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Hermite–Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications

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Abstract

In this paper, we introduce the notion of exponentially p-convex function and exponentially s-convex function in the second sense. We establish several Hermite–Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in second sense. The present investigation is an extension of several well known results.

MSC: 26A51; 26D15

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1 Introduction

Recently, the study of convex functions has become more important due to variety of their nature. Many generalizations of this notion have been established. For more details see [1–6, 13, 16–19].

Convex functions satisfy many integral inequalities. Among these, the Hermite–Hadamard inequality is well known. The Hermite–Hadamard inequality [14, 15] for a convex function $\psi : \mathcal{K} \to \mathbb{R}$ on an interval $\mathcal{K}$ is

$$\psi\left(\frac{u_1+u_2}{2}\right) \le \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \psi(w)\,dw \le \frac{\psi(u_1)+\psi(u_2)}{2}, \tag{1.1}$$

for all $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$. Many authors have made generalizations to inequality (1.1). For more results and details, see [3, 4, 6, 7, 17–19, 22, 24–26].

Definition 1.1 ([19, 20]) Consider an interval $\mathcal{K} \subset (0,\infty) = \mathbb{R}_+$, and $p \in \mathbb{R}\backslash\{0\}$. A function $\psi : \mathcal{K} \to \mathbb{R}$ is called p-convex, if

$$\psi\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right) \le r\psi(u_1) + (1-r)\psi(u_2), \tag{1.2}$$

for all $u_1, u_2 \in \mathcal{K}$ and $r \in [0,1]$. If the inequality in (1.2) is reversed, then ψ is called p-concave.

Example 1.1 A function $\psi : (0,\infty) \to \mathbb{R}$, defined by $\psi(u) = u^p$ for $p \in \mathbb{R}\backslash\{0\}$, is p-convex as well as p-concave.

Iscan [19] gave the following results.

Theorem 1.2 ([19]) *Consider an interval $\mathcal{K} \subset (0,\infty)$, and $p \in \mathbb{R}\backslash\{0\}$. Let $\psi : \mathcal{K} \to \mathbb{R}$ be p-convex and $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$. If $\psi \in L_1[u_1, u_2]$, then we have*

$$\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}}\,dw \le \frac{\psi(u_1) + \psi(u_2)}{2}. \tag{1.3}$$

Lemma 1.1 ([19]) *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$, i.e., the interior of $\mathcal{K}$, and $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, and $p \in \mathbb{R}\backslash\{0\}$. If $\psi' \in L_1[u_1, u_2]$, then*

$$\begin{aligned} &\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}}\,dw \\ &\quad = \frac{u_2^p - u_1^p}{2p}\int_0^1 \frac{1-2r}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\psi'\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right)dr. \end{aligned} \tag{1.4}$$

Definition 1.2 ([16]) Let $s \in (0,1]$. A function $\psi : \mathcal{K} \subset \mathbb{R}_0 \to \mathbb{R}_0$, where $\mathbb{R}_0 = [0,\infty)$, is called s-convex in the second sense, if

$$\psi\left(ru_1 + (1-r)u_2\right) \le r^s\psi(u_1) + (1-r)^s\psi(u_2), \tag{1.5}$$

for all $u_1, u_2 \in \mathcal{K}$ and $r \in [0,1]$.

Example 1.3 A function $\psi : (0,\infty) \to (0,\infty)$, defined by $\psi(u) = u^s$ for $s \in (0,1)$, is s-convex in the second sense.

Dragomir et al. [8, 9] gave the following important results.

Theorem 1.4 ([9]) *Let $s \in (0,1)$ and $\psi : \mathbb{R}_0 \to \mathbb{R}_0$ be s-convex in the second sense. Let $u_1, u_2 \in [0,\infty)$, $u_1 \le u_2$. If $\psi \in L_1[u_1, u_2]$, then*

$$2^{s-1}\psi\left(\frac{u_1 + u_2}{2}\right) \le \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \psi(w)\,dw \le \frac{\psi(u_1) + \psi(u_2)}{s+1}. \tag{1.6}$$

Lemma 1.2 ([8]) *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{K}^\circ$, the interior of $\mathcal{K}$, and $u_1, u_2 \in \mathcal{K}$ be two distinct points. If $\psi' \in L_1[u_1, u_2]$, then*

$$\begin{aligned} &\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \psi(w)\,dw \\ &\quad = \frac{u_2 - u_1}{2}\int_0^1 (1-2r)\psi'\left(ru_1 + (1-r)u_2\right)dr. \end{aligned} \tag{1.7}$$

Awan et al. [4] introduced the following new class of convex functions.

Definition 1.3 ([4]) A function $\psi : \mathcal{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called exponentially convex, if

$$\psi\left(ru_1 + (1-r)u_2\right) \le r\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{1.8}$$

for all $u_1, u_2 \in \mathcal{K}$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality in (1.8) is reversed, then ψ is called exponentially concave.

Example 1.5 A function $\psi : \mathbb{R} \to \mathbb{R}$, defined by $\psi(u) = -u^2$, is an exponentially convex for all $\alpha > 0$.

