beta_1)}.
\end{aligned}
$$

We now prove that $\max_{t\in[a,b]}\tilde{f}(t)\leq\max_{t\in[a,b]}f(t)$. In fact, if $\beta_1(1-\alpha_1)=0$, then $\tilde{f}(t)\equiv 0$, and the conclusion is obvious. If $\beta_1(1-\alpha_1)\neq 0$, differentiating $\tilde{f}(t)$ on (a,b), we have

$$
\begin{aligned}
\tilde{f}'(t) &= (\alpha_1+\alpha_2-1)(b-a)^{\alpha_2-(1-\beta_1)(1-\alpha_1)}(t-a)^{\alpha_1+\alpha_2-2} \\
&\quad-\bigl[\alpha_2-(1-\beta_1)(1-\alpha_1)\bigr](b-a)^{\alpha_1+\alpha_2-1}(t-a)^{\alpha_2-1-(1-\beta_1)(1-\alpha_1)},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{f}''(t) &= (\alpha_1+\alpha_2-1)(\alpha_1+\alpha_2-2)(b-a)^{\alpha_2-(1-\beta_1)(1-\alpha_1)}(t-a)^{\alpha_1+\alpha_2-3} \\
&\quad-\bigl[\alpha_2-(1-\beta_1)(1-\alpha_1)\bigr]\bigl[\alpha_2-1-(1-\beta_1)(1-\alpha_1)\bigr] \\
&\quad\times(b-a)^{\alpha_1+\alpha_2-1}(t-a)^{\alpha_2-2-(1-\beta_1)(1-\alpha_1)}.
\end{aligned}
\tag{3.12}
$$

By calculating, we get $\tilde{f}'(t) = 0$ has a unique zero in (a,b) as follows:

$$t = t_* = a + \left[\frac{\alpha_1 + \alpha_2 - 1}{\alpha_2 - (1-\alpha_1)(1-\beta_1)}\right]^{1/\beta_1(1-\alpha_1)}(b-a) \in (a,b). \tag{3.13}$$

Submitting (3.13) into (3.12), we have

$$\begin{aligned}\tilde{f}''(t_*) &= -\big[\alpha_2 - (1-\alpha_1)(1-\beta_1)\big]\beta_1(1-\alpha_1)(b-a)^{-2(1-\alpha_2)-(2-\beta_1)(1-\alpha_1)} \\ &\quad \times \left[\frac{\alpha_1+\alpha_2-1}{\alpha_2-(1-\alpha_1)(1-\beta_1)}\right]^{[\alpha_2-2-(1-\beta_1)(1-\alpha_1)]/\beta_1(1-\alpha_1)} \\ &\le 0.\end{aligned}$$

Thus,

$$\begin{aligned}\max_{t\in[a,b]}\tilde{f}(t) &= \max_{t\in[a,b]}\tilde{f}(t_*) \\ &= \frac{\beta_1(1-\alpha_1)(b-a)^{2(\alpha_1+\alpha_2-1)+\beta_1(1-\alpha_1)}}{\alpha_2-(1-\alpha_1)(1-\beta_1)} \\ &\quad \times \left[\frac{\alpha_1+\alpha_2-1}{\alpha_2-(1-\alpha_1)(1-\beta_1)}\right]^{(\alpha_1+\alpha_2-1)/\beta_1(1-\alpha_1)}.\end{aligned}$$

Take $\omega = \frac{2\alpha_2-(1-\alpha_1)(2-\beta_1)}{\alpha_2-(1-\alpha_1)(1-\beta_1)}$, then $1 < \omega < 2$. It follows from Lemma 3.3 that

$$\begin{aligned}\max_{t\in[a,b]}\tilde{f}(t) &= (b-a)^{2(\alpha_1+\alpha_2-1)+\beta_1(1-\alpha_1)}(2-\omega)(\omega-1)^{(\omega-1)/(2-\omega)} \\ &\le (b-a)^{2(\alpha_1+\alpha_2-1)+\beta_1(1-\alpha_1)}\frac{(\omega-1)^{(\omega-1)}}{\omega^\omega} \\ &= \left\{\frac{(\alpha_1+\alpha_2-1)^{(\alpha_1+\alpha_2-1)}[\alpha_2-(1-\alpha_1)(1-\beta_1)]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{[2\alpha_2-(1-\alpha_1)(2-\beta_1)]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}}\right\}^{\frac{1}{\alpha_2-(1-\alpha_1)(1-\beta_1)}} \\ &\quad \times (b-a)^{2(\alpha_1+\alpha_2-1)+\beta_1(1-\alpha_1)} \\ &\le (b-a)^{2(\alpha_1+\alpha_2-1)+\beta_1(1-\alpha_1)}(\alpha_1+\alpha_2-1)^{(\alpha_1+\alpha_2-1)} \\ &\quad \times \frac{[\alpha_2-(1-\alpha_1)(1-\beta_1)]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{[2\alpha_2-(1-\alpha_1)(2-\beta_1)]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}} \\ &= \max_{t\in[a,b]} f(t).\end{aligned}$$

