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Asymptotic estimates for n-width of fuzzy numbers

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Abstract

n-widths in approximation theory characterize how well one can approximate a subset by some "good" subsets of a normed linear space. Especially, n-widths of sets of $\mathbb{R}^N$ have been studied deeply. Now the following problem is posed: we know that $\mathbb{R}^N$ can be embedded in the fuzzy number space E^N. Is it then possible to define n-widths of set A in E^N and obtain asymptotic estimates for these n-widths?

In this paper, we shall introduce four n-widths of A in E^N and determine these n-widths of Zadeh's extension of diagonal matrices.

Keywords: Fuzzy numbers; Approximation theory; n-widths; Zadeh's extension; Diagonal matrix

1 Introduction

Let A and B be two subsets of a normed linear space X. One may ask: how well A can be approximated by B? In the theory of n-widths of A in X, B will be a simple subspace of X. We will consider the possibility of allowing the simple subspaces to vary within X and find the one best adjusted to A. In many cases very simple sets may approximate A in an asymptotically optimal manner. It is then possible to judge whether it is worthwhile or not to spend additional time and money in using better but more complicated subspaces. The results of n-widths of A in $\mathbb{R}^N$ may be found in [1–8].

It is well known that $\mathbb{R}^N$ can be embedded in E^N. Thus if we restrict d_s^p-metric (see Sect. 4) convergence and level convergence on $\mathbb{R}^N$, then these types of convergence both become l_p-metric (induced by $\|\cdot\|_p$) convergence. Motivated by the study of n-widths of A in $\mathbb{R}^N$, we introduce definitions of four n-widths of A in E^N. Moreover, asymptotic estimates of these n-widths of Zadeh's extension of diagonal matrices are obtained.

2 Preliminaries

2.1 Fuzzy numbers

For a fuzzy set $u:\mathbb{R}^N\to[0,1]$, suppose that:

(1) u is normal, i.e., there exists $x\in\mathbb{R}^N$ such that $u(x)=1$;
(2) u is upper semi-continuous;
(3) $\operatorname{supp} u=\operatorname{cl}\{x\in\mathbb{R}^N:u(x)>0\}$ is compact;
(4) u is fuzzy convex, i.e.,

$$u\big(\lambda x+(1-\lambda)y\big)\ge\min\big\{u(x),u(y)\big\},\quad 0\le\lambda\le1,$$

for all $x, y \in \mathbb{R}^N$. Then u is called a fuzzy number. Let E^N be the family of all fuzzy numbers. $\mathbb{R}^N$ can be embedded in $\mathbb{E}^N$, as any $u \in \mathbb{R}^N$ can be viewed as the fuzzy number

$$\hat{u}(x) = \begin{cases} 1, & u = x; \\ 0, & u \neq x. \end{cases}$$

For $u, v \in E^n$, $\alpha \in [0,1]$, the α-cut of u is defined as follows:

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R}^N : u(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1; \\ \operatorname{supp} u, & \text{if } \alpha = 0; \end{cases}$$

and the algebraic operations on E^N are defined as

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha, \quad k \in \mathbb{R}, \alpha \in [0,1].$$

If $f : \mathbb{R}^N \to \mathbb{R}^N$ is a function, we define Zadeh's extension of f by

$$\widetilde{f} : E^N \to E^N,$$
$$\widetilde{f}(u)(x) = \begin{cases} \sup_{z \in f^{-1}(x)} u(z), & \text{if } f^{-1}(x) \neq \emptyset; \\ 0, & \text{if } f^{-1}(x) = \emptyset. \end{cases}$$

Lemma 1 ([9]) *If $f : \mathbb{R}^N \to \mathbb{R}^N$ is continuous, then $\widetilde{f}$ is a well-defined function and*

$$\bigl[\widetilde{f}(u)\bigr]^\alpha = f\bigl([u]^\alpha\bigr), \quad \forall \alpha \in [0,1], \forall u \in E^N.$$

2.2 *n*-Widths of diagonal matrix

Definition 1 ([10]) Let $(X, \|\cdot\|)$ be a normed linear space, and $A \subseteq X$.

(1) The Kolmogorov n-width of A in X is defined by

$$d_n(A;X) = \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|,$$

where the left-most infimum is taken over all n-dimensional subspace X_n of X.

(2) The Bernstein n-width of A in X is defined by

$$\begin{aligned} b_n(A;X) &= \sup_{X_{n+1}} \sup\bigl\{\lambda : \lambda S(X_{n+1}) \subseteq A\bigr\} \\ &= \sup_{X_{n+1}} \inf_{x \in \partial(A \cap X_{n+1})} \|x\|, \end{aligned}$$

where X_{n+1} is any $(n+1)$-dimensional subspace of X, and $S(X_{n+1})$ is the unit ball of X_{n+1}.