The Beta and Hypergeometric functions are defined as:

$$\beta(u_1, u_2) = \int_0^1 w^{u_1-1}(1-w)^{u_2-1}\, dw, \quad u_1, u_2 > 0,$$

and

$${}_2F_1(u_1, u_2; t; z) = \frac{1}{\beta(u_2, t-u_2)} \int_0^1 w^{u_2-1}(1-w)^{t-u_2-1}(1-zw)^{-u_1}\, dw, \quad t > u_2 > 0, |z| < 1,$$

respectively, see [21].

2 Exponentially p-convex functions

Now we introduce exponentially p-convex functions.

Definition 2.1 Consider an interval $\mathcal{K} \subset (0,\infty) = \mathbb{R}_+$ and $p \in \mathbb{R}\backslash\{0\}$. A function $\psi : \mathcal{K} \to \mathbb{R}$ is called exponentially p-convex, if

$$\psi\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right) \le r\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{2.1}$$

for all $u_1, u_2 \in \mathcal{K}$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality in (2.1) is reversed, then ψ is called exponentially p-concave.

It is easy to note that, by taking $\alpha = 0$, an exponentially p-convex function becomes p-convex.

Example 2.1 Consider a function $\psi : (\sqrt{2}, \infty) \to \mathbb{R}$, defined by $\psi(u) = (\ln(u))^p$ for $p \ge 2$. Then ψ is exponentially p-convex for all $\alpha < 0$, and not p-convex.

Note that ψ satisfies inequality (2.1) for all $\alpha < 0$. But for $u_1 = 2$, $u_2 = 3$ and $p = 5$, inequality (1.2) does not hold.

2.1 Integral inequalities

Throughout this section, we denote by $\mathcal{K} \subset (0,\infty) = \mathbb{R}_+$ an interval with interior $\mathcal{K}^\circ$ and $p \in \mathbb{R}\backslash\{0\}$. We start with our results for exponentially p-convex functions.

Theorem 2.2 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be an integrable exponentially p-convex function. Let $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$. Then for $\alpha \in \mathbb{R}$, we have*

$$\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}e^{\alpha w}}\,dw \le A_1(r)\frac{\psi(u_1)}{e^{\alpha u_1}} + A_2(r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{2.2}$$

where

$$A_1(r) = \int_0^1 \frac{r\,dr}{e^{\alpha(ru_1^p+(1-r)u_2^p)^{\frac{1}{p}}}} \quad \text{and} \quad A_2(r) = \int_0^1 \frac{(1-r)\,dr}{e^{\alpha(ru_1^p+(1-r)u_2^p)^{\frac{1}{p}}}}.$$

Proof By using the exponential p-convexity of ψ, we have

$$2\psi\left(\left[\frac{w^p + z^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{\psi(w)}{e^{\alpha w}} + \frac{\psi(z)}{e^{\alpha z}}. \tag{2.3}$$

Letting $w^p = ru_1^p + (1-r)u_2^p$ and $z^p = (1-r)u_1^p + ru_2^p$, we get

$$2\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{\psi([ru_1^p + (1-r)u_2^p]^{\frac{1}{p}})}{e^{\alpha(ru_1^p+(1-r)u_2^p)^{\frac{1}{p}}}} + \frac{\psi([(1-r)u_1^p + ru_2^p]^{\frac{1}{p}})}{e^{\alpha((1-r)u_1^p+ru_2^p)^{\frac{1}{p}}}}. \tag{2.4}$$

Integrating with respect to $r \in [0,1]$ and applying a change of variable, we find

$$\psi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}e^{\alpha w}}\,dw. \tag{2.5}$$

Hence the first inequality of (2.2) has been established. For the next inequality, again using the exponential p-convexity of ψ, we have

$$\frac{\psi([ru_1^p + (1-r)u_2^p]^{\frac{1}{p}})}{e^{\alpha(ru_1^p+(1-r)u_2^p)^{\frac{1}{p}}}} \le \frac{r\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)\frac{\psi(u_2)}{e^{\alpha u_2}}}{e^{\alpha(ru_1^p+(1-r)u_2^p)^{\frac{1}{p}}}}. \tag{2.6}$$

Integrating with respect to $r \in [0,1]$, we get

$$\begin{aligned} &\frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}e^{\alpha w}}\,dw \\ &\quad \le \frac{\psi(u_1)}{e^{\alpha u_1}}\int_0^1 \frac{r\,dr}{e^{\alpha(ru_1^p+(1-r)u_2^p)^{\frac{1}{p}}}} + \frac{\psi(u_2)}{e^{\alpha u_2}}\int_0^1 \frac{(1-r)\,dr}{e^{\alpha(ru_1^p+(1-r)u_2^p)^{\frac{1}{p}}}}. \end{aligned} \tag{2.7}$$

By combining (2.5) and (2.7), we get (2.2). □

Remark 2.1 In Theorem 2.2, by taking $\alpha = 0$, we attain inequality (1.3) in Theorem 1.2.

