Journal of Inequalities and Applications
a SpringerOpen Journal

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Minkowski's inequality for the AB-fractional integral operator

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Abstract

Recently, AB-fractional calculus has been introduced by Atangana and Baleanu and attracted a large number of scientists in different scientific fields for the exploration of diverse topics. An interesting aspect is the generalization of classical inequalities via AB-fractional integral operators. In this paper, we aim to generalize Minkowski inequality using the AB-fractional integral operator.

Keywords: AB-fractional integral operator; Minkowski inequality

1 Introduction

Nowadays the fractional calculus has an important role in diverse scientific fields due to its several applications in dynamical problems including signals, hydrodynamics, dynamics, fluid, viscoelastic theory, biology, control theory, image processing, computer networking, and many others [1–5]. A large number of scientists have worked on generalizations of existing results including theorems, definitions, models, and many more. A generalization of classical inequalities by means of fractional-order integral operators is considered as an interesting subject area. For instance, recently, Agarwal et al. [6] proved Hermite–Hadamard-type inequalities by using generalized k-fractional-integrals. Aldhaifallah et al. [7] used the (k,s)-fractional integral operator to generalize the inequalities for a family/class of n positive functions. Set et al. [8] studied Hermite–Hadamard-type inequalities for a generalized fractional integral operator for functions with convex absolute values of derivatives. Khan et al. [9] produced the Minkowski inequality by using the Hahn integral operator. On the other hand, noninteger-order calculus, usually referred to as fractional calculus, is used to generalize integrals and derivatives, in particular, integrals involving inequalities. Recently, Dumitru and Arran [10] introduced a new formula for fractional derivatives and integrals by using the Mittag-Leffler kernel. More theoretical concepts regarding fractional operators with Mittag-Leffler kernels (Atangana–Baleanu operators) and the higher-order case have been discussed in [11, 12], whereas the generalization to the generalized Mittag-Leffler kernels to gain a semigroup property have been recently initiated in [13, 14]. Khan [15] studied inequalities for a class of n functions by means of Saigo fractional calculus. Jarad et al. [16] presented a Gronwall-type inequality for the analysis of the fractional-order Atangana–Baleanu differential equation and in [17] for generalized fractional derivatives.

Shuang and Qi [18] proved some Hermite–Hadamard-type inequalities for a class of s-convex functions and studied special means. Mehrez and Agarwal [19] produced new integral inequalities by means of classical Hermite–Hadamard inequalities and obtained particular cases of their results with applications to special means. Park et al. [20] investigated new generalized inequalities, which then were utilized for stability analysis. Sarikaya et al. [21] established fractional integral inequalities generalizing the classical results by using the local fractional approach.

The integral inequalities with Mittag-Leffler functions have been studied as a generalization of the classical inequalities. For instance, Farid et al. [22] generalized several classical inequalities using an extended Mittag-Leffler function and evaluated particular cases of their results. More related work can be found in [23–25].

In this paper, we use the AB-fractional integral operator for generalization of classical Minkowski inequalities. Our results are more general and applicable than those in the classical case. There are many definitions of fractional integrals, for example, Riemann–Liouville, Hadamard, Liouville, Weyl, Erdelyi–Kober, and Katugampola [26–29], which can be considered for getting the same results. Now we give some definitions and lemma related to the AB-fractional operator.

Definition 1.1 ([30]) The fractional ABC-derivative in the Caputo sense of a function $f \in H^*(a,b)$ is defined by

$$ {}^{ABC}{}_{a}\mathcal{D}_{\tau}^{\nu} f(\tau) = \frac{\mathbb{B}(\nu)}{1-\nu} \int_{a}^{\tau} f^{'}(s) E_{\nu}\left[\frac{-\nu(\tau-s)^{\mu}}{1-\nu}\right] ds, \tag{1.1}$$

where $b > a$ and $\nu \in [0,1]$, and $\mathbb{B}(\nu) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Definition 1.2 The fractional ABC-derivative in the Riemann–Liouville sense of a function $f \in H^*(a,b)$ is defined by

$$ {}^{ABR}{}_{a}\mathcal{D}_{\tau}^{\nu} f(\tau) = \frac{\mathbb{B}(\nu)}{1-\nu} \frac{d}{d\tau} \int_{a}^{\tau} f(s) E_{\nu}\left[\frac{-\nu(\tau-s)^{\nu}}{1-\nu}\right] ds, \tag{1.2}$$

where $b > a$ and $\nu \in [0,1]$.