(3) The Gelfand n-width of A in X is defined as

$$d^n(A;X) = \inf_{L^n} \sup_{x \in A \cap L^n} \|x\|,$$

where the infimum is taken over all subspaces L^n of X of codimension n.

(4) The linear n-width is given by

$$\delta_n(A;X) = \inf_{P_n(A)} \sup_{x\in A} \|x - P_n(x)\|,$$

where the infimum is taken over all continuous linear operators P_n of X into X of rank n.

Lemma 2 ([11]) *Let X_{n+1} be any $(n+1)$-dimensional subspace of a normed linear space $(X, \|\cdot\|)$, and let $S(X_{n+1})$ denote the unit ball of X_{n+1}. Then*

$$d_k\bigl(S(X_{n+1});X\bigr) = 1, \quad k = 1,2,\dots,n.$$

Let l_p^N be the N-dimensional normed spaces of $x = (x_1,\dots,x_N) \in \mathbb{R}^N$, with the normed

$$\|x\|_p = \begin{cases} (\sum_{i=1}^N |x_i|^p)^{1/p}, & 1 \le p < \infty, \\ \max_{1\le i\le N} |x_i|, & p = \infty. \end{cases}$$

Let $D = \operatorname{diag}\{D_1,\dots,D_N\}$ be an $N \times N$ diagonal matrix. Without loss of generality, we assume that

$$D_1 \ge D_2 \ge \cdots \ge D_N > 0.$$

n-widths of $\mathfrak{D}_p = \{Dx : \|x\|_p \le 1\}$ can be found in [1, 2, 10].

Theorem A ([2, 10]) *For $1 \le p \le \infty$,*

$$d_n\bigl(\mathfrak{D}_p;l_p^N\bigr) = d^n\bigl(\mathfrak{D}_p;l_p^N\bigr) = b_n\bigl(\mathfrak{D}_p;l_p^N\bigr) = \delta_n\bigl(\mathfrak{D}_p;l_p^N\bigr) = D_{n+1}.$$

Theorem B ([1]) *Given $1 \le q \le p \le \infty$. Let $1/r = 1/q - 1/p$. Then*

$$d_n\bigl(\mathfrak{D}_p;l_q^N\bigr) = d^n\bigl(\mathfrak{D}_p;l_q^N\bigr) = b_n\bigl(\mathfrak{D}_p;l_q^N\bigr) = \delta_n\bigl(\mathfrak{D}_p;l_q^N\bigr) = \left(\sum_{k=n+1}^N D_k^r\right)^{1/r}.$$

3 *n*-Widths of fuzzy numbers

The following notation will be used throughout this paper. Let X_n be an n-dimensional subspace of $\mathbb{R}^N$, L^n be subspaces of $\mathbb{R}^N$ of codimension n. $S(X_n)$ denotes the unit ball of X_n. Set

$$\widetilde{X_n} = \bigl\{u : u \in E^N, [u]^0 \subseteq X_n\bigr\},$$
$$\widetilde{L^n} = \bigl\{u : u \in E^N, [u]^0 \subseteq L^n\bigr\},$$
$$S(\widetilde{X_{n+1}}) = \bigl\{u : d(u,\hat{0}) \le 1, u \in \widetilde{X_{n+1}}\bigr\}.$$

Let $\widetilde{P_n}$ be Zadeh's extension of the continuous linear operators P_n of $\mathbb{R}^N$ into $\mathbb{R}^N$ of rank n.

Definition 2 Let (E^N, d) be a metric space, and $A \subseteq E^N$.

(1) The Kolmogorov n-width of A in E^N is defined by

$$d_n\left(A;E^N\right) = \inf_{\widetilde{X_n}} \sup_{u\in A} \inf_{v\in\widetilde{X_n}} d(u,v),$$

where the left-most infimum is taken over all $\widetilde{X_n} \subseteq \mathbb{E}^N$.

(2) The Bernstein n-width of A in E^N is defined by

$$b_n\left(A;E^N\right) = \sup_{\widetilde{X_{n+1}}} \sup\left\{\lambda : \lambda \geq 0, \lambda S(\widetilde{X_{n+1}}) \subseteq A\right\}.$$

(3) The Gelfand n-width of A in E^N is defined as

$$d^n\left(A;E^N\right) = \inf_{\widetilde{L^n}} \sup_{u\in A\cap\widetilde{L^n}} d(u,\hat{0}),$$

where the infimum is taken over all subspaces $\widetilde{L^n}$ of E^N.