Theorem 2.3 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\psi' \in L_1[u_1, u_2]$. If $|\psi'|^q$ is exponentially p-convex on $[u_1, u_2]$ for $q \geq 1$ and $\alpha \in \mathbb{R}$, then*

$$\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}}\,dw\right| \\
&\quad \leq \frac{u_2^p - u_1^p}{2p} B_1^{1-\frac{1}{q}} \left[B_2 \left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_3 \left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}},
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
B_1 &= B_1(u_1, u_2; p) = \frac{1}{4}\left(\frac{u_1^p + u_2^p}{2}\right)^{\frac{1}{p}-1} \\
&\quad \times \left[{}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{u_1^p - u_2^p}{u_1^p + u_2^p}\right) + {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{u_2^p - u_1^p}{u_1^p + u_2^p}\right)\right], \\
B_2 &= B_2(u_1, u_2; p) = \frac{1}{24}\left(\frac{u_1^p + u_2^p}{2}\right)^{\frac{1}{p}-1} \left[{}_2F_1\left(1 - \frac{1}{p}, 2; 4; \frac{u_1^p - u_2^p}{u_1^p + u_2^p}\right)\right. \\
&\quad \left. + 6\,{}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{u_2^p - u_1^p}{u_1^p + u_2^p}\right) + {}_2F_1\left(1 - \frac{1}{p}, 2; 4; \frac{u_2^p - u_1^p}{u_1^p + u_2^p}\right)\right], \\
B_3 &= B_3(u_1, u_2; p) = B_1 - B_2.
\end{aligned}$$

Proof Applying the power mean inequality to (1.4) of Lemma 1.1, we get

$$\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}}\,dw\right| \\
&\quad \leq \frac{u_2^p - u_1^p}{2p} \int_0^1 \left|\frac{1-2r}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\right| \left|\psi'\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right)\right|\,dr \\
&\quad \leq \frac{u_2^p - u_1^p}{2p} \left(\int_0^1 \frac{|1-2r|}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\,dr\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \frac{|1-2r|}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}} \left|\psi'\left(\left[ru_1^p + (1-r)u_2^p\right]^{\frac{1}{p}}\right)\right|^q\,dr\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.9}$$

Since $|\psi'|^q$ is exponentially p-convex on $[u_1, u_2]$, we have

$$\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p - u_1^p}\int_{u_1}^{u_2} \frac{\psi(w)}{w^{1-p}}\,dw\right| \\
&\quad \leq \frac{u_2^p - u_1^p}{2p} \left(\int_0^1 \frac{|1-2r|}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\,dr\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \frac{||1-2r||[r|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q + (1-r)|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q]}{[ru_1^p + (1-r)u_2^p]^{1-\frac{1}{p}}}\,dr\right)^{\frac{1}{q}} \\
&\quad \leq \frac{u_2^p - u_1^p}{2p} B_1^{1-\frac{1}{q}} \left[B_2 \left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_3 \left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.10}$$

It is easy to note that

$$\int_0^1 \frac{|1-2r|}{[ru_1^p+(1-r)u_2^p]^{1-\frac{1}{p}}}\,dr = B_1(u_1,u_2;p),$$
$$\int_0^1 \frac{|1-2r|r}{[ru_1^p+(1-r)u_2^p]^{1-\frac{1}{p}}}\,dr = B_2(u_1,u_2;p),$$
$$\int_0^1 \frac{|1-2r|(1-r)}{[ru_1^p+(1-r)u_2^p]^{1-\frac{1}{p}}}\,dr = B_1(u_1,u_2;p)-B_2(u_1,u_2;p).$$

Hence the proof is completed. □

Remark 2.2 In Theorem 2.3,
(a) by taking $\alpha = 0$, we attain Theorem 7 in [19];
(b) by taking $p = 1$, we attain Theorem 5 in [4].

Corollary 2.4 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, and $\psi' \in L_1[u_1,u_2]$. If $|\psi'|$ is exponentially p-convex on $[u_1,u_2]$, then*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \le \frac{u_2^p-u_1^p}{2p}\left[B_2\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + B_3\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right], \tag{2.11}$$

where B_2 and B_3 are given in Theorem 2.3.

Remark 2.3 In Corollary 2.4,
(a) by taking $\alpha = 0$, we attain Corollary 1 in [19];
(b) by taking $p = 1$, we attain Theorem 3 in [4].

Theorem 2.5 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$. Let $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, and $\psi' \in L_1[u_1,u_2]$. If $|\psi'|^q$ is exponentially p-convex on $[u_1,u_2]$, and $q, l > 1$, $1/q + 1/l = 1$, and $\alpha \in \mathbb{R}$, then*

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \le \frac{u_2^p-u_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left[B_4\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_5\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}, \tag{2.12}$$

where

$$B_4 = B_4(u_1,u_2;p;q) = \begin{cases} \frac{1}{2u_1^{qp-q}}\,{}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_2}{u_1})^p), & p<0,\\ \frac{1}{2u_2^{qp-q}}\,{}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_1}{u_2})^p), & p>0,\end{cases}$$

$$
\begin{aligned}
B_5 &= B_5(u_1,u_2;p;q) \\
&= \begin{cases} \frac{1}{2u_1^{qp-q}}\, {}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_2}{u_1})^p), & p<0, \\ \frac{1}{2u_2^{qp-q}}\, {}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_1}{u_2})^p), & p>0. \end{cases}
\end{aligned}
$$