Definition 1.3 ([31, 32]) The fractional AB-integral of the function $f \in H^*(a,b)$ is given by

$$ {}^{AB}{}_{a}\mathcal{I}_{\tau}^{\nu} f(\tau) = \frac{1-\nu}{\mathbb{B}(\nu)} f(\tau) + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{a}^{\tau} f(s)(\tau-s)^{\nu-1} ds, \tag{1.3}$$

where $b > a$ and $0 < \nu < 1$.

Remark 1.4 Since the normalization function $\mathbb{B}(\nu) > 0$ is positive, it immediately follows that the AB-integral of a positive function is positive. We will rely on this fact throughout the proofs of the main results.

Lemma 1.5 ([33]) *The ABC-fractional derivative and AB-fractional integral of a function f satisfy the Newton–Leibnitz formula*

$$ {}^{AB}{}_{a}\mathcal{I}_{\tau}^{\nu}\left({}^{ABC}{}_{a}\mathcal{D}_{\tau}^{\nu} f(\tau)\right) = f(\tau) - f(a). \tag{1.4}$$

Organization of the paper. This paper includes four sections. Introduction is given in Sect. 1, with a literature review, important definitions, and a lemma, which we will use in the proofs. In Sect. 2, we prove Minkowski's inequality for the AB-fractional integral operator. Other AB-fractional integral inequalities are proved in Sect. 3. The summary is given in Sect. 4.

2 The AB-fractional Minkowski inequality

Theorem 2.1 *Let $\nu > 0$ and $p \geq 1$. Let $u, v \in C_\nu[a,b]$ be two positive functions in $[0,\infty[$ such that ${}^{AB}{}_a\mathcal{I}_t^\nu u(t) < \infty$ and ${}^{AB}{}_a\mathcal{I}_t^\nu v(t) < \infty$ for all $t > a$. If $0 < \alpha \leq \frac{u(t)}{v(t)} \leq \theta$ for some $\alpha, \theta \in \mathbb{R}_+^*$ and all $t \in [a,b]$, then*

$$\left({}^{AB}{}_a\mathcal{I}_t^\nu u^p(t)\right)^{\frac{1}{p}} + \left({}^{AB}{}_a\mathcal{I}_t^\nu v^p(t)\right)^{\frac{1}{p}} \leq \mathcal{A}\left[{}^{AB}{}_a\mathcal{I}_t^\nu \left(u(t)+v(t)\right)^p\right]^{\frac{1}{p}}, \tag{2.1}$$

where

$$\mathcal{A} = \frac{\theta(1+\alpha)+(\theta+1)}{(1+\alpha)(\theta+1)}.$$

Proof From the condition $\frac{u(t)}{v(t)} \leq \theta$ we obtain

$$u(t) \leq \left(\frac{\theta}{\theta+1}\right)\left(u(t)+v(t)\right). \tag{2.2}$$

Taking the pth power of both sides of Eq. (2.2), we have

$$u^p(t) \leq \left(\frac{\theta}{\theta+1}\right)^p \left(u(t)+v(t)\right)^p. \tag{2.3}$$

Multiplying both sides of (2.3) by $\frac{1-\nu}{\mathbb{B}(\nu)}$, we get

$$\frac{1-\nu}{\mathbb{B}(\nu)} u^p(t) \leq \left(\frac{\theta}{\theta+1}\right)^p \frac{1-\nu}{\mathbb{B}(\nu)}\left(u(t)+v(t)\right)^p. \tag{2.4}$$

Also, replacing t by s in Eq. (2.3) and multiplying both sides by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we get

$$\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^p(s) \leq \left(\frac{\theta}{\theta+1}\right)^p \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left(u(s)+v(s)\right)^p. \tag{2.5}$$

Integrating both sides of Eq. (2.4) with respect to s, we have

$$\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^p(s)\,ds \leq \left(\frac{\theta}{\theta+1}\right)^p \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left(u(s)+v(s)\right)^p\,ds. \tag{2.6}$$

Adding (2.4) and (2.6), we obtain

$$\begin{aligned}\frac{1-\nu}{\mathbb{B}(\nu)} u^p(t) + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^p(s)\,ds \leq \left(\frac{\theta}{\theta+1}\right)^p \Bigg[&\frac{1-\nu}{\mathbb{B}(\nu)}\left(u(t)+v(t)\right)^p \\ &+ \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left(u(s)+v(s)\right)^p\,ds\Bigg].\end{aligned}$$

This implies

$$ {}^{AB}{}_{a}\mathcal{I}_{t}^{\nu} u^{p}(s) \leq \left(\frac{\theta}{\theta+1}\right)^{p} {}^{AB}{}_{a}\mathcal{I}_{t}^{\nu}\big(u(s)+v(s)\big)^{p}. \tag{2.7} $$