(4) The linear n-width of A in E^N is given by

$$\delta_n\left(A;E^N\right) = \inf_{\widetilde{P_n}} \sup_{u\in A} d\left(u,\widetilde{P_n}(u)\right),$$

where the infimum is taken over all $\widetilde{P_n}$.

Proposition 1 *Let (E^N,d) be a metric space, and $A \subseteq E^N$.*

(i) $\delta_n(A;E^N) \geq d_n(A;E^N)$.

(ii) $\delta_n(A;E^N) \geq d^n(A;E^N)$.

Proof Let $\widetilde{P_n}$ be Zadeh's extension of the continuous linear operators P_n of $\mathbb{R}^N$ into $\mathbb{R}^N$ of rank n.

(i) From Lemma 1 and rank $P_n = n$, we know that there exists an n-dimensional subspace X_n of $\mathbb{R}^N$ subject to the following relation:

$$\left[\widetilde{P_n}(u)\right]^0 = P_n\left([u]^0\right) \subseteq X_n, \quad u \in A \subseteq E^N.$$

Then $\widetilde{P_n}(u) \in \widetilde{X_n}$, i.e., $\widetilde{P_n}(A) \subseteq \widetilde{X_n}$. By the definitions of $d_n(A;E^N)$ and $\delta_n(A;E^N)$

$$\delta_n\left(A;E^N\right) \geq d_n\left(A;E^N\right).$$

(ii) If $\widetilde{P_n}(u) = \hat{0}$, then

$$\left[\widetilde{P_n}(u)\right]^0 = P_n\left([u]^0\right) = \{0\},$$

and $[u]^0 \subseteq L^n$, i.e., $u \in \widetilde{L^n}$. Therefore

$$\sup_{u\in A} d\left(u,\widetilde{P_n}(u)\right) \geq \sup_{\substack{u\in A \\ \widetilde{P_n}(u)=0}} d(u,\hat{0}),$$

whence it follows that $\delta_n(A;E^N) \geq d^n(A;E^N)$. □

4 n-Widths of $\widetilde{D}$

We first choose a suitable metric $d(\cdot,\cdot)$ on E^N to establish a relation between $\|x-y\|_p$ and $d(u,v)$, $x,y\in l_p^N$, $u,v\subset E^N$. Then the distance between subsets of E^N can be estimated by $\|x-y\|_p$.

Let $\mathcal{K}(\mathbb{R}^N)$ be the family of nonempty compact subsets of l_p^N. If $A,B\in\mathcal{K}(\mathbb{R}^N)$, $1\le p<\infty$, the Hausdorff distance between A and B is defined by

$$d_H^p(A,B)=\max\Bigl\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_p,\sup_{b\in B}\inf_{a\in A}\|a-b\|_p\Bigr\}.$$

For $u,v\in E^N$, $1\le s<\infty$, $\alpha\in[0,1]$, we define

$$d_s^p(u,v)=\biggl(\int_0^1 d_H^p\bigl([u]^\alpha,[v]^\alpha\bigr)^s\,d\alpha\biggr)^{1/s},$$

then d_s^p is called the L_s-metric on E^N [12]. Let $L_{s,p}^N:=(E^N,d_s^p)$.

Proposition 2 ([12]) *(E^N,d_s^p) is a metric space for $1\le s,p<\infty$.*

Let $D=\operatorname{diag}\{D_1,\dots,D_N\}$, $D_1\ge D_2\ge\cdots\ge D_N>0$, be an $N\times N$ diagonal matrix, and $\widetilde{D}$ be Zadeh's extension of D.

Lemma 3 *Let $u\in E^N$, $k\in\mathbb{R}$, $1\le s<\infty$, $\alpha\in[0,1]$. Then*

$$d_s^p(\widetilde{D}u,k\hat{0})=\biggl(\int_0^1\Bigl(\sup_{a\in[u]^\alpha}\|Da\|_p\Bigr)^s\,d\alpha\biggr)^{1/s},$$

and

$$d_s^p\bigl(\widetilde{D}(ku),\hat{0}\bigr)=|k|d_s^p\bigl(\widetilde{D}(u),\hat{0}\bigr).$$