Proof Using Hölder's inequality on (1.4) of Lemma 1.1 and then applying the exponential p-convexity of $|\psi'|^q$ on $[u_1,u_2]$, we get

$$
\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\
&\leq \frac{u_2^p-u_1^p}{2p}\left(\int_0^1 |1-2r|^l\,dr\right)^{\frac{1}{l}} \\
&\quad\times\left(\int_0^1 \frac{1}{[ru_1^p+(1-r)u_2^p]^{q(1-\frac{1}{p})}}\left|\psi'\left(\left[ru_1^p+(1-r)u_2^p\right]^{\frac{1}{p}}\right)\right|^q dr\right)^{\frac{1}{q}} \\
&\leq \frac{u_2^p-u_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left(\int_0^1 \frac{r|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q+(1-r)|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{[ru_1^p+(1-r)u_2^p]^{q-\frac{q}{p}}}\,dr\right)^{\frac{1}{q}} \\
&\leq \frac{u_2^p-u_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left[B_4\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + B_5\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}},
\end{aligned} \tag{2.13}
$$

where after calculations, we have

$$
\begin{aligned}
B_4 &= \int_0^1 \frac{r}{[ru_1^p+(1-r)u_2^p]^{q-\frac{q}{p}}}\,dr \\
&= \begin{cases} \frac{1}{2u_1^{qp-q}}\, {}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_2}{u_1})^p), & p<0, \\ \frac{1}{2u_2^{qp-q}}\, {}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_1}{u_2})^p), & p>0, \end{cases} \\
B_5 &= \int_0^1 \frac{1-r}{[ru_1^p+(1-r)u_2^p]^{q-\frac{q}{p}}}\,dr \\
&= \begin{cases} \frac{1}{2u_1^{qp-q}}\, {}_2F_1(q-\frac{q}{p},2;3;1-(\frac{u_2}{u_1})^p), & p<0, \\ \frac{1}{2u_2^{qp-q}}\, {}_2F_1(q-\frac{q}{p},1;3;1-(\frac{u_1}{u_2})^p), & p>0. \end{cases}
\end{aligned}
$$

□

Remark 2.4 In Theorem 2.5,
(a) by letting $\alpha = 0$, we attain Theorem 8 in [19];
(b) by letting $p = 1$, we attain Theorem 4 in [4].

Theorem 2.6 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1,u_2 \in \mathcal{K}$, $u_1 < u_2$, and $\psi' \in L_1[u_1,u_2]$. If $|\psi'|^q$ is exponentially p-convex on $[u_1,u_2]$, and $q,l > 1$, $1/q+1/l = 1$, and $\alpha \in \mathbb{R}$, then*

$$
\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\
&\leq \frac{u_2^p-u_1^p}{2p}B_6^{\frac{1}{l}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q+|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{2}\right)^{\frac{1}{q}},
\end{aligned} \tag{2.14}
$$

where

$$B_6 = B_6(u_1,u_2;p;l) = \begin{cases} \frac{1}{2u_1^{pl-l}}\,{}_2F_1(l-\frac{l}{p},1;2;1-(\frac{u_2}{u_1})^p), & p<0,\\ \frac{1}{2u_2^{pl-l}}\,{}_2F_1(l-\frac{l}{p},1;2;1-(\frac{u_1}{u_2})^p), & p>0. \end{cases}$$

Proof Using Hölder's inequality on (1.4) of Lemma 1.1 and then applying the exponential p-convexity of $|\psi'|^q$ on $[u_1,u_2]$, we get

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{p}{u_2^p-u_1^p}\int_{u_1}^{u_2}\frac{\psi(w)}{w^{1-p}}\,dw\right| \\ &\leq \frac{u_2^p-u_1^p}{2p}\left(\int_0^1 \frac{1}{[ru_1^p+(1-r)u_2^p]^{l-\frac{l}{p}}}\,dr\right)^{\frac{1}{l}} \\ &\quad\times\left(\int_0^1 |1-2r|^q\left|\psi'\left(\left[ru_1^p+(1-r)u_2^p\right]^{\frac{1}{p}}\right)\right|^q dr\right)^{\frac{1}{q}} \\ &\leq \frac{u_2^p-u_1^p}{2p}B_6^{\frac{1}{l}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q+|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{2}\right)^{\frac{1}{q}}, \end{aligned} \tag{2.15}$$

where a simple calculation implies

$$\begin{aligned} B_6(u_1,u_2;p;l) &= \int_0^1 \frac{1}{[ru_1^p+(1-r)u_1^p]^{l-\frac{l}{p}}}\,dr \\ &= \begin{cases} \frac{1}{2u_1^{pl-l}}\,{}_2F_1(l-\frac{l}{p},1;2;1-(\frac{u_2}{u_1})^p), & p<0,\\ \frac{1}{2u_2^{pl-l}}\,{}_2F_1(l-\frac{l}{p},1;2;1-(\frac{u_1}{u_2})^p), & p>0, \end{cases} \end{aligned} \tag{2.16}$$

and

$$\int_0^1 r|1-2r|^q\,dr = \int_0^1 (1-r)|1-2r|^q\,dr = \frac{1}{2(q+1)}. \tag{2.17}$$

By substituting (2.16) and (2.17) into (2.15), we get (2.14). □

Remark 2.5 In Theorem 2.6, by letting $\alpha=0$, we attain Theorem 9 in [19].