From the above we get

$$\begin{aligned}\big|k_1(t,s)\big| &\le \max_{t\in[a,b]} k_1(t,t) = \max_{s\in[a,b]} k_2(s,s) \\ &= \big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}(\alpha_1+\alpha_2-1)^{\alpha_1+\alpha_2-1} \\ &\quad \times \left[\frac{b-a}{2\alpha_2-(1-\alpha_1)(2-\beta_1)}\right]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|K(t,s)\right| &\le \frac{\max_{t\in[a,b]} k_1(t,t)}{\Gamma(\alpha_1+\alpha_2)(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}} \\ &= \frac{[(b-a)(\alpha_1+\alpha_2-1)]^{\alpha_1+\alpha_2-1}[\alpha_2-(1-\alpha_1)(1-\beta_1)]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Gamma(\alpha_1+\alpha_2)[2\alpha_2-(1-\alpha_1)(2-\beta_1)]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}}.\end{aligned}$$

To prove (iii), for $(t,s)\in[a,b]\times[a,b]$, obviously, we have that the following inequalities hold:

$$0\le h_2(t,s)\le h_2(s,s)=h_1(s,s).$$

Differentiating $h_1(t,s)$ with respect to t, we have

$$\begin{aligned}\frac{\partial h_1(t,s)}{\partial t} &= (\alpha_1+\alpha_2-1)\big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big] \\ &\quad\times (b-a)^{2-(\alpha_1+\alpha_2)-\beta_1(1-\alpha_1)}(t-a)^{\alpha_2-1-(1-\alpha_1)(1-\beta_1)} \\ &\quad-\big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big](\alpha_1+\alpha_2-1)\frac{(t-s)^{\alpha_1+\alpha_2-2}}{(b-s)^{\alpha_1+\alpha_2-2}} \\ &= (\alpha_1+\alpha_2-1)\big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big] \\ &\quad\times\left[-\left(\frac{b-s}{t-s}\right)^{2-(\alpha_1+\alpha_2)}+\left(\frac{b-a}{t-a}\right)^{1-\alpha_2+(1-\beta_1)(1-\alpha_1)}\right] \\ &\le 0.\end{aligned}$$

Hence, $h_1(t,s)$ is a decreasing function of $t\in[s,b]$, which implies that

$$\begin{aligned}&h_1(b,s)\le h_1(t,s)\le h_1(s,s)=h_2(s,s), \\ &h_1(s,s)=(\alpha_1+\alpha_2-1)(b-a)^{2-(\alpha_1+\alpha_2)-\beta_1(1-\alpha_1)}(s-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)} \\ &\qquad\quad\le(\alpha_1+\alpha_2-1)(b-a), \\ &h_1(b,s)=(\alpha_1+\alpha_2-1)(b-a)-\big[\alpha_2-(1-\alpha_1)(1-\beta_1)\big](b-s).\end{aligned}\tag{3.14}$$

It is trivial to show that $h_1(t,s)$ is an increasing function with respect to $s\in[a,b]$. Thus,

$$h_1(b,a)\le h_1(b,s)\le h_1(b,b).$$

Note that

$$\begin{aligned}&h_1(b,b)=(\alpha_1+\alpha_2-1)(b-a)>0, \\ &h_1(b,a)=-\beta_1(1-\alpha_1)(b-a)<0.\end{aligned}$$

We get

$$\left|h_1(b,s)\right|\le(b-a)\max\big\{\alpha_1+\alpha_2-1,\beta_1(1-\alpha_1)\big\}.\tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\left|h_1(t,s)\right| \le (b-a)\max\left\{\alpha_1+\alpha_2-1, \beta_1(1-\alpha_1)\right\}.$$

Therefore,

$$\left|H_1(t,s)\right| \le (b-a)\max\left\{\alpha_1+\alpha_2-1, \beta_1(1-\alpha_1)\right\}.$$