Proof Since $\operatorname{supp}\hat{0}=\{0\}$, it follows that

$$\begin{aligned}d_H^p\bigl([\widetilde{D}u]^\alpha,k[\hat{0}]^\alpha\bigr)&=d_H^p\bigl(D\bigl([u]^\alpha\bigr),k[\hat{0}]^\alpha\bigr)\\&=\max\Bigl\{\sup_{a\in[u]^\alpha}\inf_{b\in k[\hat{0}]^\alpha}\|Da-b\|_p,\sup_{b\in k[\hat{0}]^\alpha}\inf_{a\in[u]^\alpha}\|Da-b\|_p\Bigr\}\\&=\sup_{a\in[u]^\alpha}\|Da\|_p.\end{aligned}$$

Hence

$$d_s^p(\widetilde{D}u,k\hat{0})=\biggl(\int_0^1\Bigl(\sup_{a\in[u]^\alpha}\|Da\|_p\Bigr)^s\,d\alpha\biggr)^{1/s}.$$

Similarly, we can get the second equation. □

In this paper we are concerned with the estimate of n-widths of

$$\widetilde{\mathfrak{D}_{s,p}}=\bigl\{\widetilde{D}u:u\in E^N,d_s^p(u,\hat{0})\le 1\bigr\}$$

and

$$\widetilde{\mathfrak{D}_{s,p}^{-1}} = \{\widetilde{D^{-1}}u : u \in E^N, d_s^p(u,\hat{0}) \leq 1\}.$$

It is often the case, see examples in [10], that a very simple form of $b_n(A;X) = \sup_{X_{n+1}} \inf_{x\in\partial(A\cap X_{n+1})} \|x\|$ is used. We introduce a similar definition of $b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N)$ for easy computation.

Definition 3 For $1 \leq s,p < \infty$,

$$b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) = \sup_{\widetilde{X_{n+1}}} \inf_{\substack{u\in\widetilde{X_{n+1}}\\ d_s^p(u,\hat{0})=1}} d(\widetilde{D}u, \hat{0}).$$

Now we state our main results.

Theorem 1 *For* $1 \leq s,p < \infty$,

$$d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) = d^n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) = b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) = \delta_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) = D_{n+1}.$$

Theorem 2 *Given* $1 \leq s < \infty$, $1 \leq q \leq p < \infty$. *Let* $1/r = 1/q - 1/p$. *Then*

$$d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,q}^N) = d^n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,q}^N) = \delta_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,q}^N) = \left(\sum_{k=n+1}^{N} D_k^r\right)^{1/r}.$$

Remark For $1 \leq q \leq p < \infty$, Theorem 1 and 2 are obvious generalizations of Theorems A and B.

Before proving these two theorems, we need some lemmas.

Lemma 4 *For* $1 \leq s,p < \infty$,
(i) $\delta_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N)$.
(ii) $\delta_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq d^n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N)$.

Proof From Proposition 1 the following results are known:

$$\delta_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N), \qquad \delta_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq d^n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N).$$

Now we prove that $d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N)$. If $\lambda S(\widetilde{X_{n+1}}) \subseteq \widetilde{\mathfrak{D}_{s,p}}$, then from the definition of $d_n(A;E^N)$

$$d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq d_n(\lambda S(\widetilde{X_{n+1}}); L_{s,p}^N) \geq d_n(\lambda S(X_{n+1}); L_{s,p}^N). \tag{1}$$

For $\hat{a} \in \lambda S(X_{n+1})$, $\nu \in \widetilde{X_n}$, a direct computation shows that

$$d_H^p(\{a\}, [\nu]^\alpha) = \sup_{b\in[\nu]^\alpha} \|a-b\|_p \geq \inf_{b\in X_n} \|a-b\|_p$$

and

$$d_s^p(\hat{a}, v) = \left(\int_0^1 d_H^p\left(\{a\}, [v]^\alpha\right)^s d\alpha\right)^{1/s} \geq \inf_{b \in X_n} \|a - b\|_p.$$

Therefore

$$\sup_{\hat{a} \in \lambda S(X_{n+1})} \inf_{v \in \widetilde{X_{n+1}}} d_s^p(\hat{a}, v) \geq \sup_{a \in \lambda S(X_{n+1})} \inf_{b \in X_n} \|a - b\|_p,$$

which implies that

$$d_n\left(\lambda S(X_{n+1}); L_{s,p}^N\right) \geq d_n\left(\lambda S(X_{n+1}); l_p^N\right).$$

From (1) and Lemma 2

$$d_n\left(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N\right) \geq \lambda.$$

By the definition of $b_n(A; E^N)$ in Definition 2 (2), we have $d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N)$.