2.2 Applications

Consider some special means of two positive numbers u_1, u_2, $u_1<u_2$:

(1) The arithmetic mean

$$A = A(u_1,u_2) = \frac{u_1+u_2}{2};$$

(2) The harmonic mean

$$H = H(u_1,u_2) = \frac{2u_1u_2}{u_1+u_2};$$

(3) The p-logarithmic mean

$$L_p = L_p(u_1,u_2) = \left(\frac{u_2^{p+1}-u_1^{p+1}}{(p+1)(u_2-u_1)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\setminus\{-1,0\}.$$

In the next three propositions we consider $0 < u_1 < u_2$ and $q > 1$.

Proposition 2.1 *Let $\alpha \in \mathbb{R}$ and $p < 1$. Then we have*

$$\left|L_{p-1}^{p-1} - HL_{p-2}^{p-2}\right| \le \frac{u_2^p - u_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left|\frac{1}{u_1^2 e^{\alpha u_1}}\right|^q, \left|\frac{1}{u_2^2 e^{\alpha u_2}}\right|^q\right) HL_{p-1}^{p-1},$$

where B_6 is defined as in Theorem 2.6.

Proof The proof ensues from Theorem 2.6, for a function $\psi:(0,\infty)\to\mathbb{R}$, $\psi(w)=\frac{1}{w}$. Here note that $|\psi'(w)|^q = |\frac{1}{w^2}|^q$ is exponentially p-convex for all $p<1$ and $\alpha\in\mathbb{R}$. □

Proposition 2.2 *Let $\alpha \le 0$ and $p > 1$. Then we have*

$$\left|L_{p-1}^{p-1} A\left(u_1^p,u_2^p\right) - L_{2p-1}^{2p-1}\right| \le \frac{u_2^p - u_1^p}{2} B_6^{\frac{1}{l}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left|\frac{1}{u_1^{p-1} e^{\alpha u_1}}\right|^q, \left|\frac{1}{u_2^{p-1} e^{\alpha u_2}}\right|^q\right) L_{p-1}^{p-1},$$

where B_6 is defined as in Theorem 2.6.

Proof The proof ensues from Theorem 2.6, for $\psi:(0,\infty)\to\mathbb{R}$, $\psi(w)=w^p$. Here note that $|\psi'(w)|^q = |pw^{p-1}|^q$ is exponentially p-convex for all $p>1$ and $\alpha\le 0$. □

Proposition 2.3 *Let $\alpha \le 0$ and $p > 1$. Then we have*

$$\left|L_{p-1}^{p-1} A - L_p^p\right| \le \frac{u_2^p - u_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left|\frac{1}{e^{\alpha u_1}}\right|^q, \left|\frac{1}{e^{\alpha u_2}}\right|^q\right) L_{p-1}^{p-1},$$

where B_6 is given as in Theorem 2.6.

Proof The proof ensues from Theorem 2.6, for $\psi:(0,\infty)\to\mathbb{R}$, $\psi(w)=w$. Here note that $|\psi'(w)|^q = 1$ is exponentially p-convex for all $p>1$ and $\alpha\le 0$. □

3 Exponentially s-convex functions in the second sense

We first generalize Definition 1.2.

Definition 3.1 Let $s\in(0,1]$ and $\mathcal{K}\subset\mathbb{R}_0$ be an interval. A function $\psi:\mathcal{K}\to\mathbb{R}$ is called exponentially s-convex in the second sense, if

$$\psi\left(ru_1+(1-r)u_2\right) \le r^s \frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)^s \frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{3.1}$$

for all $u_1,u_2\in\mathcal{K}$, $r\in[0,1]$ and $\alpha\in\mathbb{R}$. If the inequality in (3.1) is reversed then ψ is called exponentially s-concave.

Observe that, by taking $\alpha = 0$, an exponentially s-convex function becomes s-convex.

Example 3.1 Consider a function $\psi : [0,\infty) \to \mathbb{R}$, defined by $\psi(u) = \ln(u)$ for $s \in (0,1)$. Then ψ is exponentially s-convex, for all $\alpha \leq -1$, but not s-convex in the second sense.

3.1 Integral inequalities

Throughout this section, we denote by $\mathcal{K} \subset \mathbb{R}_0$ an interval with nonempty interior $\mathcal{K}^\circ$ and $s \in (0,1]$. We start our new results with the following theorem.

Theorem 3.2 *Let $\psi : \mathcal{K} \subset \mathbb{R}_0 \to \mathbb{R}$ be an integrable exponentially s-convex function in the second sense on $\mathcal{K}^\circ$. Then for $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\alpha \in \mathbb{R}$, we have*

$$2^{s-1}\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\frac{\psi(w)}{e^{\alpha w}}\,dw \leq A_3(r)\frac{\psi(u_1)}{e^{\alpha u_1}} + A_4(r)\frac{\psi(u_2)}{e^{\alpha u_2}}, \tag{3.2}$$

where

$$A_3(r) = \int_0^1 \frac{r^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}} \quad and \quad A_4(r) = \int_0^1 \frac{(1-r)^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}}.$$

Proof Applying exponential s-convexity of ψ, we have

$$2^s\psi\left(\frac{w+z}{2}\right) \leq \frac{\psi(w)}{e^{\alpha w}} + \frac{\psi(z)}{e^{\alpha z}}. \tag{3.3}$$