Taking the $\frac{1}{p}$th power of both sides of Eq. (2.7), we find

$$ \big({}^{AB}{}_{a}\mathcal{I}_{t}^{\nu} u^{p}(t)\big)^{\frac{1}{p}} \leq \frac{\theta}{\theta+1}\big[{}^{AB}{}_{a}\mathcal{I}_{t}^{\nu}\big(u(t)+v(t)\big)^{p}\big]^{\frac{1}{p}}. \tag{2.8} $$

On the other hand, by using the condition $0<\alpha\leq\frac{u(t)}{v(t)}$ we directly get

$$ v^{p}(t) \leq \frac{1}{(1+\alpha)^{p}}\big(u(t)+v(t)\big)^{p}. \tag{2.9} $$

Multiplying Eq. (2.9) by $\frac{1-\nu}{\mathbb{B}(\nu)}$, we get

$$ \frac{1-\nu}{\mathbb{B}(\nu)} v^{p}(t) \leq \frac{1}{(1+\alpha)^{p}}\frac{1-\nu}{\mathbb{B}(\nu)}\big(u(t)+v(t)\big)^{p}. \tag{2.10} $$

Also, replacing t by s in Eq. (2.9) and multiplying both sides by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we get

$$ \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} v^{p}(s) \leq \frac{1}{(1+\alpha)^{p}}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u(s)+v(s)\big)^{p}. \tag{2.11} $$

Integrating both sides of Eq. (2.11) with respect to s, we have

$$ \int_{a}^{t}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} v^{p}(s)\,ds \leq \frac{1}{(1+\alpha)^{p}}\int_{a}^{t}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u(s)+v(s)\big)^{p}\,ds. \tag{2.12} $$

Adding (2.10) and (2.12), we obtain

$$ \begin{aligned} \frac{1-\nu}{\mathbb{B}(\nu)} v^{p}(t) + \int_{a}^{t}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} v^{p}(s)\,ds \leq{}& \frac{1}{(1+\alpha)^{p}}\frac{1-\nu}{\mathbb{B}(\nu)}\big(u(t)+v(t)\big)^{p} \\ &+ \int_{a}^{t}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u(s)+v(s)\big)^{p}\,ds]. \end{aligned} \tag{2.13} $$

This leads to the AB-fractional integral inequality

$$ {}^{AB}{}_{a}\mathcal{I}_{t}^{\nu} v^{p}(t) \leq \frac{1}{(1+\alpha)^{p}} {}^{AB}{}_{a}\mathcal{I}_{t}^{\nu}\big(u(t)+v(t)\big)^{p}. \tag{2.14} $$

Taking the $\frac{1}{p}$th power of both sides of Eq. (2.14), we find

$$ \big({}^{AB}{}_{a}\mathcal{I}_{t}^{\nu} v^{p}(t)\big)^{\frac{1}{p}} \leq \frac{1}{1+\alpha}\big[{}^{AB}{}_{a}\mathcal{I}_{t}^{\nu}\big(u(t)+v(t)\big)^{p}\big]^{\frac{1}{p}}. \tag{2.15} $$

By Eqs. (2.8) and (2.15) we obtain

$$ \big({}^{AB}{}_{a}\mathcal{I}_{t}^{\nu} u^{p}(t)\big)^{\frac{1}{p}} + \big({}^{AB}{}_{a}\mathcal{I}_{t}^{\nu} v^{p}(t)\big)^{\frac{1}{p}} \leq \mathcal{A}\big[{}^{AB}{}_{a}\mathcal{I}_{t}^{\nu}\big(u(t)+v(t)\big)^{p}\big]^{\frac{1}{p}}. \tag{2.16} $$

Thus, the proof of the AB-fractional integral inequality is completed. □

3 Other types of inequalities

Theorem 3.1 *Let $\nu > 0$ and $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$. Let $u, v \in C_\nu[a,b]$ be two positive functions in $[0,\infty[$ such that ${}^{AB}{}_{a}\mathcal{I}_t^\nu u(t) < \infty$ and ${}^{AB}{}_{a}\mathcal{I}_t^\nu v(t) < \infty$ for all $t > a$. If $0 < \alpha \le \frac{u(t)}{v(t)} \le \theta$ for some $\alpha, \theta \in \mathbb{R}_+^*$ and all $t \in [a,b]$, then*

$$\left({}^{AB}{}_{a}\mathcal{I}_t^\nu u(t)\right)^{\frac{1}{p}}\left({}^{AB}{}_{a}\mathcal{I}_t^\nu v(t)\right)^{\frac{1}{q}} \le \left(\frac{\theta}{\alpha}\right)^{\frac{1}{pq}}\left[{}^{AB}{}_{a}\mathcal{I}_t^\nu\left(u^{\frac{1}{p}}(t)v^{\frac{1}{q}}(t)\right)\right]. \tag{3.1}$$