The proof of $d^n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N)$ is totally analogous to the proof of $d_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N) \geq b_n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,p}^N)$, here we omit it. The theorem is proved. □

Let $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, $A, B \subseteq \mathbb{R}^N$, we write $a \perp A$ if $\sum_{i=1}^N a_i x_i = 0$ for all $x = (x_1, \ldots, x_N) \in A$ and $A \perp B$ if $\sum_{i=1}^N a_i b_i = 0$ for $\forall a = (a_1, \ldots, a_N) \in A$, $b = (b_1, \ldots, b_N) \in B$. Set

$$A^\perp = \left\{ y \in \mathbb{R}^N : \sum_{i=1}^N x_i y_i = 0, \forall x = (x_1, \ldots, x_N) \in A \right\}.$$

Let $u \in E^N$, $C \subseteq E^N$, we write $u \perp C$ if $[u]^0 \perp [v]^0$, $\forall v \in C$.

Lemma 5 *For $u \in E^N$, the following properties are equivalent.*

(i) $u \perp \widetilde{X_{N-n}}$.
(ii) $u \in \widetilde{X_n}$.

Proof (i) ⇒ (ii). Since $u \perp \widetilde{X_{N-n}}$, we know that for $\forall v \in \widetilde{X_{N-n}}$, i.e., $[v]^0 \subseteq X_{N-n}$, we have $[u]^0 \perp [v]^0$.

$$[u]^0 \subset X_{N-n}^\perp.$$

Note that $X_{N-n}^\perp = X_n$. Then (ii) follows.

(ii) ⇒ (i). If $u \in \widetilde{X_n}$, i.e., $[u]^0 \subseteq X_n$, then for all $x \in [u]^0$ there is an $(N-n)$-dimensional subspace X_{N-n} such that $x \perp X_{N-n}$. Consequently, we have $[u]^0 \perp X_{N-n}$, i.e., $u \perp \widetilde{X_{N-n}}$. □

Lemma 6 *For $1 \leq s, p < \infty$, $n < N$,*

$$b_n\left(\widetilde{\mathfrak{D}_{s,p}}, L_{s,q}^N\right) d^{N-n-1}\left(\widetilde{\mathfrak{D}_{s,q}^{-1}}, L_{s,p}^N\right) = 1.$$

Proof By the definition of $d^{N-n-1}(\widetilde{\mathfrak{D}^{-1}_{s,q}}, L^N_{s,p})$ and Lemma 5, we have

$$\begin{aligned} d^{N-n-1}\big(\widetilde{\mathfrak{D}^{-1}_{s,p}}, L^N_{s,q}\big) &= \min_{\widetilde{X_{N-n-1}}} \max_{\substack{u\in \widetilde{X_{n+1}} \\ d_s^p(u,\hat{0})\le 1}} d_s^q\big(\widetilde{D^{-1}}u, \hat{0}\big) \\ &= \min_{\widetilde{X_{N-n-1}}} \max_{\substack{u\perp \widetilde{X_{N-n-1}} \\ u\ne\hat{0}}} \frac{d_s^q(\widetilde{D^{-1}}u,\hat{0})}{d_s^p(u,\hat{0})} \\ &= \min_{\widetilde{X_{N-n-1}}} \max_{\substack{u\perp \widetilde{X_{N-n-1}} \\ u\ne\hat{0}}} \left[\frac{d_s^p(u,\hat{0})}{d_s^q(\widetilde{D^{-1}}u,\hat{0})}\right]^{-1}. \end{aligned}$$

Setting $v = \widetilde{D^{-1}}u$, by Lemma 1 we have

$$[v]^\alpha = D^{-1}[u]^\alpha, \quad \forall\alpha\in[0,1],$$

and

$$[u]^\alpha = DD^{-1}[u]^\alpha = D\big[\widetilde{D^{-1}}u\big]^\alpha = D[v]^\alpha = [\widetilde{D}v]^\alpha, \quad \forall\alpha\in[0,1].$$

Since D is invertible, hence $v\perp \widetilde{X_{N-n-1}}$. Now

$$\begin{aligned} d^{N-n-1}\big(\widetilde{\mathfrak{D}^{-1}_{s,q}}, L^N_{s,p}\big) &= \min_{\widetilde{X_{N-n-1}}} \max_{\substack{v\perp \widetilde{X_{N-n-1}} \\ v\ne\hat{0}}} \left[\frac{d_{s,p}(\widetilde{D}v,\hat{0})}{d_s^q(v,\hat{0})}\right]^{-1} \\ &= \left[\max_{\widetilde{X_{n+1}}} \min_{\substack{v\in \widetilde{X_{n+1}} \\ v\ne\hat{0}}} \frac{d_s^p(\widetilde{D}v,\hat{0})}{d_s^q(v,\hat{0})}\right]^{-1} \\ &= \big[b_n\big(\widetilde{\mathfrak{D}_{s,p}}, L^N_{s,q}\big)\big]^{-1}. \end{aligned}$$