Then we complete the proof of Lemma 3.4. □

Remark 3.1 From the proof of Lemma 3.4, we have the following conclusions:

(i) $K(t,s)$ has a unique maximum, given by

$$\begin{aligned}
&\max_{(t,s)\in[a,b]^2}\left|K(t,s)\right| \\
&\quad = K\left(t^*,t^*\right) \\
&\quad = \frac{[(b-a)(\alpha_1+\alpha_2-1)]^{\alpha_1+\alpha_2-1}[\alpha_2-(1-\alpha_1)(1-\beta_1)]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Gamma(\alpha_1+\alpha_2)[2\alpha_2-(1-\alpha_1)(2-\beta_1)]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}},
\end{aligned}$$

where t^* is defined by (3.11);

(ii) $\max_{(t,s)\in[a,b]^2}|H_1(t,s)| = (b-a)\max\{\alpha_1+\alpha_2-1, \beta_1(1-\alpha_1)\}$ and

$$\left|H_1(t,s)\right| = \begin{cases} (b-a)(\alpha_1+\alpha_2-1), & \text{if and only if } t=s=b, \\ (b-a)\beta_1(1-\alpha_1), & \text{if and only if } t=b, s=a. \end{cases}$$

Theorem 3.1 *Assume that* (A_1) *holds. If the fractional BVP* (1.14), (1.15) *has a nontrivial continuous solution for a real-valued continuous function q, then*

$$\begin{aligned}
&\int_a^b \left|q(s)\right| ds \\
&\quad > \frac{\Gamma(\alpha_1+\alpha_2)[2\alpha_2-(1-\alpha_1)(2-\beta_1)]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}\Delta_1}{(b-a)^{\alpha_1+\alpha_2-1}(\alpha_1+\alpha_2-1)^{\alpha_1+\alpha_2-1}[\alpha_2-(1-\alpha_1)(1-\beta_1)]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}\tilde{\Delta}_1},
\end{aligned} \tag{3.16}$$

where

$$\tilde{\Delta}_1 := \Delta_1 + \sum_{i=1}^{m-2}\sigma_i(b-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}.$$

Proof Assume $x(t)$ is a nontrivial solution of BVP (1.14), (1.15), then

$$x(t) = \int_a^b K(t,s)q(s)x(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_1}\int_a^b \sum_{i=1}^{m-2}\sigma_i K(\xi_i,s)q(s)x(s)\,ds,$$

and

$$\begin{aligned}|x(t)| &\leq \int_a^b |K(t,s)|\cdot|q(s)|\cdot|x(s)|\,ds \\ &\quad + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_1}\int_a^b \sum_{i=1}^{m-2}\sigma_i|K(\xi_i,s)|\cdot|q(s)|\cdot|x(s)|\,ds, \quad t\in[a,b].\end{aligned}$$

Since x is a nontrivial solution, which will require $q(s)\not\equiv 0$ on $[a,b]$. Moreover, from $q(s)\in C[a,b]$, we obtain that there exists an interval $[a_1,b_1]\subset[a,b]$ such that $|q(s)|>0$ on $[a_1,b_1]$. Then, by Lemma 3.4 and Remark 3.1, we have

$$\begin{aligned}\|x\|_\infty &< \frac{[(b-a)(\alpha_1+\alpha_2-1)]^{\alpha_1+\alpha_2-1}[\alpha_2-(1-\alpha_1)(1-\beta_1)]^{\alpha_2-(1-\alpha_1)(1-\beta_1)}\tilde{\Delta}_1}{\Gamma(\alpha_1+\alpha_2)[2\alpha_2-(1-\alpha_1)(2-\beta_1)]^{2\alpha_2-(1-\alpha_1)(2-\beta_1)}\Delta_1} \\ &\quad\times\int_a^b |q(s)|\,ds\|x\|_\infty.\end{aligned}$$

Thus, inequality (3.16) holds. This completes the proof of Theorem 3.1. □

Theorem 3.2 *Assume that* (A_2) *holds. If the fractional BVP* (1.14), (1.16) *has a nontrivial continuous solution for a real-valued continuous function* q, *then*

$$\int_a^b (b-s)^{\alpha_1+\alpha_2-2}|q(s)|\,ds > \frac{\Gamma(\alpha_1+\alpha_2)[\alpha_2-(1-\alpha_1)(1-\beta_1)]\Delta_2}{\max\{\alpha_1+\alpha_2-1,\beta_1(1-\alpha_1)\}\tilde{\Delta}_2}, \tag{3.17}$$

where

$$\tilde{\Delta}_2 := \Delta_2(b-a)+\sum_{i=1}^{m-2}\delta_i(b-a)^{\alpha_1+\alpha_2+\beta_1(1-\alpha_1)}.$$