Proof Using the condition $\frac{u(t)}{v(t)} \le \theta$, we get

$$u^{\frac{1}{q}} \le \theta^{\frac{1}{q}} v^{\frac{1}{q}}. \tag{3.2}$$

Multiplying (3.2) by $u^{\frac{1}{p}}$ and using the condition $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$u \le \theta^{\frac{1}{q}} u^{\frac{1}{p}} v^{\frac{1}{q}}. \tag{3.3}$$

Now let us use (3.3) twice. First, multiplying by $\frac{1-\nu}{\mathbb{B}(\nu)}$, we get

$$\frac{1-\nu}{\mathbb{B}(\nu)} u \le \theta^{\frac{1}{q}} \frac{1-\nu}{\mathbb{B}(\nu)} u^{\frac{1}{p}} v^{\frac{1}{q}}. \tag{3.4}$$

Second, multiplying by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we obtain

$$\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u \le \theta^{\frac{1}{q}} \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^{\frac{1}{p}} v^{\frac{1}{q}}. \tag{3.5}$$

Integrating both sides of Eq. (3.5) from 0 to t, we have

$$\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u(s)\,ds \le \theta^{\frac{1}{q}} \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u(s)^{\frac{1}{p}} v(s)^{\frac{1}{q}}\,ds. \tag{3.6}$$

Now, by adding Eq. (3.4) and Eq. (3.6) we find

$$\begin{aligned}\frac{1-\nu}{\mathbb{B}(\nu)} u(t) + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u(s)\,ds \le \theta^{\frac{1}{q}} \biggl[&\frac{1-\nu}{\mathbb{B}(\nu)} u(t)^{\frac{1}{p}} v(t)^{\frac{1}{q}} \\ &+ \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u(s)^{\frac{1}{p}} v(s)^{\frac{1}{q}}\,ds \biggr].\end{aligned}$$

This implies

$${}^{AB}{}_{a}\mathcal{I}_t^\nu u(t) \le \theta^{\frac{1}{q}}\left[{}^{AB}{}_{a}\mathcal{I}_t^\nu\left(u^{\frac{1}{p}}(t)v^{\frac{1}{q}}(t)\right)\right]. \tag{3.7}$$

Taking the $\frac{1}{p}$th power of both sides of (3.7), we have

$$\left[{}^{AB}{}_{a}\mathcal{I}_t^\nu u(t)\right]^{\frac{1}{p}} \le \theta^{\frac{1}{pq}}\left[{}^{AB}{}_{a}\mathcal{I}_t^\nu\left(u^{\frac{1}{p}}(t)v^{\frac{1}{q}}(t)\right)\right]^{\frac{1}{p}}. \tag{3.8}$$

Now, by the condition $\alpha \leq \frac{u(t)}{v(t)}$ we have

$$v^{\frac{1}{p}} \leq \alpha^{\frac{-1}{p}} u^{\frac{1}{p}}. \tag{3.9}$$

Multiplying Eq. (3.9) by $v^{\frac{1}{q}}$, we get

$$v \leq \alpha^{\frac{-1}{p}} u^{\frac{1}{p}} v^{\frac{1}{q}}. \tag{3.10}$$

Now let us use (3.10) twice. First, multiplying by $\frac{1-\nu}{\mathbb{B}(\nu)}$, we get

$$\frac{1-\nu}{\mathbb{B}(\nu)} v \leq \alpha^{\frac{-1}{p}} \frac{1-\nu}{\mathbb{B}(\nu)} u^{\frac{1}{p}} v^{\frac{1}{q}}. \tag{3.11}$$

Second, multiplying by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we obtain

$$\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} v \leq \alpha^{\frac{-1}{p}} \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^{\frac{1}{p}} v^{\frac{1}{q}}. \tag{3.12}$$

Integrating both sides of Eq. (3.12) from 0 to t, we have

$$\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} v(s)\,ds \leq \alpha^{\frac{-1}{p}} \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u(s)^{\frac{1}{p}} v(s)^{\frac{1}{q}}\,ds. \tag{3.13}$$