□

Proof of Theorem 1 Let $P_n = \operatorname{diag}(D_1,\dots,D_n,0,\dots,0)$. For any $u\in E^N$,

$$\begin{aligned} d_H^p\big([\widetilde{D}u]^\alpha\big), [\widetilde{P_n}u]^\alpha) &= d_H^p\big(D\big([u]^\alpha\big), P_n\big([u]^\alpha\big)\big) \\ &= \max\Big\{\sup_{a\in[u]^\alpha}\inf_{b\in[u]^\alpha}\|Da-P_nb\|_p, \sup_{b\in[u]^\alpha}\inf_{a\in[u]^\alpha}\|Da-P_nb\|_p\Big\} \\ &= \max_{a\in[u]^\alpha}\big\|(D-P_n)a\big\|_p. \end{aligned} \tag{2}$$

Then

$$\begin{aligned} \delta_n\big(\widetilde{\mathfrak{D}_{s,p}}; L^N_{s,p}\big) &\le \max_{d_s^p(u,\hat{0})\le 1} d_s^p(\widetilde{D}u, \widetilde{P_n}u) \\ &= \max_{u\ne\hat{0}} \frac{d_s^p(\widetilde{D}u,\widetilde{P_n}u)}{d_s^p(u,\hat{0})} \\ &= \max_{u\ne\hat{0}} \frac{(\int_0^1(\max_{a\in[u]^\alpha}\|(D-P_n)a\|_p)^s\,d\alpha)^{1/s}}{(\int_0^1(\max_{a\in[u]^\alpha}\|a\|_p)^s\,d\alpha)^{1/s}} \\ &= \max_{u\ne\hat{0}} \frac{(\int_0^1(\max_{a\in[u]^\alpha}(\sum_{i=n+1}^N|D_ia_i|^p)^{1/p})^s\,d\alpha)^{1/s}}{(\int_0^1(\max_{a\in[u]^\alpha}\|a\|_p)^s\,d\alpha)^{1/s}} \end{aligned}$$

$$\begin{aligned}&\le \max_{u\neq\hat{0}} \frac{D_{n+1}(\int_0^1(\max_{a\in[u]^\alpha}(\sum_{i=n+1}^N |a_i|^p)^{1/p})^s\,d\alpha)^{1/s}}{(\int_0^1(\max_{a\in[u]^\alpha}\|a\|_p)^s\,d\alpha)^{1/s}}\\ &\le D_{n+1}.\end{aligned}$$

In a similar way,

$$\delta_n\bigl(\widetilde{\mathfrak{D}^{-1}_{s,p}};L^N_{s,p}\bigr) \le 1/D_{n+1}.$$

From Lemma 6

$$\begin{aligned}b_n\bigl(\widetilde{\mathfrak{D}_{s,p}};L^N_{s,p}\bigr) &= \bigl(d^{M-n-1}\bigl(\widetilde{\mathfrak{D}^{-1}_{s,p}};L^N_{s,p}\bigr)\bigr)^{-1}\\ &\ge \bigl(\delta_n\bigl(\widetilde{\mathfrak{D}^{-1}_{s,p}};L^N_{s,p}\bigr)\bigr)^{-1}.\end{aligned}$$

Thus

$$b_n\bigl(\widetilde{\mathfrak{D}_{s,p}};L^N_{s,p}\bigr) \ge D_{n+1}.$$

By Lemma 4, we prove that these four n-widths equal D_{n+1}. □

Let $1 \le p,q < \infty$, and $1/p + 1/p' = 1/q + 1/q' = 1$. We use the notation $x'(x) = \langle x,x'\rangle$ for $x \in l^N_p$, $x' \in l^N_{p'}$, and similarly for l^N_q. Diagonal matrix $D : l^N_p \to l^N_q$ has an adjoint $D' : l^N_{q'} \to l^N_{p'}$, defined by $\langle Dx,y'\rangle = \langle x,D'y'\rangle$ for $x \in l^N_p$ and $y' \in l^N_{q'}$. It is well known that $D = D'$. Let L^n be subspaces of l^N_p of codimension n and $\widetilde{L^n} = \{u \in E^N : [u]^0 \subseteq L^n\}$, set

$$\begin{aligned}L^n_\perp &= \bigl\{x' : x' \in l^N_{p'}, \langle x,x'\rangle = 0, \text{all } x \in L^n\bigr\},\\ \widetilde{L^n_\perp} &= \bigl\{v : v \in E^N, [v]^0 \subset L^n_\perp\bigr\}.\end{aligned}$$

It is well known that $\dim L^n_\perp = n$.