Proof Assume that $x(t)$ is a nontrivial solution of BVP (1.14), (1.16), then

$$\begin{aligned}x(t) &= \int_a^b H(t,s)q(s)x(s)\,ds + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}}{\Delta_2}\int_a^b\sum_{i=1}^{m-2}\delta_i H(\eta_i,s)q(s)x(s)\,ds \\ &= \frac{\int_a^b (b-s)^{\alpha_1+\alpha_2-2}H_1(t,s)q(s)x(s)\,ds}{\Gamma(\alpha_1+\alpha_2)[\alpha_2-(1-\alpha_1)(1-\beta_1)]} \\ &\quad + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}\int_a^b\sum_{i=1}^{m-2}\delta_i(b-s)^{\alpha_1+\alpha_2-2}H_1(\eta_i,s)q(s)x(s)\,ds}{\Delta_2\Gamma(\alpha_1+\alpha_2)[\alpha_2-(1-\alpha_1)(1-\beta_1)]},\end{aligned}$$

and

$$\begin{aligned}|x(t)| &\leq \frac{\int_a^b (b-s)^{\alpha_1+\alpha_2-2}|H_1(t,s)|\cdot|q(s)|\cdot|x(s)|\,ds}{\Gamma(\alpha_1+\alpha_2)[\alpha_2-(1-\alpha_1)(1-\beta_1)]} \\ &\quad + \frac{(t-a)^{\alpha_2-(1-\alpha_1)(1-\beta_1)}\int_a^b\sum_{i=1}^{m-2}\delta_i(b-s)^{\alpha_1+\alpha_2-2}|H_1(\eta_i,s)|\cdot|q(s)|\cdot|x(s)|\,ds}{\Delta_2\Gamma(\alpha_1+\alpha_2)[\alpha_2-(1-\alpha_1)(1-\beta_1)]}.\end{aligned}$$

An argument similar to the one used in Theorem 3.1 shows that there exists an interval $[a_2,b_2]\subset[a,b]$ such that $|q(s)|>0$ on $[a_2,b_2]$. Now, applying Lemma 3.4 and Remark 3.1, we have

$$\|x\|_\infty < \frac{\max\{\alpha_1+\alpha_2-1, \beta_1(1-\alpha_1)\}\tilde{\Delta}_2}{\Gamma(\alpha_1+\alpha_2)[\alpha_2-(1-\alpha_1)(1-\beta_1)]\Delta_2}\int_a^b (b-s)^{\alpha_1+\alpha_2-2}\bigl|q(s)\bigr|\,ds\|x\|_\infty,$$

from which inequality in (3.17) follows. The proof is completed. □

Theorem 3.1 gives the following corollaries.

Corollary 3.1 *The necessary condition for the existence of a nontrivial solution for BVP* (1.1) *is*

$$\int_a^b \bigl|q(s)\bigr|\,ds > \frac{4}{(b-a)}. \tag{3.18}$$

Proof Apply Theorem 3.1 for $\alpha_1 = \alpha_2 = 1$, $\sigma_i = 0$ ($i = 1,2,\dots,m-2$), then (3.18) holds. Obviously, (3.18) coincides with the classical Lyapunov inequality, i.e., inequality (1.2). □

Corollary 3.2 *The necessary condition for the existence of a nontrivial solution for BVP* (1.5) *of case* (i) *is*

$$\int_a^b \bigl|q(s)\bigr|\,ds > \frac{\Gamma(\alpha+\beta)2^{2(\alpha+\beta-1)}}{(b-a)^{\alpha+\beta-1}}. \tag{3.19}$$

Proof Apply Theorem 3.1 for $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\beta_1 = \beta_2 = 0$, $\sigma_i = 0$ ($i = 1,2,\dots,m-2$), then (3.19) holds. Obviously, (3.19) coincides with inequality (1.6). □

Corollary 3.3 *The necessary condition for the existence of a nontrivial solution for BVP* (1.3) *is*

$$\int_a^b \bigl|q(s)\bigr|\,ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{3.20}$$