Now, by adding Eq. (3.11) and Eq. (3.13) we find

$$\begin{aligned}\frac{1-\nu}{\mathbb{B}(\nu)} v(t) + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} v(s)\,ds \leq \alpha^{\frac{-1}{p}} \Bigg[&\frac{1-\nu}{\mathbb{B}(\nu)} u(t)^{\frac{1}{p}} v(t)^{\frac{1}{q}} \\ &+ \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u(s)^{\frac{1}{p}} v(s)^{\frac{1}{q}}\,ds\Bigg].\end{aligned} \tag{3.14}$$

This implies

$$^{AB}{}_a\mathcal{I}_t^{\nu} v(t) \leq \alpha^{\frac{-1}{p}} \left[{}^{AB}{}_a\mathcal{I}_t^{\nu}\left(u^{\frac{1}{p}}(t) v^{\frac{1}{q}}(t)\right)\right]. \tag{3.15}$$

Taking the $\frac{1}{q}$th power of both sides of (3.15), we have

$$\left[{}^{AB}{}_a\mathcal{I}_t^{\nu} v(s)\right]^{\frac{1}{q}} \leq \alpha^{\frac{-1}{pq}} \left[{}^{AB}{}_a\mathcal{I}_t^{\nu}\left(u^{\frac{1}{p}}(t) v^{\frac{1}{q}}(t)\right)\right]^{\frac{1}{q}}. \tag{3.16}$$

Finally, multiplying Eq. (3.8) and Eq. (3.16), we obtain

$$\left({}^{AB}{}_a\mathcal{I}_t^{\nu} u(t)\right)^{\frac{1}{p}} \left({}^{AB}{}_a\mathcal{I}_t^{\nu} v(t)\right)^{\frac{1}{q}} \leq \left(\frac{\theta}{\alpha}\right)^{\frac{1}{pq}} \left[{}^{AB}{}_a\mathcal{I}_t^{\nu}\left(u^{\frac{1}{p}}(t) v^{\frac{1}{q}}(t)\right)\right]. \tag{3.17}$$

□

Theorem 3.2 *Let $\nu > 0$ and $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$. Let $u, v \in C_\nu[a,b]$ be two positive functions in $[0,\infty[$ such that ${}^{AB}{}_a\mathcal{I}_t^{\nu} u^p(t) < \infty$, ${}^{AB}{}_a\mathcal{I}_t^{\nu} u^q(t) < \infty$, ${}^{AB}{}_a\mathcal{I}_t^{\nu} v^p(t) < \infty$, and ${}^{AB}{}_a\mathcal{I}_t^{\nu} v^q(t) < \infty$ for all $t > a$. If $0 < \alpha \leq \frac{u(t)}{v(t)} \leq \theta$ for some $\alpha, \theta \in \mathbb{R}_+^*$ and all $t \in [a,b]$, then*

$$^{AB}{}_a\mathcal{I}_t^{\nu}\left(u(t) + v(t)\right) \leq \mathcal{A}^{*\,AB}{}_a\mathcal{I}_t^{\nu}\left(u^p(t) + v^p(t)\right) + \mathcal{B}_m^{*\,AB}{}_a\mathcal{I}_t^{\nu}\left(u^q(t) + v^q(t)\right), \tag{3.18}$$

where

$$\mathcal{A}^* = \frac{2^{p-1}\theta^p}{p(\theta+1)^p}, \qquad \mathcal{B}_m^* = \frac{2^{q-1}}{q(1+\alpha)^q}.$$

Proof Using the condition $\frac{u(t)}{v(t)} \leq \theta$, we obtain

$$u(t) \leq \left(\frac{\theta(u(t)+v(t))}{1+\theta}\right). \tag{3.19}$$

Taking the pth power of both sides of Eq. (2.2), we have

$$u^p(t) \leq \left(\frac{\theta}{\theta+1}\right)^p \bigl(u(t)+v(t)\bigr)^p. \tag{3.20}$$

Multiplying both sides of (3.20) by $\frac{1-\nu}{\mathbb{B}(\nu)}$, we get

$$\frac{1-\nu}{\mathbb{B}(\nu)} u^p(t) \leq \left(\frac{\theta}{\theta+1}\right)^p \frac{1-\nu}{\mathbb{B}(\nu)} \bigl(u(t)+v(t)\bigr)^p. \tag{3.21}$$

Also, replacing t by s in Eq. (3.20) and multiplying both sides by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we get

$$\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^p(s) \leq \left(\frac{\theta}{\theta+1}\right)^p \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} \bigl(u(s)+v(s)\bigr)^p. \tag{3.22}$$