Lemma 7 *Let* $1 \le s,p,q < \infty$, *and* $1/p + 1/p' = 1/q + 1/q' = 1$. *Then*

$$d_n\bigl(\widetilde{\mathfrak{D}_{s,p}};L^N_{s,q}\bigr) \ge d^n\bigl(\widetilde{\mathfrak{D}_{s,q'}};L^N_{s,p'}\bigr).$$

Proof Let $\widetilde{A_p} = \{u \in E^N : d^p_s(u,\hat{0}) \le 1\}$. Then

$$\begin{aligned}\sup_{u\in\widetilde{A_p}\cap\widetilde{L^n}}\ \sup_{a\in[u]^0}\|Da\|_q &= \sup_{u\in\widetilde{A_p}\cap\widetilde{L^n}}\ \sup_{a\in[u]^0}\ \sup_{\|b'\|_{q'}\le 1}\bigl\langle Da,b'\bigr\rangle\\ &\le \sup_{u\in\widetilde{A_p}\cap\widetilde{L^n}}\ \sup_{a\in[u]^0}\ \sup_{v\in\widetilde{A_{q'}}}\ \sup_{b'\in[v]^\alpha}\bigl\langle Da,b'\bigr\rangle\\ &= \sup_{v\in\widetilde{A_{q'}}}\ \sup_{u\in\widetilde{A_p}\cap\widetilde{L^n}}\ \sup_{b'\in[v]^\alpha}\ \sup_{a\in[u]^0}\bigl\langle a,D'b'\bigr\rangle.\end{aligned} \tag{3}$$

Let $u \in \widetilde{A_p} \cap \widetilde{L^n}$ and $a' \in L^n_\perp$. Then $\langle a,a'\rangle = 0$, $\forall a \in [u]^0$. Hence

$$\bigl\langle a,D'b'\bigr\rangle = \bigl\langle a,D'b'-a'\bigr\rangle \le \|a\|_p\bigl\|D'b'-a'\bigr\|_{p'} \le \bigl\|D'b'-a'\bigr\|_{p'}.$$

For $u' \in \widetilde{L^n_\perp}$, we have

$$\sup_{b' \in [v]^\alpha} \sup_{a \in [u]^0} \langle a, D'b' \rangle \le \sup_{b' \in [v]^\alpha} \inf_{a' \in [u']^\alpha} \| Db' - a' \|_{p'}. \tag{4}$$

Following (3), (4), and $D = D'$

$$\begin{aligned} \sup_{v \in \widetilde{A_{q'}}} \sup_{u \in \widetilde{A_p} \cap \widetilde{L^n}} \sup_{b' \in [v]^\alpha} \sup_{a \in [u]^0} \langle a, D'b' \rangle &\le \sup_{v \in \widetilde{A_{q'}}} \inf_{u' \in \widetilde{L^n_\perp}} \sup_{b' \in [v]^\alpha} \inf_{a' \in [u']^\alpha} \| Db' - a' \|_{p'} \\ &\le \sup_{v \in \widetilde{A_{q'}}} \inf_{u' \in \widetilde{L^n_\perp}} d_s^{p'} (u', \widetilde{D}v). \end{aligned} \tag{5}$$

Combining Lemma 3, (3), and (5)

$$\sup_{u \in \widetilde{A_p} \cap \widetilde{L^n}} d_s^q (\widetilde{D}u, \hat{0}) \le \sup_{v \in \widetilde{A_{q'}}} \inf_{u' \in \widetilde{L^n_\perp}} d_s^{p'} (u', \widetilde{D}v),$$

then taking the infimum over $\widetilde{L^n}$, we have the result. □

Proof of Theorem 2 We first prove that $\delta_n(\widetilde{\mathfrak{D}_{s,p}}; L^N_{s,q}) \le (\sum_{k=n+1}^N D_k^r)^{1/r}$.