Proof Apply Theorem 3.1 for $\alpha_1 = 1$, $\beta_2 = 0$, $\alpha = 1+\alpha_2$, $\sigma_i = 0$ ($i = 1,2,\dots,m-2$), then (3.20) holds. Obviously, (3.20) coincides with inequality (1.4). □

Corollary 3.4 *The necessary condition for the existence of a nontrivial solution for BVP* (1.5) *of case* (ii) *is*

$$\int_a^b \bigl|q(s)\bigr|\,ds > \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta}. \tag{3.21}$$

Proof Apply Theorem 3.1 for $\alpha_1 = \alpha, \alpha_2 = \beta, \beta_1 = \beta_2 = 1, \sigma_i = 0$ ($i = 1,2,\dots,m-2$), then (3.21) holds, which coincides with inequality (1.7). □

Corollary 3.5 *Consider the following fractional BVP*:

$$\begin{cases} {}^C D_{a+}^\alpha x(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x(b) = 0, \end{cases} \tag{3.22}$$

where $q \in C([a,b],\mathbb{R})$, ${}^{C}D_{a+}^{\alpha}$ is the left Caputo fractional derivative. If (3.22) *has a nontrivial continuous solution in* $[a,b]$, *then*

$$\int_a^b |q(s)|\,ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(b-a)(\alpha-1)]^{\alpha-1}}. \tag{3.23}$$

Proof Apply Theorem 3.1 for $\alpha_2 = 1$, $\beta_1 = 1$, $\alpha = \alpha_1 + 1$, $\sigma_i = 0$ ($i = 1,2,\ldots,m-2T$, then (3.23) holds. Corollary 3.5 coincides with [14, Theorem 1]. □

Corollary 3.6 *The necessary condition for the existence of a nontrivial solution for BVP* (1.8), (1.9) *is*

$$\int_a^b |q(s)|\,ds > \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha-1}(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}. \tag{3.24}$$

Proof Apply Theorem 3.1 for $\alpha_2 = 1$, $\alpha = \alpha_1 + 1$, $\beta = \beta_1$, $\sigma_i = 0$ ($i = 1,2,\ldots,m-2$), then (3.24) holds, which coincides with inequality (1.11). By Remark 3.1, we show that the non-strict inequality (1.11) can be replaced by strict inequality (3.24). □

Corollary 3.7 *Assume that the following boundary value problem*

$$\begin{cases} D_{a+}^{\alpha,\beta}x(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = 0, & x(b) = \sum_{i=1}^{m-2}\sigma_i x(\xi_i), \end{cases}$$

where $q \in C([a,b],\mathbb{R})$, $D_{a+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type $\beta \in [0,1]$, has a nontrivial continuous solution in $[a,b]$, *then*

$$\int_a^b |q(s)|\,ds > \frac{\Gamma(\alpha)[2-(2-\alpha)(2-\beta)]^{2-(2-\alpha)(2-\beta)}\Delta_{11}}{(b-a)^{\alpha-1}(\alpha-1)^{\alpha-1}[1-(2-\alpha)(1-\beta)]^{1-(2-\alpha)(1-\beta)}\tilde{\Delta}_{11}}, \tag{3.25}$$

where

$$\Delta_{11} := (b-a)^{1-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\sigma_i(\xi_i - a)^{1-(2-\alpha)(1-\beta)},$$

$$\tilde{\Delta}_{11} := \Delta_{11} + \sum_{i=1}^{m-2}\sigma_i(b-a)^{1-(2-\alpha)(1-\beta)}.$$

Proof Apply Theorem 3.1 for $\alpha_2 = 1$, $\alpha = \alpha_1 + 1$, $\beta = \beta_1$, then (3.25) holds, which coincides with [30, Theorem 3.1]. □

Theorem 3.2 gives the following corollaries.