Letting $w = ru_1 + (1-r)u_2$ and $z = (1-r)u_1 + ru_2$, we get

$$2^s\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{\psi(ru_1+(1-r)u_2)}{e^{\alpha(ru_1+(1-r)u_2)}} + \frac{\psi((1-r)u_1+ru_2)}{e^{\alpha((1-r)u_1+ru_2)}}. \tag{3.4}$$

Integrating with respect to $r \in [0,1]$ and applying a change of variable, we find

$$2^{s-1}\psi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\frac{\psi(w)}{e^{\alpha w}}\,dw. \tag{3.5}$$

Hence the proof of the first inequality of (3.2) has been completed. For the next inequality, again using the exponential s-convexity of ψ, we have

$$\frac{\psi(ru_1+(1-r)u_2)}{e^{\alpha(ru_1+(1-r)u_2)}} \leq \frac{r^s\frac{\psi(u_1)}{e^{\alpha u_1}} + (1-r)^s\frac{\psi(u_2)}{e^{\alpha u_2}}}{e^{\alpha(ru_1+(1-r)u_2)}}. \tag{3.6}$$

Integrating with respect to $r \in [0,1]$, we get

$$\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\frac{\psi(w)}{e^{\alpha w}}\,dw \leq \frac{\psi(u_1)}{e^{\alpha u_1}}\int_0^1\frac{r^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}} + \frac{\psi(u_2)}{e^{\alpha u_2}}\int_0^1\frac{(1-r)^s\,dr}{e^{\alpha(ru_1+(1-r)u_2)}}. \tag{3.7}$$

By combining (3.5) and (3.7), we get (3.2). □

Remark 3.1 In Theorem 3.2, by letting $\alpha = 0$, we get inequality (1.6) in Theorem 1.4.

Theorem 3.3 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\psi' \in L_1[u_1, u_2]$. If $|\psi'|$ is exponentially s-convex in the second sense on $[u_1, u_2]$, then*

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\ &\quad\le \frac{u_2-u_1}{2(s+1)(s+2)}\left[(3s+4)\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + (s+4)\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]. \end{aligned} \tag{3.8}$$

Proof From Lemma 1.2, we have

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\ &\quad= \frac{u_2-u_1}{2}\left|\int_0^1 (1-2r)\psi'\bigl(ru_1+(1-r)u_2\bigr)\,dr\right| \\ &\quad\le \frac{u_2-u_1}{2}\int_0^1 |1-2r|\bigl|\psi'\bigl(ru_1+(1-r)u_2\bigr)\bigr|\,dr. \end{aligned} \tag{3.9}$$

Using the exponential s-convexity of ψ', we get

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\ &\quad\le \frac{u_2-u_1}{2}\int_0^1 |1-2r|\left[r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + (1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]dr \\ &\quad\le \frac{u_2-u_1}{2}\int_0^1 (1+2r)\left[r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + (1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]dr \\ &\quad= \frac{u_2-u_1}{2}\int_0^1 \left[(1+2r)r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + (1+2r)(1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]dr. \end{aligned} \tag{3.10}$$

Since

$$\int_0^1 (1+2r)r^s\,dr = \frac{3s+4}{(s+1)(s+2)}, \tag{3.11}$$

$$\int_0^1 (1+2r)(1-r)^s\,dr = \frac{s+4}{(s+1)(s+2)}, \tag{3.12}$$

by substituting equalities (3.11) and (3.12) into (3.10), we get inequality (3.8). □

Corollary 3.4 *Under the assumptions of Theorem 3.3, we have the following:*

(a) *If $s = 1$, then*

$$\begin{aligned} &\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\ &\quad\le \frac{u_2-u_1}{12}\left[7\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + 5\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right]. \end{aligned} \tag{3.13}$$

(b) *If* $\alpha = 0$, *then*

$$\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\
&\quad \le \frac{u_2-u_1}{2(s+1)(s+2)}\bigl[(3s+4)\bigl|\psi'(u_1)\bigr| + (s+4)\bigl|\psi'(u_2)\bigr|\bigr].
\end{aligned} \tag{3.14}$$

Theorem 3.5 *Let* $\psi : \mathcal{K} \to \mathbb{R}$ *be a differentiable function on* $\mathcal{K}^\circ$ *and* $u_1, u_2 \in \mathcal{K}$, $u_1 < u_2$, *and* $\psi' \in L_1[u_1,u_2]$. *If* $|\psi'|$ *is exponentially* s*-convex in the second sense on* $[u_1,u_2]$, *then*

$$\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\
&\quad \le \frac{u_2-u_1}{2}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + \left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right].
\end{aligned} \tag{3.15}$$

Proof From Lemma 1.2, we have

$$\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\
&\quad = \frac{u_2-u_1}{2}\left|\int_0^1 (1-2r)\psi'\bigl(ru_1+(1-r)u_2\bigr)\,dr\right| \\
&\quad \le \frac{u_2-u_1}{2}\int_0^1 |1-2r|\bigl|\psi'\bigl(ru_1+(1-r)u_2\bigr)\bigr|\,dr.
\end{aligned} \tag{3.16}$$