Let $P_n = \operatorname{diag}(D_1, \dots, D_n, 0, \dots, 0)$. For any $u \in E^N$, as (2) in the proof of Theorem 1

$$\begin{aligned} d_H^q ([\widetilde{D}u]^\alpha), [\widetilde{P}_n u]^\alpha) &= \sup_{a \in [u]^\alpha} \| (D - P_n) a \|_q \\ &= \sup_{a \in [u]^\alpha} \left(\sum_{k=n+1}^N |D_k a_k|^q \right)^{1/q}. \end{aligned}$$

From $1/r = 1/q - 1/p$ and Hölder's inequality

$$\left(\sum_{k=n+1}^N |D_k a_k|^q \right)^{1/q} \le \left(\sum_{k=n+1}^N |D_k|^r \right)^{1/r} \left(\sum_{k=n+1}^N |a_k|^p \right)^{1/p},$$

we have

$$\begin{aligned} d_H^q ([\widetilde{D}u]^\alpha), [\widetilde{P}_n u]^\alpha) &\le \sup_{a \in [u]^\alpha} \left(\sum_{k=n+1}^N |D_k|^r \right)^{1/r} \left(\sum_{k=n+1}^N |a_k|^p \right)^{1/p} \\ &\le \left(\sum_{k=n+1}^N |D_k|^r \right)^{1/r} \sup_{a \in [u]^\alpha} \left(\sum_{k=1}^N |a_k|^p \right)^{1/p} \\ &= \left(\sum_{k=n+1}^N |D_k|^r \right)^{1/r} d_H^p ([u]^\alpha, [\hat{0}]^\alpha). \end{aligned}$$

Then

$$\begin{aligned} \delta_n (\widetilde{\mathfrak{D}_{s,p}}; L^N_{s,q}) &\le \max_{d_s^p(u,\hat{0}) \le 1} d_s^q ([\widetilde{D}u]^\alpha), [\widetilde{P}_n u]^\alpha) \\ &\le \max_{d_s^p(u,\hat{0}) \le 1} \left(\int_0^1 d_H^q (D([u]^\alpha), P_n([u]^\alpha))^s \, d\alpha \right)^{1/s} \end{aligned}$$

$$
\begin{aligned}
&\leq \max_{d_s^p(u,\hat{0})\leq 1} \left(\sum_{k=n+1}^{N} |D_k|^r \right)^{1/r} d_s^p(u,\hat{0}) \\
&\leq \left(\sum_{k=n+1}^{N} |D_k|^r \right)^{1/r}. \qquad (6)
\end{aligned}
$$

Now we are to prove $d^n(\widetilde{\mathfrak{D}_{s,p}}; L_{s,q}^N) \geq (\sum_{k=n+1}^{N} |D_k|^r)^{1/r}$. From Lemma 3

$$
\begin{aligned}
d_H^q\big([\widetilde{D}u]^\alpha), [\hat{0}]^\alpha\big) &= d_H^q\big(D\big([u]^\alpha\big), [\hat{0}]^\alpha\big) \\
&= \max_{a\in[u]^\alpha} \|Da\|_q \\
&= \max_{a\in[u]^\alpha} \left(\sum_{k=1}^{N} |D_k a_k|^q \right)^{1/q}
\end{aligned}
$$

and

$$
d_H^p\big([u]^\alpha), [\hat{0}]^\alpha\big) = \max_{a\in[u]^\alpha} \left(\sum_{k=1}^{N} |a_k|^p \right)^{1/p}.
$$

By the definition of $d^n(\widetilde{\mathfrak{D}_{s,p}}, L_{s,q}^N)$ and Theorem B

$$
d^n\big(\widetilde{\mathfrak{D}_{s,p}}, L_{s,q}^N\big) \geq d^n\big(\mathfrak{D}_p, L_{s,q}^N\big) = d^n\big(\mathfrak{D}_p, l_q^N\big) = \left(\sum_{k=n+1}^{N} D_k^r \right)^{1/r}. \qquad (7)
$$

Combine (6), (7), and Proposition 1(ii),

$$
\delta_n\big(\widetilde{\mathfrak{D}_{s,p}}; L_{s,q}^N\big) = d^n\big(\widetilde{\mathfrak{D}_{s,p}}, L_{s,q}^N\big) = \left(\sum_{k=n+1}^{N} D_k^r \right)^{1/r}.
$$

Similarly, we can have

$$
\delta_n\big(\widetilde{\mathfrak{D}_{s,q'}}; L_{s,p'}^N\big) = \left(\sum_{k=n+1}^{N} D_k^r \right)^{1/r}
$$

and

$$
d^n\big(\widetilde{\mathfrak{D}_{s,q'}}, L_{s,p'}^N\big) \geq \left(\sum_{k=n+1}^{N} D_k^r \right)^{1/r}.
$$

By Proposition 1(i) and Lemma 7

$$
d_n\big(\widetilde{\mathfrak{D}_{s,p}}; L_{s,q}^N\big) = \left(\sum_{k=n+1}^{N} D_k^r \right)^{1/r}.
$$

The theorem is proved. □

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