Corollary 3.8 *If a nontrivial continuous solution of the following boundary value problem*

$$\begin{cases} x''(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x'(b) = 0, \end{cases}$$

exists, where $q \in C([a,b],\mathbb{R})$, *then*

$$\int_a^b |q(s)|\,ds > \frac{1}{b-a}. \tag{3.26}$$

Proof Apply Theorem 3.2 for $\alpha_1 = \alpha_2 = 1$, $\delta_i = 0$ $(i = 1,2,\ldots,m-2)$, then (3.26) holds, which coincides with [16, Corollary 5]. □

Corollary 3.9 *Suppose that the following boundary value problem*

$$\begin{cases} D_{a+}^{\alpha_1} D_{a+}^{\alpha_2} x(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x'(b) = 0, \end{cases}$$

where $q \in C([a,b],\mathbb{R})$, $D_{a+}^{(\cdot)}$ *is the left Riemann–Liouville fractional derivative, has a nontrivial continuous solution in* $[a,b]$, *then*

$$\int_a^b (b-s)^{\alpha_1+\alpha_2-2} |q(s)|\,ds > \frac{\Gamma(\alpha_1+\alpha_2)}{b-a}. \tag{3.27}$$

Proof Apply Theorem 3.2 for $\beta_1 = \beta_2 = 0$, $\delta_i = 0$ $(i = 1,2,\ldots,m-2)$, then (3.27) holds. □

Corollary 3.10 *Consider the following fractional BVP*:

$$\begin{cases} D_{a+}^{\alpha} x(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x'(b) = 0, \end{cases} \tag{3.28}$$

where $q \in C([a,b],\mathbb{R})$, D_{a+}^{α} *is the Riemann–Liouville fractional derivative of fractional order* α. *If* (3.28) *has a nontrivial continuous solution in* $[a,b]$, *then*

$$\int_a^b (b-s)^{\alpha-2} |q(s)|\,ds > \frac{\Gamma(\alpha)}{b-a}. \tag{3.29}$$

Proof Apply Theorem 3.2 for $\alpha_1 = 1$, $\beta_2 = 0$, $\alpha = 1 + \alpha_2$, $\delta_i = 0$ $(i = 1,2,\ldots,m-2)$, then (3.29) holds. □

Corollary 3.11 *Consider the following fractional BVP*:

$$\begin{cases} {}^{C}D_{a+}^{\alpha_1}\, {}^{C}D_{a+}^{\alpha_2} x(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x'(b) = 0, \end{cases} \tag{3.30}$$

where $q \in C([a,b],\mathbb{R})$, ${}^{C}D_{a+}^{(\cdot)}$ *is the left Caputo fractional derivative. If* (3.30) *has a nontrivial continuous solution in* $[a,b]$, *then*

$$\int_a^b (b-s)^{\alpha_1+\alpha_2-2} |q(s)|\,ds > \frac{\Gamma(\alpha_1+\alpha_2)\alpha_2}{(b-a)\max\{\alpha_1+\alpha_2-1, 1-\alpha_1\}}. \tag{3.31}$$

Proof Apply Theorem 3.2 for $\beta_1 = \beta_2 = 1$, $\delta_i = 0$ $(i = 1,2,\ldots,m-2)$, then (3.31) holds. □

Corollary 3.12 *Consider the following fractional BVP*:

$$\begin{cases} {}^{C}D_{a+}^{\alpha}x(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x'(b) = 0, \end{cases} \tag{3.32}$$

where $q \in C([a,b],\mathbb{R})$, ${}^{C}D_{a+}^{\alpha}$ *is the left Caputo fractional derivative of order* α. *If* (3.32) *has a nontrivial continuous solution in* $[a,b]$, *then*

$$\int_a^b (b-s)^{\alpha-2}\bigl|q(s)\bigr|\,ds > \frac{\Gamma(\alpha)}{(b-a)\max\{\alpha-1, 2-\alpha\}}. \tag{3.33}$$

Proof Apply Theorem 3.2 for $\alpha_2 = 1, \beta_1 = 1, \alpha = \alpha_1 + 1, \delta_i = 0$ $(i = 1,2,\ldots,m-2)$, then (3.33) holds, which coincides with [17] and [23, Theorem 3]. □

Corollary 3.13 *The necessary condition for the existence of a nontrivial solution for BVP* (1.8), (1.10) *is*

$$\int_a^b (b-s)^{\alpha-2}\bigl|q(s)\bigr|\,ds > \frac{\Gamma(\alpha)[\alpha-1+\beta(2-\alpha)]}{(b-a)\max\{\alpha-1, \beta(2-\alpha)\}}, \tag{3.34}$$

Proof Apply Theorem 3.2 for $\alpha_2 = 1, \alpha = \alpha_1 + 1, \beta = \beta_1, \delta_i = 0$ $(i = 1,2,\ldots,m-2)$, then (3.34) holds, which coincides with inequality (1.12). By Remark 3.1, we show that the non-strict inequality (1.12) can be replaced by strict inequality (3.34). □