Using the exponential s-convexity of ψ', we get

$$\begin{aligned}
&\left|\frac{\psi(u_1)+\psi(u_2)}{2} - \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \\
&\quad \le \frac{u_2-u_1}{2}\int_0^1 |1-2r|\left[r^s\left|\frac{\psi(u_1)}{e^{\alpha u_1}}\right| + (1-r)^s\left|\frac{\psi(u_2)}{e^{\alpha u_2}}\right|\right]dr \\
&\quad = \frac{u_2-u_1}{2}\int_0^1 \left[|1-2r|r^s\left|\frac{\psi(u_1)}{e^{\alpha u_1}}\right| + |1+2r|(1-r)^s\left|\frac{\psi(u_2)}{e^{\alpha u_2}}\right|\right]dr \\
&\quad = \frac{u_2-u_1}{2}\left[C_1(s)\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right| + C_2(s)\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right].
\end{aligned} \tag{3.17}$$

It is easily seen that

$$C_1(s) = \int_0^1 |1-2r|r^s\,dr = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}, \tag{3.18}$$

$$C_2(s) = \int_0^1 |1-2r|(1-r)^s\,dr = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}. \tag{3.19}$$

Thus by substituting equalities (3.18) and (3.19) into (3.17), we achieve inequality (3.15). □

Remark 3.2 In Theorem 3.5,

(a) by taking $\alpha = 0$, we obtain Theorem 1, for $q = 1$, in [23];

(b) by taking $s = 1$, we obtain Theorem 3 in [4].

Theorem 3.6 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\psi' \in L_1[u_1, u_2]$. If $|\psi'|^q$ is exponentially s-convex in the second sense on $[u_1, u_2]$ with $q > 1$, then we have*

$$\left|\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \leq \frac{u_2 - u_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{s + \frac{1}{2^s}}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left[\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + \left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]^{\frac{1}{q}}. \tag{3.20}$$

Proof From Lemma 1.2, we have

$$\begin{aligned}\left|\frac{\psi(u_1) + \psi(u_2)}{2} - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| &= \frac{u_2 - u_1}{2}\left|\int_0^1 (1-2r)\psi'\big(ru_1 + (1-r)u_2\big)\,dr\right| \\ &\leq \frac{u_2 - u_1}{2}\int_0^1 |1-2r|\big|\psi'\big(ru_1 + (1-r)u_2\big)\big|\,dr. \end{aligned} \tag{3.21}$$

Applying the power-mean inequality, we find

$$\begin{aligned}&\frac{u_2 - u_1}{2}\int_0^1 |1-2r|\big|\psi'\big(ru_1 + (1-r)u_2\big)\big|\,dr \\ &\quad\leq \frac{u_2 - u_1}{2}\left(\int_0^1 |1-2r|\,dr\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2r|\big|\psi'\big(ru_1 + (1-r)u_2\big)\big|^q\,dr\right)^{\frac{1}{q}}.\end{aligned} \tag{3.22}$$

Since $|\psi'|^q$ is exponentially s-convex, we get

$$\begin{aligned}&\int_0^1 |1-2r|\big|\psi'\big(ru_1 + (1-r)u_2\big)\big|^q\,dr \\ &\quad\leq \int_0^1 |1-2r|\left[r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + (1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]dr \\ &\quad= \left[C_1(s)\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q + C_2(s)\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right],\end{aligned} \tag{3.23}$$

where

$$\int_0^1 |1-2r|\,dr = \frac{1}{2}. \tag{3.24}$$

Using (3.22)–(3.24) in (3.21), we get (3.20). □

Remark 3.3 In Theorem 3.6,
(a) by putting $\alpha = 0$, we get Theorem 1, for $q > 1$, in [23];
(b) by putting $s = 1$, we get Theorem 5 in [4].

Theorem 3.7 *Let $\psi : \mathcal{K} \to \mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1, u_2 \in \mathcal{K}$ with $u_1 < u_2$ and $\psi' \in L_1[u_1, u_2]$. If $|\psi'|^q$ is exponentially s-convex in the second sense on $[u_1, u_2]$ and*

$q,l>1$, $\frac{1}{l}+\frac{1}{q}=1$, then we have

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \leq \frac{u_2-u_1}{2(l+1)^{\frac{1}{l}}}\left[\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q+|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{s+1}\right]^{\frac{1}{q}}. \tag{3.25}$$

Proof From Lemma 1.2 and using Hölder's inequality, we have

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \leq \frac{u_2-u_1}{2}\left(\int_0^1|1-2r|^l\,dr\right)^{\frac{1}{l}}\left(\int_0^1\left|\psi'\bigl(ru_1+(1-r)u_2\bigr)\right|^q\,dr\right)^{\frac{1}{q}}. \tag{3.26}$$

Since $|\psi'|^q$ is exponentially s-convex, we get

$$\begin{aligned}\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| &\leq \frac{u_2-u_1}{2}\left(\int_0^1|1-2r|^l\,dr\right)^{\frac{1}{l}}\left(\int_0^1\left[r^s\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|^q+(1-r)^s\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|^q\right]\right)^{\frac{1}{q}} \\ &= \frac{u_2-u_1}{2(l+1)^{\frac{1}{l}}}\left[\frac{|\frac{\psi'(u_1)}{e^{\alpha u_1}}|^q+|\frac{\psi'(u_2)}{e^{\alpha u_2}}|^q}{s+1}\right]^{\frac{1}{q}}.\end{aligned} \tag{3.27}$$

Hence the proof is completed. □

Remark 3.4 In Theorem 3.7,

(a) by letting $\alpha=0$, we get

$$\left|\frac{\psi(u_1)+\psi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\psi(w)\,dw\right| \leq \frac{u_2-u_1}{2(l+1)^{\frac{1}{l}}}\left[\frac{|\psi'(u_1)|^q+|\psi'(u_2)|^q}{s+1}\right]^{\frac{1}{q}}; \tag{3.28}$$

(b) by letting $s=1$, we get Theorem 4 in [4].