Integrating both sides of Eq. (3.21) with respect to s, we have

$$\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^p(s)\,ds \leq \left(\frac{\theta}{\theta+1}\right)^p \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} \bigl(u(s)+v(s)\bigr)^p\,ds. \tag{3.23}$$

Adding (3.21) and (3.23), we obtain

$$\begin{aligned} &\frac{1-\nu}{\mathbb{B}(\nu)} u^p(t) + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} u^p(s)\,ds \\ &\quad \leq \left(\frac{\theta}{\theta+1}\right)^p \left[\frac{1-\nu}{\mathbb{B}(\nu)} \bigl(u(t)+v(t)\bigr)^p + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)} \bigl(u(s)+v(s)\bigr)^p\,ds\right]. \end{aligned}$$

This implies

$${}^{AB}{}_a\mathcal{I}_t^{\nu} u^p(t) \leq \left(\frac{\theta}{\theta+1}\right)^p {}^{AB}{}_a\mathcal{I}_t^{\nu} \bigl(u(t)+v(t)\bigr)^p. \tag{3.24}$$

Multiplying (2.7) by the constant $\frac{1}{p}$, we find

$$\frac{1}{p}\bigl({}^{AB}{}_a\mathcal{I}_t^{\nu} u^p(t)\bigr) \leq \frac{1}{p}\left(\frac{\theta}{\theta+1}\right)^p \bigl[{}^{AB}{}_a\mathcal{I}_t^{\nu} \bigl(u(t)+v(t)\bigr)^p\bigr]. \tag{3.25}$$

On the other hand, by using the condition $0 < \alpha \leq \frac{u(t)}{v(t)}$ we directly get

$$v^q(t) \leq \frac{1}{(1+\alpha)^q} \bigl(u(t)+v(t)\bigr)^q. \tag{3.26}$$

Multiplying (3.26) by $\frac{1-\nu}{\mathbb{B}(\nu)}$, we get

$$\frac{1-\nu}{\mathbb{B}(\nu)}v^q(t) \leq \frac{1}{(1+\alpha)^q}\frac{1-\nu}{\mathbb{B}(\nu)}\big(u(t)+v(t)\big)^q. \tag{3.27}$$

Also, replacing t by s in Eq. (3.26) and multiplying both sides by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we get

$$\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}v^q(s) \leq \frac{1}{(1+\alpha)^q}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u(s)+v(s)\big)^q. \tag{3.28}$$

Integrating both sides of Eq. (3.28) with respect to s, we have

$$\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}v^q(s)\,ds \leq \frac{1}{(1+\alpha)^q}\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u(s)+v(s)\big)^q\,ds. \tag{3.29}$$

Adding (3.27) and (3.29), we obtain

$$\begin{aligned}\frac{1-\nu}{\mathbb{B}(\nu)}v^q(t) + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}v^q(s)\,ds \leq \frac{1}{(1+\alpha)^q}\bigg[&\frac{1-\nu}{\mathbb{B}(\nu)}\big(u(t)+v(t)\big)^q \\ &+ \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u(s)+v(s)\big)^q\,ds\bigg].\end{aligned}$$

This implies

$$ {}^{AB}_{\ a}\mathcal{I}^{\nu}_t v^q(t) \leq \frac{1}{(1+\alpha)^q}\,{}^{AB}_{\ a}\mathcal{I}^{\nu}_t\big(u(t)+v(t)\big)^q. \tag{3.30}$$

Multiplying (2.14) by $\frac{1}{q}$, we have

$$\frac{1}{q}\big({}^{AB}_{\ a}\mathcal{I}^{\nu}_t v^q(t)\big) \leq \frac{1}{q}\frac{1}{(1+\alpha)^q}\big[{}^{AB}_{\ a}\mathcal{I}^{\nu}_t\big(u(t)+v(t)\big)^q\big]. \tag{3.31}$$

By means of Eqs. (3.25) and (3.31) we get

$$\begin{aligned}&\frac{1}{p}\big({}^{AB}_{\ a}\mathcal{I}^{\nu}_t u^p(t)\big) + \frac{1}{q}\big({}^{AB}_{\ a}\mathcal{I}^{\nu}_t v^q(t)\big) \\ &\quad\leq \frac{1}{p}\left(\frac{\theta}{\theta+1}\right)^p\big[{}^{AB}_{\ a}\mathcal{I}^{\nu}_t\big(u(t)+v(t)\big)^p\big] + \frac{1}{q}\frac{1}{(1+\alpha)^q}\big[{}^{AB}_{\ a}\mathcal{I}^{\nu}_t\big(u(t)+v(t)\big)^q\big].\end{aligned} \tag{3.32}$$