Corollary 3.14 *If a nontrivial continuous solution of the following fractional boundary value problem*

$$\begin{cases} D_{a+}^{\alpha,\beta}x(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = 0, \qquad x'(b) = \sum_{i=1}^{m-2}\delta_i x(\eta_i), \end{cases}$$

exists, $q \in C([a,b],\mathbb{R})$, $D_{a+}^{\alpha,\beta}$ *is the Hilfer fractional derivative of order* α *and type* $\beta \in [0,1]$, *then*

$$\int_a^b (b-s)^{\alpha-2}\bigl|q(s)\bigr|\,ds > \frac{\Gamma(\alpha)[\alpha-1+\beta(2-\alpha)]\Delta_{22}}{\max\{\alpha-1, \beta(2-\alpha)\}\tilde{\Delta}_{22}}, \tag{3.35}$$

where

$$\Delta_{22} := \bigl[\alpha-1+\beta(2-\alpha)\bigr](b-a)^{-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\delta_i(\eta_i-a)^{\alpha-1+\beta(2-\alpha)},$$

$$\tilde{\Delta}_{22} := \Delta_2(b-a) + \sum_{i=1}^{m-2}\delta_i(b-a)^{\alpha+\beta(2-\alpha)}.$$

Proof Apply Theorem 3.2 for $\alpha_2 = 1, \alpha = \alpha_1 + 1, \beta = \beta_1$, then (3.35) holds. □

4 Conclusion

In this paper, the Lyapunov-type inequalities of sequential Hilfer fractional BVPs were investigated for the first time. Since the Hilfer fractional derivative is a generalization of both Riemann–Liouville and Caputo types fractional derivatives, this requires that our results can be reduced to the corresponding classical results, and we do it. So our work is meaningful and the results we obtained are more general. There is some work to be done in the future such as: finding Lyapunov-type inequalities for higher order Hilfer fractional BVPs; studying Lyapunov-type inequalities for Hilfer fractional p-Laplacian equation, and so on.

Acknowledgements
The authors wish to express their sincere appreciation to the editor and two anonymous referees for valuable comments and suggestions.

Funding
This research is supported by the National Natural Science Foundation of China (No. 11271364).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made equal contributions to each part of this paper. All the authors read and approved the final manuscript.

Publisher's Note

Received: 25 January 2019 Accepted: 1 April 2019 Published online: 11 April 2019

References

1. Lyapunov, A.M.: Problème général de la stabilité du mouvement. Ann. Fac. Sci. Univ. Tolouse **2**, 203–407 (1907)
2. Agarwal, R.P., Çetin, E., Özbekler, A.: Lyapunov type inequalities for second-order forced dynamic equations with mixed nonlinearities on time scales. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **111**, 231–246 (2017)
3. Agarwal, R.P., Özbekler, A.: Lyapunov type inequalities for nth order forced differential equations with mixed nonlinearities. Commun. Pure Appl. Anal. **15**, 2281–2300 (2016)
4. Çakmak, D., Tiryaki, A.: On Lyapunov-type inequality for quasilinear systems. Appl. Math. Comput. **216**, 3584–3591 (2010)
5. Yang, X.J., Kim, Y.I., Lo, K.M.: Lyapunov-type inequalities for a class of higher-order linear differential equations. Appl. Math. Lett. **34**, 86–89 (2014)
6. Agarwal, R.P., Özbekler, A.: Lyapunov type inequalities for mixed nonlinear Riemann–Liouville fractional differential equations with a forcing term. J. Comput. Appl. Math. **314**, 69–78 (2017)
7. Al Arifi, N., Altun, I., Jleli, M., Lashin, A., Samet, B.: Lyapunov-type inequalities for a fractional p-Laplacian equation. J. Inequal. Appl. **2016**, 189 (2016)
8. Cabrera, I.J., Rocha, J., Sadarangani, K.B.: Lyapunov type inequalities for a fractional thermostat model. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**, 17–24 (2018)
9. Chidouh, A., Torres, D.F.M.: A generalized Lyapunov's inequality for a fractional boundary value problem. J. Comput. Appl. Math. **312**, 192–197 (2017)
10. Dhar, S., Kong, Q.K., McCabe, M.: Fractional boundary value problems and Lyapunov-type inequalities with fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2016**, 43 (2016)
11. Ferreira, R.A.C.: A Lyapunov-type inequality for a fractional boundary value problem. Fract. Calc. Appl. Anal. **16**, 978–984 (2013)
12. Ferreira, R.A.C.: Lyapunov-type inequalities for some sequential fractional boundary value problems. Adv. Dyn. Syst. Appl. **11**, 33–43 (2016)
13. Ferreira, R.A.C.: Lyapunov-type inequality for an anti-periodic fractional boundary value problem. Fract. Calc. Appl. Anal. **20**, 284–291 (2017)
14. Ferreira, R.A.C.: On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function. J. Math. Anal. Appl. **412**, 1058–1063 (2014)
15. Ferreira, R.A.C.: Novel Lyapunov-type inequalities for sequential fractional boundary value problems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**, 171–179 (2019)
16. Guezane-Lakoud, A., Khaldi, R., Torres, D.F.M.: Lyapunov-type inequality for a fractional boundary value problem with natural conditions. SeMA J. **75**, 157–162 (2018)