3.2 Applications

Suppose d is a partition of the interval $[u_1,u_2]$, that is, $d: u_1=w_0<w_1<\cdots<w_{m-1}<w_m=u_2$, then the trapezoidal formula is given as

$$T(\psi,d)=\sum_{n=0}^{m-1}\frac{\psi(w_n)+\psi(w_{n+1})}{2}(w_{n+1}-w_n).$$

We known that if $\psi:[u_1,u_2]\to\mathbb{R}$ is twice differentiable on (u_1,u_2) and $\mathcal{M}=\max_{w\in(u_1,u_2)}|\psi''(w)|<\infty$, then

$$\int_{u_1}^{u_2}\psi(w)\,dw=T(\psi,d)+R(\psi,d), \tag{3.29}$$

where the remainder term is given as

$$\left|R(\psi,d)\right| \leq \frac{\mathcal{M}}{12}\sum_{n=0}^{m-1}(w_{n+1}-w_n)^3. \tag{3.30}$$

It is noticed that if ψ'' does not exist or ψ'' is unbounded, then (3.29) is invalid. However, Dragomir and Wang [10–12] have shown that the term $R(\psi,d)$ can be obtained by using the first derivative only. These estimates surely have several applications. In this section, we estimate the remainder term $R(\psi,d)$ in a new sense.

Proposition 3.1 *Let $\psi:\mathcal{K}\subseteq\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$. Let $u_1,u_2\in\mathcal{K}$, $u_1<u_2$. If $|\psi'|$ is exponentially s-convex in the second sense on $[u_1,u_2]$ and $s\in(0,1]$, then in (3.29), for every partition d of $[u_1,u_2]$, we have*

$$\begin{aligned}\left|R(\psi,d)\right| &\leq \frac{1}{2}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2\left[\left|\frac{\psi'(w_n)}{e^{\alpha w_n}}\right|+\left|\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}\right|\right]\\ &\leq \max\left\{\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|,\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right\}\\ &\quad\times\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2.\end{aligned} \tag{3.31}$$

Proof Applying Theorem 3.5 on the subinterval $[w_n,w_{n+1}]$ $(n=0,1,\dots,m-1)$ of the partition d, we obtain

$$\begin{aligned}&\left|\frac{\psi(w_n)+\psi(w_{n+1})}{2}(w_{n+1}-w_n)-\int_{w_n}^{w_{n+1}}\psi(w)\,dw\right|\\ &\quad\leq\frac{(w_{n+1}-w_n)^2}{2}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\psi'(w_n)}{e^{\alpha w_n}}\right|+\left|\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}\right|\right].\end{aligned} \tag{3.32}$$

Summing over n from 0 to $m-1$, we get

$$\begin{aligned}&\left|T(\psi,d)-\int_{u_1}^{u_2}\psi(w)\,dw\right|\\ &\quad\leq\frac{1}{2}\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\psi'(w_n)}{e^{\alpha w_n}}\right|+\left|\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}\right|\right]\\ &\quad\leq\max\left\{\left|\frac{\psi'(u_1)}{e^{\alpha u_1}}\right|,\left|\frac{\psi'(u_2)}{e^{\alpha u_2}}\right|\right\}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\sum_{n=0}^{m-1}(w_{n+1}-w_n)^2.\end{aligned} \tag{3.33}$$

□

Proposition 3.2 *Let $\psi:\mathcal{K}\subseteq\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathcal{K}^\circ$ and $u_1,u_2\in\mathcal{K}$ with $u_1<u_2$. If $|\psi'|^q$ is exponentially s-convex in the second sense on $[u_1,u_2]$ and $s\in(0,1]$*

and $q, l > 1$ such that $\frac{1}{l} + \frac{1}{q} = 1$, then in (3.29), for every partition d of $[u_1, u_2]$, we have

$$\begin{aligned}\left|R(\psi, d)\right| &\leq \frac{1}{2(l+1)^{\frac{1}{l}}} \sum_{n=0}^{m-1} (w_{n+1} - w_n)^2 \left[\frac{|\frac{\psi'(w_n)}{e^{\alpha w_n}}|^q + |\frac{\psi'(w_{n+1})}{e^{\alpha w_{n+1}}}|^q}{s+1}\right]^{\frac{1}{q}} \\ &\leq \frac{\max\{\frac{2|\frac{\psi'(u_1)}{e^{\alpha u_1}}|}{s+1}, \frac{2|\frac{\psi'(u_2)}{e^{\alpha u_2}}|}{s+1}\}}{2(l+1)^{\frac{1}{l}}} \sum_{n=0}^{m-1} (w_{n+1} - w_n)^2. \end{aligned} \tag{3.34}$$

Proof Using Theorem 3.7 and similar arguments as in Proposition 3.1, we get the required result. □

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