To complete our proof, we have to use Young's inequality

$$u(t)v(t) \leq \frac{u^p(t)}{p} + \frac{v^q(t)}{q}. \tag{3.33}$$

Multiplying (3.33) by $\frac{1-\nu}{\mathbb{B}(\nu)}$, we get

$$\frac{1-\nu}{\mathbb{B}(\nu)}u(t)v(t) \leq \frac{1-\nu}{\mathbb{B}(\nu)}\left(\frac{u^p(t)}{p} + \frac{v^q(t)}{q}\right). \tag{3.34}$$

Also, replacing t by s in Eq. (3.33) and multiplying both sides by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we get

$$\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}u(s)v(s) \leq \frac{\nu(t-s)^{\nu-1}}{p\mathbb{B}(\nu)\Gamma(\nu)}u^p(s) + \frac{\nu(t-s)^{\nu-1}}{q\mathbb{B}(\nu)\Gamma(\nu)}v^q(s). \tag{3.35}$$

Integrating both sides of Eq. (3.35) with respect to s, we have

$$\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}u(s)v(s)\,ds \leq \int_a^t \frac{\nu(t-s)^{\nu-1}}{p\mathbb{B}(\nu)\Gamma(\nu)}u^p(s)\,ds + \int_a^t \frac{\nu(t-s)^{\nu-1}}{q\mathbb{B}(\nu)\Gamma(\nu)}v^q(s)\,ds. \tag{3.36}$$

Adding (3.34) and (3.36), we obtain

$$\begin{aligned}
&\frac{1-\nu}{\mathbb{B}(\nu)}u(t)v(t) + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}u(s)v(s)\,ds \\
&\quad\leq \frac{1-\nu}{\mathbb{B}(\nu)}\left(\frac{u^p(t)}{p} + \frac{v^q(t)}{q}\right) \\
&\qquad + \int_a^t \frac{\nu(t-s)^{\nu-1}}{p\mathbb{B}(\nu)\Gamma(\nu)}u^p(s)\,ds + \int_a^t \frac{\nu(t-s)^{\nu-1}}{q\mathbb{B}(\nu)\Gamma(\nu)}v^q(s)\,ds.
\end{aligned} \tag{3.37}$$

This implies

$${}^{AB}{}_a\mathcal{I}_t^\nu u(t)v(t) \leq \frac{1}{p}{}^{AB}{}_a\mathcal{I}_t^\nu u^p(t) + \frac{1}{q}{}^{AB}{}_a\mathcal{I}_t^\nu v^q(t). \tag{3.38}$$

Using (3.32) and (3.38), we have

$$\begin{aligned}
&{}^{AB}{}_a\mathcal{I}_t^\nu u(t)v(t) \\
&\quad\leq \frac{1}{p}\left(\frac{\theta}{\theta+1}\right)^p \left[{}^{AB}{}_a\mathcal{I}_t^\nu \left(u(t)+v(t)\right)^p\right] + \frac{1}{q}\frac{1}{(1+\alpha)^q}\left[{}^{AB}{}_a\mathcal{I}_t^\nu \left(u(t)+v(t)\right)^q\right].
\end{aligned} \tag{3.39}$$

Using the inequality

$$(u+v)^r \leq 2^{r-1}\left(u^r + v^r\right), \quad u,v \geq 0, r > 1, \tag{3.40}$$

with $r = p$ and multiplying (3.40) by the constant $\frac{1-\nu}{\mathbb{B}(\nu)}$, we find

$$\frac{1-\nu}{\mathbb{B}(\nu)}\left(u(t)+v(t)\right)^p \leq 2^{p-1}\frac{1-\nu}{\mathbb{B}(\nu)}\left(u(t)^p + v(t)^p\right). \tag{3.41}$$

Then multiplying Eq. (3.40) with $r = p$ by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we get

$$\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}(u+v)^p \leq 2^{p-1}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left(u^p + v^p\right). \tag{3.42}$$

Integrating Eq. (3.42) from a to t, we have

$$\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left(u(s)+v(s)\right)^p ds \leq 2^{p-1}\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left(u^p(s) + v^p(s)\right)ds. \tag{3.43}$$

Adding Eq. (3.41) and Eq. (3.43), we obtain

$$\begin{aligned} &\frac{1-\nu}{\mathbb{B}(\nu)}\big(u(t)+v(t)\big)^p + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u(s)+v(s)\big)^p\,ds \\ &\quad \le 2^{p-1}\left(\frac{1-\nu}{\mathbb{B}(\nu)}\big(u(t)^p+v(t)^p\big) + \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\big(u^p(s)+v^p(s)\big)\,ds\right). \end{aligned} \tag{3.44}$$