17. Jleli, M., Samet, B.: Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions. Math. Inequal. Appl. **18**, 443–451 (2015)
18. Liu, Y., Xie, D.P., Yang, D.D., Bai, C.Z.: Two generalized Lyapunov-type inequalities for a fractional *p*-Laplacian equation with fractional boundary conditions. J. Inequal. Appl. **2017**, 98 (2017)
19. López, B., Rocha, J., Sadarangani, K.: Lyapunov type inequality for a nonlinear fractional hybrid boundary value problem. Z. Anal. Anwend. **38**, 97–106 (2019)
20. López, B., Rocha, J., Sadarangani, K.: Lyapunov-type inequalities for a class of fractional boundary value problems with integral boundary conditions. Math. Methods Appl. Sci. **42**, 49–58 (2019)
21. Łupińska, B., Odzijewicz, T.: A Lyapunov-type inequality with the Katugampola fractional derivative. Math. Methods Appl. Sci. **41**, 1–12 (2018)
22. Ma, D.X.: A generalized Lyapunov inequality for a higher-order fractional boundary value problem. J. Inequal. Appl. **2016**, 261 (2016)
23. Ntouyas, S.K., Ahmad, B., Horikis, T.P.: Recent developments of Lyapunov-type inequalities for fractional differential equations (2018). arXiv:1804.10760
24. O'Regan, D., Samet, B.: Lyapunov-type inequalities for a class of fractional differential equations. J. Inequal. Appl. **2015**, 247 (2015)
25. Pathak, N.: Lyapunov-type inequality for fractional boundary value problems with Hilfer derivative. Math. Inequal. Appl. **21**, 179–200 (2018)
26. Rong, J., Bai, C.Z.: Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions. Adv. Differ. Equ. **2015**, 82 (2015)
27. Samet, B., Aydi, H.: Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving Ψ-Caputo fractional derivative. J. Inequal. Appl. **2018**, 286 (2018)
28. Wang, Y.Y., Liang, S.L., Xia, C.X.: A Lyapunov-type inequality for a fractional differential equation under Sturm–Liouville boundary conditions. Math. Inequal. Appl. **20**, 139–148 (2017)
29. Wang, Y.Y., Wang, Q.C.: Lyapunov-type inequalities for nonlinear differential equation with Hilfer fractional derivative operator. J. Math. Inequal. **12**, 709–717 (2018)
30. Wang, Y.Y., Wang, Q.C.: Lyapunov-type inequalities for nonlinear fractional differential equation with Hilfer fractional derivative under multi-point boundary conditions. Fract. Calc. Appl. Anal. **21**, 833–843 (2018)
31. Hilfer, R.: Fractional Calculus and Regular Variation in Thermodynamics. Applications of Fractional Calculus in Physics. World Scientific, River Edge (2000)
32. Hilfer, R., Luchko, Y., Tomovski, Ž.: Operational method for the solution of fractional differential equations with generalized Riemann–Liouville fractional derivatives. Fract. Calc. Appl. Anal. **12**, 299–318 (2009)
33. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, p. 71, 75. Elsevier, Amsterdam (2006)
34. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus. Models and Numerical Methods, 2nd edn. Series on Complexity, Nonlinearity and Chaos, vol. 5, p. 13. World Scientific, Hackensack (2017)
35. Podlubny, I.: Fractional Differential Equations. an Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, vol. 198, p. 70, 72, 73, 81. Academic Press, San Diego (1999)
36. Tomovski, Ž.: Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator. Nonlinear Anal. **75**(7), 3364–3384 (2012)