This implies

$$ {}^{AB}{}_a\mathcal{I}_t^\nu \big(u(t)+v(t)\big)^p \le 2^{p-1AB}{}_a\mathcal{I}_t^\nu \big(u^p(t)+v^p(t)\big). \tag{3.45}$$

Repeating the same process with $r = q$, we get

$$ {}^{AB}{}_a\mathcal{I}_t^\nu \big(u(t)+v(t)\big)^q \le 2^{q-1AB}{}_a\mathcal{I}_t^\nu \big(u^q(t)+v^q(t)\big). \tag{3.46}$$

Substituting by (3.45) and (3.46) into Eq. (3.39), the proof completed. □

Theorem 3.3 *Let $\nu > 0$, and let $u, v \in C_\nu[a,b]$ be two positive functions in $[0,\infty[$ such that ${}^{AB}{}_a\mathcal{I}_t^\nu u(t) < \infty$ and ${}^{AB}{}_a\mathcal{I}_t^\nu v(t) < \infty$ for all $t > a$, If $0 < \alpha \le \frac{u(t)}{v(t)} \le \theta$ for some $\alpha, \theta \in \mathbb{R}_+^*$ and all $t \in [a,b]$, then*

$$\frac{1}{\theta}{}^{AB}{}_a\mathcal{I}_t^\nu \big(u(t)v(t)\big) \le {}^{AB}{}_a\mathcal{I}_t^\nu \big(u(t)+v(t)\big)^2 \le \frac{1}{\alpha}{}^{AB}{}_a\mathcal{I}_t^\nu \big(u(t)v(t)\big), \tag{3.47}$$

Proof Using the condition

$$0 < \alpha \le \frac{u(t)}{v(t)} \le \theta, \tag{3.48}$$

we conclude that

$$(1+\alpha)v(t) \le \big(u(t)+v(t)\big) \le (\theta+1)v(t), \tag{3.49}$$

$$\frac{\theta+1}{\theta}u(t) \le \big(u(t)+v(t)\big) \le \frac{1+\alpha}{\alpha}u(t). \tag{3.50}$$

By (3.49) and (3.50) we obtain

$$\frac{1}{\theta}u(t)v(t) \le \frac{(u(t)+v(t))^2}{(1+\alpha)(\theta+1)} \le \frac{1}{\alpha}u(t)v(t). \tag{3.51}$$

Multiplying (3.51) by $\frac{1-\nu}{\mathbb{B}(\nu)}$ and then by $\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}$, we get

$$\frac{1}{\theta}\frac{1-\nu}{\mathbb{B}(\nu)}u(t)v(t) \le \frac{1-\nu}{\mathbb{B}(\nu)}\frac{(u(t)+v(t))^2}{(1+\alpha)(\theta+1)} \le \frac{1}{\alpha}\frac{1-\nu}{\mathbb{B}(\nu)}u(t)v(t), \tag{3.52}$$

$$\frac{1}{\theta}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}u(t)v(t) \le \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\frac{(u(t)+v(t))^2}{(1+\alpha)(\theta+1)} \le \frac{1}{\alpha}\frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}u(t)v(t). \tag{3.53}$$

Integrating Eq. (3.53) from 0 to t with respect to s, we have

$$\frac{1}{\theta}\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}u(s)v(s)\,ds \le \int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}\frac{(u(s)+v(s))^2}{(1+\alpha)(\theta+1)}\,ds \tag{3.54}$$

$$\leq \frac{1}{\alpha}\int_a^t \frac{\nu(t-s)^{\nu-1}}{\mathbb{B}(\nu)\Gamma(\nu)}u(s)v(s)\,ds.$$

Adding Eqs. (3.52) and Eq. (3.54), we obtain the required inequality. □

4 Conclusion

In this paper, we have considered Minkowski's inequality for the AB-fractional integral operator. We have also obtained some other types of integral inequalities for the AB-fractional integral operator. By the help of this work we obtained more general inequalities than in the classical cases. For possible further work, we suggest to apply the obtained inequalities to prove the existence of solutions of fractional differential equations.

Acknowledgements
The author Thabet Abdeljawad would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM), group number RG-DES-2017-01-17. All the authors are very grateful to the editorial board and the reviewers, whose comments improved the quality of the paper.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this paper as no datasets were generated during the current study.

Competing interests
The authors have no conflict of interests regarding the publication of this paper.

Authors' contributions
All the authors have equal contributions in this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 4 February 2019 Accepted: 27 March 2019 Published online: 08 April 2019

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